Chapter 17

Interest-Rate Models

Learning Objectives

After reading this chapter, you will understand

- what an interest-rate model is
- how an interest-rate model is represented mathematically
- the characteristics of an interest-rate model: drift, volatility, and mean reversion
- what a one-factor interest-rate model is
- the difference between an arbitrage-free model and an equilibrium model
- the different types of arbitrage-free models and why they are used in practice
- the difference between a normal model and a lognormal model
- the empirical evidence on interest-rate changes
- considerations in selecting an interest-rate model
- how to calculate historical volatility

In implementing bond portfolio strategies, there are two important activities that a manager will undertake. One will be the determination of whether the bonds that are purchase and sale candidates are fairly priced. The same applies to any interest-rate derivatives that the manager may want to employ to control interest-rate risk or potentially enhance returns. Second, a manager will want to assess the performance of a portfolio over realistic future interest-rate scenarios. For both of these activities, the manager will have to rely on an interest-rate model.

Future interest rates are, of course, unknown. The description of the uncertainty about future interest rates is mathematically described by an interest-rate model. More specifically, an **interest-rate model** is a probabilistic description of how interest rates can change over time. In this chapter, we provide an overview of interest-rate models. Our focus will be on *nominal* interest rates rather than *real* interest rates (i.e., the nominal interest rate reduced by the inflation rate). At the end of this chapter, we will see how interest-rate volatility is computed using historical data.

Mathematical Description of One-Factor Interest-Rate Models

Interest-rate models must incorporate statistical properties of interest-rate movements. These properties are (1) drift, (2) volatility, and (3) mean reversion. We will describe each property next. The commonly used mathematical tool for describing the movement of interest rates that can incorporate these properties is **stochastic differential equations** (SDEs). A rigorous treatment of interest-rate modeling requires an understanding of this specialized topic in mathematics. Because SDEs are typically not covered in finance courses (except in financial engineering programs), we provide only the basic elements of the subject here. It is also worth noting that SDEs are used in the pricing of options, the most well-known model being the Black-Scholes model that we will describe in Chapter 30.

The most common interest-rate model used to describe the behavior of interest rates assumes that short-term interest rates follow some statistical process and that other interest rates in the term structure are related to short-term rates. The short-term interest rate (i.e., short rate) is the only one that is assumed to drive the rates of all other maturities. Hence, these models are referred to as **one-factor models**. The other rates are not randomly determined once the short rate is specified. Using arbitrage arguments, the rates for all other maturities are determined.

Multi-factor models have also been proposed in the literature. The most common multi-factor model is a two-factor model where a long-term rate is the second factor. In practice, however, one-factor models are used because of the difficulty of applying even a two-factor model. The high correlation between rate changes for different maturities provides some support for the use of a one-factor model. Empirical evidence also supports the position that a level shift in interest rates accounts for the major portion of the change in the yield curve.[1] Consequently, our focus is on one-factor models.

Although the value of the short rate at some future time is uncertain, the pattern by which it changes over time can be assumed. In statistical terminology, this pattern or behavior is called a **stochastic process.** Thus, describing the dynamics of the short rate means specifying the stochastic process that describes the movement of the short rate. It is assumed that the short rate is a continuous random variable and therefore the stochastic process used is a **continuous-time stochastic process.**

There are different types of continuous-time stochastic processes, and we describe those used in interest-rate modeling next. In all of these models, because time is a continuous variable, the letter d is used to denote the "change in" some variable. Specifically, in the models below, we will let

r = the short rate and, therefore, dr denotes the change in the short rate
t = time, and therefore, dt denotes the change in time or equivalently the length of the
time interval (dt is a very small interval of time)
z = a random term and dz denotes a random process

A Basic Continuous-Time Stochastic Process

Let's start with a basic continuous-time stochastic process for describing the dynamics of the short rate given by

$dr = bdt + \sigma\, dz$ (17.1)

where dr, dt, and dz were defined above and

σ = standard deviation of the changes in the short rate
b = expected direction of rate change

The expected direction of the change in the short rate (b) is called the **drift term** and σ is called the **volatility term**.[2]

In words, equation (17.1) says that the change in the short rate (dr) over the time interval (dt) depends on

1. the expected direction of the change in the short rate (b) and
2. a random process (dz) that is affected by volatility

The random nature of the change in the short rate comes from the random process dz.

The assumptions are that

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[1] Note that a one-factor model should not be used in valuing financial instruments where the payoff depends on the shape of the spot rate curve rather than simply the level of interest rates. Examples would be dual index floaters and yield curve options.

[2] A special case of the SDE described by equation (17.1) when b is equal to zero and σ is equal to one is called a standard Wiener process and is the building block for constructing models in continuous time.

1. the random term z follows a normal distribution with a mean of zero and a standard deviation of one (i.e., is a standardized normal distribution)
2. the change in the short rate is proportional to the value of the random term, which depends on the standard deviation of the change in the short rate
3. the change in the short rate for any two different short intervals of time are independent

Based on the assumptions above, important properties can be shown for equation (17.1). The expected value of the change in the short rate is equal to b, the drift term. Notice that in the special case where the drift term is equal to zero, equation (17.1) tells us that expected value of the change in the short rate is zero. This means that the expected value for the short rate is its current value. Note that in the special case where the drift term is zero and the variance is one, it can be shown that the variance of the change in the short rate over some interval of length T is equal to T and, therefore, the standard deviation is the square root of T.

Itô Process

Notice that in equation (17.1) neither the drift term nor the standard deviation of the change in the short rate depend on either the level of the short rate and time. So, for example, suppose the current short rate is 3%, then the SDE given by equation (17.1) assumes that b is the same if the current short rate is 12%. Economic reasons might suggest that the expected direction of the rate change will depend on the level of the current short rate. The same is true for σ.

We can change the dynamics of the drift term and the dynamics of the volatility term by allowing these two parameters to depend on the level of the short rate and/or time. We can denote that the drift term depends on both the level of the short rate and time by $b(r,t)$.

Similarly, we can denote the volatility term by $\sigma(r,t)$. We can then write

$$dr = b(r,t)dt + \sigma(r,t)dz \quad (17.2)$$

The continuous-time stochastic model given by equation (17.2) is called an **Itô process**.

Specifying the Dynamics of the Drift Term

In specifying the dynamics of the drift term, one can specify that the drift term depends on the level of the short rate by assuming it follows a **mean-reversion process**. By mean reversion, it is meant that some long-run stable mean value for the short rate is assumed.

We will denote this value by $\bar{r}$. So, if r is greater than $\bar{r}$, the direction of change in the short rate will move down in the direction of the long-run stable value and vice versa. However, in specifying the mean-reversion process, it is necessary to indicate the speed at which the short rate will move or converge to the long-run stable mean value. This parameter is called the **speed of adjustment** and we will denote it by α. Thus, the mean-reversion process that specifies the dynamics of the drift term is

$$b(r,t) = -\alpha(r - \bar{r}) \quad (17.3)$$

Specifying the Dynamics of the Volatility Term

There have been several formulations of the dynamics of the volatility term. If volatility is not assumed to depend on time, then $\sigma(r,t) = \sigma(r)$. In general, the dynamics of the volatility term can be specified as follows:

$$\sigma r^{\gamma} dz \quad (17.4)$$

where γ is equal to the **constant elasticity of variance**. Equation (17.4) is called **the constant elasticity of variance model** (CEV model). The CEV model allows us to distinguish between the different specifications of the dynamics of the volatility term for the various interest-rate models suggested by researchers.

Let's look at three cases for γ: 0, 1, and 1/2. Substituting these values for γ into equation (17.4) we get the following models identified by the researchers who first proposed them:

$\gamma = 0$: $\sigma(r,t) = \sigma$	Vasicek specification[3]
$\gamma = 1$: $\sigma(r,t) = \sigma r$	Dothan specification[4]
$\gamma = ½$: $\sigma(r,t) = \sigma\sqrt{r}$	Cox-Ingersoll-Ross specification[5]

In the Vasicek specification, volatility is independent of the level of the short rate as in equation (17.1) and is referred to as the **normal model**. In the normal model, it is possible for negative interest rates to be generated. In the Dothan specification, volatility is proportional to the short rate. This model is referred to as the **proportional volatility model**. The Cox-Ingersoll-Ross (CIR) specification, referred to for obvious reasons as the **square-root model**, makes the volatility proportional to the square root of the short rate. Negative interest rates are not possible in the square-root model.

One can combine the dynamics of the drift term and volatility term to create the following commonly used interest-rate model:

$dr = -a(r - \bar{r})dt + a\sqrt{r}\ dz$ (17.5)

Notice that this model specifies a mean-reversion process for the drift term and the square-root model for volatility. The model given by equation (17.5) is referred to as the **mean-reverting square-root model**.

Arbitrage-Free Versus Equilibrium Models

Interest-rate models fall into two general categories: arbitrage-free models and equilibrium models. We describe both in this section.

Arbitrage-Free Models

In arbitrage-free models, also referred to as **no-arbitrage models**, the analysis begins with the observed market price of a set of financial instruments. The financial instruments can include cash market instruments and interest-rate derivatives, and they are referred to as the **benchmark instruments** or **reference set**. The underlying assumption is that the benchmark instruments are fairly priced. A random process for the generation of the term structure is assumed. The random process assumes a drift term for interest rates and volatility of interest rates. Based on the random process and the assumed value for the parameter that represents the drift term, a computational procedure is used to calculate the term structure of interest rates (i.e., the spot rate curve) such that the valuation process generates the observed market prices for the benchmark instruments. The model is referred to as arbitrage-free because it matches the observed prices of the benchmark instruments. In other words, one cannot realize an arbitrage profit by pursuing a strategy based on the value of the securities generated by the model and the observed market price. Non-benchmark instruments are then valued using the term structure of interest rates estimated and the volatility assumed.

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[3] Oldrich A. Vasicek, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics* (1977), pp. 177–188.
[4] L. Uri Dothan, "On the Term Structure of Interest Rates," *Journal of Financial Economics* (1978), pp. 59–69.
[5] John C. Cox, Jonathan E. Ingersoll, Jr, and Stephen A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica* (1985), pp. 385–407.

We will describe how this is done in the next chapter, where we will start with the price of benchmark bonds, generate a spot rate curve that matches the market prices of the benchmark bonds, and then use the model to generate the theoretical price of non-benchmark bonds.

The arbitrage-free model is also used to value certain derivatives (options, caps, floors, and swaptions) using a consistent framework for valuing cash market instruments. In Chapter 30, we will see how the arbitrage-free model is used to value option-type derivatives.

The most popular arbitrage-free interest-rate models used for valuation are:[6]

- the Ho-Lee model
- the Hull-White model
- the Kalotay-Williams-Fabozzi model
- the Black-Karasinski model
- the Black-Derman-Toy model
- the Heath-Jarrow-Morton model

The first arbitrage-free interest-rate model was introduced by Ho and Lee in 1986.[7] In the Ho-Lee model, there is no mean reversion and volatility is independent of the level of the short rate. That is, it is a normal model [i.e., γ = 0 in equation (17.4)]. The Hull-White model is also a normal model.[8] Unlike the Ho-Lee model, however, it allows for mean reversion. Thus, the Hull-White model is the first arbitrage-free, mean-reverting normal model.

The last three models just listed are lognormal models. In the Kalotay-Williams-Fabozzi (KWF) model,[9] changes in the short rate are modeled by modeling the natural logarithm of r ; no allowance for mean reversion is considered in the model. It is this model that will be used in the next chapter to value bonds with embedded options. The Black-Karasinski model[10] is a generalization of the KWF model by allowing for mean reversion. That is, the Black-Karasinski model is the logarithmic extension of the KWF model in the same way that the Hull-White model is the normal model extension of the Ho-Lee model.

The Black-Derman-Toy (BDT) model[11] allows for mean reversion. However, unlike the Black-Karasinski model, mean reversion is endogenous to the model. The mean reversion in the BDT model is determined by market conditions. The Heath-Jarrow-Morton (HJM) model is a general continuous time, multi-factor model.[12] The HJM model has received considerable attention in the industry as well as in the finance literature. Many other no-arbitrage models are shown to be special cases of the HJM model. The HJM model does not require assumptions about investor

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6 For a more detailed discussion including a discussion of the solution to these models, see Gerald W. Buetow, Frank J. Fabozzi, and James Sochacki, "A Review of No Arbitrage Interest Rate Models," Chapter 3 in Frank J. Fabozzi (ed.), *Interest Rate, Term Structure, and Valuation Modeling* (Hoboken, NJ: John Wiley & Sons, 2002).

7 Thomas Ho and Sang Lee,"Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance* (1986), pp. 1011–1029.

8 John Hull and Alan White, "Pricing Interest Rate Derivative Securities," *Review of Financial Studies* (1990), pp. 573–592.

9 Andrew Kalotay, George Williams, and Frank J. Fabozzi, "A Model for the Valuation of Bonds and Embedded Options," *Financial Analyst Journal* (May–June 1993), pp. 35–46.

10 Fisher Black and Piotr Karasinski, "Bond and Option Pricing When Short Rates Are Lognormal," *Financial Analyst Journal* (July–August 1991), pp. 52–59.

11 Fischer Black, Emanuel Derman, and William Toy, "A One Factor Model of Interest Rates and Its Application to the Treasury Bond Options," *Financial Analyst Journal* (January–February 1990), pp. 33–39.

12 David Heath, Robert A. Jarrow, and Andrew J. Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60 (1992), pp. 77–105. The Brace-Gatarek-Musiela model is a particular implementation of the HJM model, which corresponds to a specific choice of the volatility term: Alan Brace, Dariusz Gatarek, and Marcek Musiela, "The Market Model of Interest Rate Dynamics," *Mathematical Finance*, 7 (1997), pp. 127–155.

preferences but instead only requires a description of the volatility structure of forward interest rates. A special case of the one-factor HJM model is derived by Jeffrey.[13]

Equilibrium Models

A fair characterization of arbitrage-free models is that they allow one to interpolate the term structure of interest rates from a set of observed market prices at one point in time assuming that one can rely on the market prices used. **Equilibrium models**, however, are models that seek to describe the dynamics of the term structure using fundamental economic variables that are assumed to affect the interest-rate process. In the modeling process, restrictions are imposed allowing for the derivation of closed-form solutions for equilibrium prices of bonds and interest-rate derivatives. In these models, (1) a functional form of the interest-rate volatility is assumed, and (2) how the drift moves up and down over time is assumed.

In characterizing the difference between arbitrage-free and equilibrium models, one can think of the distinction being whether the model is designed to be consistent with any initial term structure, or whether the parameterization implies a particular family of term structure of interest rates. Arbitrage-free models have the deficiency that the initial term structure is an input rather than being explained by the model. Basically, equilibrium models and arbitrage models are seeking to do different things.

Although there have been many developments in equilibrium models, the best known models are the Vasicek and CIR models discussed previously and the Brennan and Schwartz,[14] and Longstaff and Schwartz models.[15] To implement these models, estimates of the parameters of the assumed interest-rate process are needed, including the parameters of the volatility function for interest rates. These estimated parameters are typically obtained using econometric techniques using historical yield curves without regard to how the final model matches any market prices.

In practice, there are two concerns with implementing and using equilibrium models. First, many economic theories start with an assumption about the class of utility functions to describe how investors make choices. Equilibrium models are no exception: the model builder must specify the assumed class of utility functions. Second, as noted previously, these models are not calibrated to the market so that the prices obtained from the model can lead to arbitrage opportunities in the current term structure.[16] These models are such that volatility is an input into the model rather than output that can be extracted from observed prices for financial instruments.

Empirical Evidence on Interest-Rate Changes

Now that we are familiar with the different types of interest-rate models, let’s look at empirical evidence regarding the historical movement of interest rates. Our motivation for doing so is to help in assessing the various arbitrage-free interest-rate models. Specifically, in our review of interest-rate models, we encountered the following issues:

1. The choice between normal models (i.e., volatility is independent of the level of interest rates) and logarithm models.
2. If interest rates are highly unlikely to be negative, then interest-rate models that allow for negative rates may be less suitable as a description of the interest-rate process.

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13 Andrew Jeffrey, “Single Factor Heath-Jarrow-Morton Term Structure Models Based on Spot Interest Rate Dynamics,” *Journal of Financial and Quantitative Analysis*, 30 (1995), pp. 619–642.

14 Michael Brennan and Eduardo Schwartz, “A Continuous Time Approach to the Pricing of Bonds,” *Journal of Banking and Finance* (1979), pp. 133–155; “An Equilibrium Model of Bond Pricing and a Test of Market Efficiency,” *Journal of Financial and Quantitative Analysis*, 1982, pp. 301–329.

15 Francis Longstaff and Eduardo Schwartz, “Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model,” *Journal of Finance* (1992), pp. 1259–1282.

16 To deal with this, Dybvig has suggested an approach that has been used by some commercial vendors of analytical systems. See Philip Dybvig, “Bond and Bond Option Pricing Based on the Current Term Structure,” in Michael A.H. Dempster and Stanley Pliska (eds.), *Mathematics of Derivatives Securities* (Cambridge, U.K.: Cambridge University Press, 1997).

Accordingly, we present evidence regarding

- the relationship between interest-rate volatility and the level of interest rates
- negative interest rates

Recall from Chapter 3 that the change in interest rates can be measured by either the absolute rate change (absolute value of the change in spread in basis points between two time periods) or percentage rate change (computed as the natural logarithm of the ratio of the yield for two time periods).

Volatility of Rates and the Level of Interest Rates

We will first look at the historical movement to examine the issue as to whether interest-rate volatility is affected by the level of interest rates or independent of the level of interest rates. In the former case, the higher the level of interest rates, the greater the interest-rate volatility. That is, there is a positive correlation between the level of interest rates and interest- rate volatility. If the two are independent, a low correlation would exist.

The dependence of volatility on the level of interest rates has been examined by several researchers. The earlier research focused on short-term rates and employed a statistical time series model called generalized autoregressive conditional heteroscedasticity (GARCH).[17] With respect to short-term rates, the findings were inconclusive.

Rather than focusing on the short-term rate, Oren Cheyette of MSCI Barra examined all the spot rates for the Treasury yield curve for the period from 1977 to early 1996, a period covering a wide range of interest rates and different Federal Reserve policies.[18] He finds that for different periods, there are different degrees of dependence of volatility on the level of interest rates. (Interest-rate changes are measured as absolute rate changes in Cheyette's study.) Specifically, in the high-interest-rate environment of the late 1970s and early 1980s where interest rates exceeded 10%, there was a positive correlation between interest-rate volatility and the level of interest rates. However, when interest rates were below 10%, the relationship was weak. Hence, the findings suggest that since the 1980s, interest-rate volatility has been independent of the level of interest rates. These conclusions were supported in a study by Levin of the Treasury 10-year rate from 1980 to 2003 and the 10-year swap rate from 1989 to 2003.[19]

The implication is that in modeling interest rates, one can assume that interest-rate volatility is independent of the level of interest rates in an environment where rates are less than double digit. That is, in modeling the dynamics of the volatility term, the normal model can be used.

Negative Interest Rates

Our focus is on nominal interest rates. Although we know that real interest rates (rates adjusted for inflation) in an economy have been negative, it is generally thought that it is impossible for the nominal interest rate to be negative. The reason is that if the nominal rate is negative, investors will simply hold cash. However, there have been time periods in countries where interest rates have been negative for a brief time period, refuting the Notion that investors would not be willing to lend at negative interest rates.

For example, during the Great Depression in the United States, financial historians have identified periods where Treasury securities traded at a negative yield. Japan provides another example. In early November 1998, Western banks charged Japanese banks interest of 3 to 6 basis points to hold 2- or 3-month yen deposits that Japanese banks were unwilling to deposit with local institutions because of the perceived instability of Japan's financial

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[17] See, for example, K.C. Chan, G.A. Karolyi, Francis A. Longstaff, and Anthony B. Sanders, "An Empirical Comparison of Alternative Models of the Short Rate," *Journal of Finance*, 47:3 (1992), pp. 1209–1227; Robin J. Brenner, Richard H. Harjes, and Kenneth F. Kroner, "Another Look at Alternative Models of the Short-Term Interest Rate," *Journal of Financial and Quantitative Analysis*, 31 (1996), pp. 85–107; and Yacine Aït-Sahalia, "Testing Continuous Time Models of the Spot Interest Rate," *Review of Financial Studies*, 9:2 (1996), pp. 385–426.

[18] Oren Cheyette, "Interest Rate Models," Chapter 1 in *Interest Rate, Term Structure, and Valuation Modeling*.

[19] Alexander Levin, "Interest Rate Model Selection," *Journal of Portfolio Management* (Winter 2004), pp. 74–86.

system. The yield on 3-month Japanese Treasury bills during one trading day in November 1998 fell to –5 basis points, although the closing yield was positive.

It is fair to say that although negative interest rates are not impossible, they are unlikely. The significance of this is that one might argue that an interest-rate model should not permit negative interest rates (or negative rates greater than a few basis points). Yet, this may occur in a model where volatility is measured in terms of basis points—as in the normal model. In contrast, if interest-rate volatility is measured in terms of the percentage yield change (i.e., logarithm of the yield ratio), interest rates cannot be negative. Hence, a stated advantage of using an interest-rate model whose volatility is dependent on the level of interest rates is that negative returns are not possible. How critical is this assumption in deciding whether to use a lognormal model rather than a normal model? We address this in the next section.

Selecting An Interest-Rate Model

Cheyette provides guidance in the selection of an interest-rate model. He writes:

> It may seem that one's major concern in choosing an interest-rate model should be the accuracy with which it represents the empirical volatility of the term structure of rates, and its ability to fit market prices of vanilla derivatives such as at-the-money caps and swaptions. These are clearly important criteria, but they are not decisive. The first criterion is hard to pin down, depending strongly on what historical period one chooses to examine. The second criterion is easy to satisfy for most commonly used models, by the simple (though unappealing) expedient of permitting predicted future volatility to be time dependent. So, while important, this concern doesn't really do much to narrow the choices.[20]

Moreover, as Cheyette notes, the ease of application is a critical issue in selecting an interest-rate model. Although our focus in this chapter has been on describing interest-rate models, there is the implementation issue. For consistency in valuation, a portfolio manager would want a model that can be used to value all financial instruments that are included in a portfolio. In practice, writing efficient algorithms to value all financial instruments that may be included in a portfolio for some interest-rate models that have been proposed in the literature is "difficult or impossible."[21]

Based on the empirical evidence, Cheyette and Levin have concluded that the normal model is a suitable model. Cheyette argues that for typical initial spot rate curves and volatility parameters, the probability that negative rates would be generated by the model is quite small.[22] What is important from a practical perspective is not just whether the normal model admits the possibility of negative interest rates but whether negative interest rates may have a significant impact on the pricing of financial instruments. Cheyette tests this by pricing a call option on a zero-coupon bond and concluded that: "The oft raised bogeyman of negative interest rates proves to have little consequence for option pricing, since negative rates occur with very low probability for reasonable values of the model parameters and initial term structure."[23]

Levin, who empirically investigated the issue for valuing mortgage-backed securities, also concluded that the normal model, in particular the Hull-White model, is appropriate. He states: "It will not lead to sizable mispricing even in the worst mortgage-irrelevant case."[24] But, as he notes, "This conclusion, however, certainly merits periodic review."[25]

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[20] Cheyette, "Interest Rate Models," p. 4.
[21] Cheyette, "Interest Rate Models," p. 4.
[22] Cheyette, "Interest Rate Models," p. 10.
[23] Cheyette, "Interest Rate Models," p. 25.
[24] Levin, "Interest Rate Model Selection," p. 85.
[25] Levin, "Interest Rate Model Selection," p. 85.

Estimating Interest-Rate Volatility Using Historical Data

As we have seen, one of the inputs into an interest-rate model is interest-rate volatility. Where does a practitioner obtain this value in order to implement an interest-rate model? Market participants estimate yield volatility in one of two ways. The first way is by estimating historical interest volatility. This method uses historical interest rates to calculate the standard deviation of interest-rate changes and for obvious reasons is referred to as **historical volatility**. The second method is more complicated to explain at this juncture of the book. It involves using models for valuing option-type derivative instruments to obtain an estimate of what the market expects interest-rate volatility to be. Basically, in any option pricing model, the only input that is not observed in the model is interest-rate volatility.

What is done in practice is to assume that the observed price for an option-type derivative is priced according to some option pricing model. The calculation then involves determining what interest-rate volatility will make the market price of the option-type derivative equal to the value generated by the option pricing model. Because the expected interest-rate volatility obtained is being "backed out" of the model, it is referred to as **implied volatility**.

We use the data in Exhibit 17-1 to explain how to calculate the historical volatility as measured by the standard deviation based on the absolute rate change and the percentage change in rates. The historical interest rates shown in Exhibit 17-1 are the weekly returns for 1-month LIBOR from 7/30/2004 to 7/29/2005. The observations are based on bid rates for Eurodollar deposits collected around 9:30 a.m. Eastern time.[26] The calculation can be performed on an electronic spreadsheet. The *weekly* standard deviation is reported in the exhibit. For the absolute rate change, it is 2.32 basis points; for the percentage rate change, it is 1.33%.

The weekly measures must be annualized. The formula for annualizing a weekly standard deviation is[27]

Weekly standard deviation × $\sqrt{52}$

Exhibit 17-1 Data for Calculating Historical Volatility: One-Month from 7/30/2004 to 7/29/2005

Date	1-Month LIBOR (%)	Absolute Rate Change (bps)	Percentage Rate Change (%)
7/30/2004	1.43		
8/6/2004	1.49	6	4.11
8/13/2004	1.51	2	1.333
8/20/2004	1.52	1	0.66
8/27/2004	1.54	2	1.307
9/3/2004	1.59	5	3.195
9/10/2004	1.67	8	4.909
9/17/2004	1.73	6	3.53
9/24/2004	1.77	4	2.286
10/1/2004	1.77	0	0

[26] The data were obtained from the Federal Reserve Statistical Release H.15.

[27] In the annualizing formula, there is an assumption made when the square root of the number of time periods in a year is used. The assumption is that the correlation between the interest-rate changes over time is not significant. The term *serial correlation* is used to describe this correlation.

10/8/2004	1.78	1	0.563
10/15/2004	1.81	3	1.671
10/22/2004	1.86	5	2.725
10/29/2004	1.9	4	2.128
11/5/2004	1.98	8	4.124
11/12/2004	2.03	5	2.494
11/19/2004	2.06	3	1.467
11/26/2004	2.11	5	2.398
12/3/2004	2.24	13	5.979
12/10/2004	2.3	6	2.643
12/17/2004	2.35	5	2.151
12/24/2004	2.34	1	-0.426
12/31/2004	2.34	0	0
1/7/2005	2.34	0	0
1/14/2005	2.39	5	2.114
1/21/2005	2.44	5	2.07
1/28/2005	2.5	6	2.429
2/4/2005	2.53	3	1.193
2/11/2005	2.53	0	0
2/18/2005	2.53	0	0
2/25/2005	2.59	6	2.344
3/4/2005	2.66	7	2.667
3/11/2005	2.71	5	1.862
3/18/2005	2.77	6	2.19
3/25/2005	2.79	2	0.719
4/1/2005	2.81	2	0.714
4/8/2005	2.85	4	1.413
4/15/2005	2.89	4	1.394
4/22/2005	2.95	6	2.055
4/29/2005	3.01	6	2.013
5/6/2005	3.04	3	0.992
5/13/2005	3.03	1	-0.329
5/20/2005	3.02	1	-0.331
5/27/2005	3.03	1	0.331
6/3/2005	3.08	5	1.637
6/10/2005	3.12	4	1.29
6/17/2005	3.19	7	2.219
6/24/2005	3.25	6	1.863
7/1/2005	3.28	3	0.919
7/8/2005	3.29	1	0.304
7/15/2005	3.32	3	0.908

7/22/2005	3.38	6	1.791
7/29/2005	3.44	6	1.76
Average		3.98	1.69
Weekly Variance		6.84	1.78
Weekly Std. Dev.		2.62	1.33
Annualized Std. Dev.		18.86	9.62

Source: *Federal Reserve Statistical Release H.15.*

Annualizing the two weekly volatility measures:

Absolute rate change: 2.62 × $\sqrt{52}$ = 18.89 basis points
Logarithm percentage change: 1.33 × $\sqrt{52}$ = 9.62%

If we use daily or monthly data to compute the standard deviation, the following formulas would be used to annualize:

Monthly standard deviation × $\sqrt{12}$
Daily standard deviation × $\sqrt{\text{Number of trading days in a year}}$

Note that annualizing of the daily volatility requires that the number of trading days in a year be determined. Market practice varies with respect to the number of trading days in the year that should be used in the annualizing formula above. Typically, either 250 days or 260 days are used. For many traders who use daily rates, the difference in the calculated historical annual volatility could be significant depending on the number of trading days assumed in a year. Specifically, the difference in the factor that the daily standard deviation will be multiplied by depending on the number of days assumed in the year is:

Days Assumed	Square Root of Days Assumed
250	15.81
260	16.12

KEY POINTS

- An interest-rate model is a probabilistic description of how interest rates can change over time. A stochastic differential equation is the most commonly used mathematical tool for describing interest-rate movements that incorporate statistical properties of interest-rate movements (drift, volatility, and mean reversion).
- In practice, one-factor models are used to describe the behavior of interest rates; they assume that short-term interest rates follow some statistical process and that other interest rates in the term structure are related to short-term rates.
- In a one-factor model, the SDE expresses the interest-rate movement in terms of the change in the short rate over the time interval based on two components: (1) the expected direction of the change in the short rate (the drift term), and (2) a random process (the volatility term).
- Interest-rate models fall into two general categories: arbitrage-free models and equilibrium models.
- For arbitrage-free models, the analysis begins with the observed market price of benchmark instruments that are assumed to be fairly priced, and using those prices one derives a term structure that is consistent with observed market prices for the benchmark instruments. The model is referred to as arbitrage-free because it matches the observed prices of the benchmark instruments.

- Equilibrium models attempt to describe the dynamics of the term structure using fundamental economic variables that are assumed to affect the interest-rate process.
- In practice, because of the difficulties of implementing equilibrium models, arbitrage-free models are used.
- The classification of a model as normal or lognormal is based on the assumed dynamics of the random component of the SDE. Normal models assume that interest-rate volatility is independent of the level of rates and therefore admits the possibility of negative interest rates. The lognormal models assume that interest-rate volatility is proportional to the level of rates, and therefore negative interest rates are not possible.
- Empirical evidence reviewed in this chapter regarding the relationship between interest-rate volatility and the level of rates suggests that the relationship is weak at interest-rate levels below 10%. However, for rates exceeding 10%, there tends to be a positive relationship. This evidence suggests that in rate environments below 10%, a normal model would be more descriptive of the behavior of interest rates than the lognormal model.
- Empirical tests also suggest that the impact of negative interest rates on pricing is minimal, and therefore one should not be overly concerned that a normal model admits the possibility of negative interest rates.
- Interest-rate volatility can be estimated using historical volatility or implied volatility.
- Historical volatility is calculated from observed rates over some period of time. When calculating historical volatility using daily observations, differences in annualized volatility occur for a given set of observations because of the different assumptions that can be made about the number of trading days in a year. Implied volatility is obtained using an option pricing model and observed prices for option-type derivative instruments.

Questions

1. What is meant by an interest-rate model?

2. Explain the following three properties of an interest-rate model:
a. drift
b. volatility
c. mean reversion

3. What is the commonly used mathematical tool for describing the movement of interest rates that can incorporate the properties of an interest-rate model?

4. a. Why is the most common interest-rate model used to describe the behavior of interest rates a one-factor model?
b. What is the one-factor in a one-factor interest-rate model?

5. What is meant by
a. a normal model of interest rates?
b. a lognormal model of interest rates?

6. Explain the treatment of the dynamics of the volatility term for the following interest-rate models:
a. Vasicek model
b. Dothan model
c. Cox-Ingersoll-Ross model

7. What is an arbitrage-free interest-rate model?

8. a. What are the general characteristics of the Ho-Lee arbitrage-free interest-rate model?
b. How does the Ho-Lee arbitrage-free interest-rate model differ from the Hull-White arbitrage-free interest-rate model?

9. What is an equilibrium interest-rate model?

10. Explain why in practice arbitrage-free models are typically used rather than equilibrium models.

11. a. What is the empirical evidence on the relationship between volatility and the level of interest rates?
b. Explain whether the historical evidence supports the use of a normal model or a lognormal model.

12. Comment on the following statement: "If an interest-rate model allows the possibility of negative interest rates, then it is not useful in practice."

13. a. What is meant by historical volatility?
b. What is meant by implied volatility?

14. Suppose that the following weekly interest-rate volatility estimates are computed: absolute rate change = 3.85 basis points percentage rate change = 2.14%
a. What is the annualized volatility for the absolute rate change?
b. What is the annualized volatility for the percentage rate change?