
Mathematics for Economics

third edition

Michael Hoy
John Livernois
Chris McKenna
Ray Rees
Thanasis Stengos

The MIT Press
Cambridge, Massachusetts
London, England

© 2011 Massachusetts Institute of Technology

All rights reserved. No part of this book may be reproduced in any form by any electronic or mechanical means (including photocopying, recording, or information storage and retrieval) without permission in writing from the publisher.

This book was set in Helvetica and Times Roman by Westchester Book Group.
Printed and bound in the United States of America.

Library of Congress Cataloging-in-Publication Data

Mathematics for economics / Michael Hoy . . . [et al.]. — 3rd ed.

p. cm.

Includes index.

ISBN 978-0-262-01507-3 (hbk. : alk. paper) — ISBN 978-0-262-51622-8

(pbk. : alk. paper) 1. Economics, Mathematical. I. Hoy, Michael, 1953 Sept. 22–

HB135 . M3698 2011

511'.8—dc22

2010022679

10 9 8 7 6 5 4 3 2

Index

- Added-worker effect, 286
- Additively separable function, 399
- Adjoint matrix, 320, 326
- Antidifferentiation, 585
- Arc elasticity, 168
 - of demand, 168
- Arrows of motion, 741
- Associative laws, 26
- Asymptote, 108
- Autonomous dynamic optimization problem, 887
- Autonomous optimization problem, 861
- Average-cost curve, 159, 186
- Average propensity to consume, 183
- Average value function, 158

- Bang-bang control, 901–905
- Base of a function, 49
- Basic variables, 248
- Basis, 356
- Bernoulli's equation, 748
- Bertrand model, 118
- “Best reply” function, 482
- “Best response” function, 482
- Better set, 54
- Bordered Hessian, 454
- Boundary conditions of maximum principle, 851
- Boundary point, 33, 36
- Bounded intervals, 33
- Bounded sequence, 65, 82
- Bounded set, 37
- Budget set, 14
- Butterfly effect, 676

- Cardinality, 23
- Cartesian product, 32
- Center, 815
- Chain rule, 148, 160, 165, 404
- Chaos, 676

- Closed economy model, 314–15
- Closed interval, 37
- Closed set, 37
- Closure, 26
- Cobb-Douglas production function, 53, 413–14
- Cobweb model, 643, 652–54, 696–701
- Codomain, 41
- Cofactor, 319
- Cofactor expansion, 319, 325
- Cofactor matrix, 325
- Column matrix, 270
- Column vector, 270, 347
- Commutative laws, 26
- Compact interval, 33
- Comparative statics, general method, 551
- Compensated-demand functions, 513
- Compensated-price effect, 514
- Complement of a set, 17
- Completeness, 28
- Completeness property, 11
- Composite mapping, 43
- Concave function, 51, 175, 178
- Concave programming, 567
- Concavity, 51
- Conformable matrices, 273, 277, 280
- Constant elasticity, 195
 - demand function, 168
 - of substitution production function, 403, 461
- Constant function, derivative of, 147
- Constant multiple of a function, derivative of, 147
- Constant of integration, 585–96
- Constrained problem, 196
- Constraint qualification, 571
- Consumer behavior, 430
- Consumer's surplus, 612
- Consumption function, 244, 464
- Consumption set of a good, 14
- Continuity, 103
- Continuity of a function, 103

- Continuous compounding, 72–76
 Continuous function, 105–11
 Control variable, 849
 Convergent sequence, 65
 Convex combination, 34, 37, 46
 Convex function, 52, 175–76
 Convex set, 38–39
 Convexity, 34, 51
 Coordinate system, 31
 Cost function, 103, 181–82
 Cost minimization, 503, 567
 short-run, 558
 Costate variable, 850
 Cournot duopoly, 481–83
 Cournot model, 473
 Cournot, Augustin, 481
 Cramer's rule, 329
 Cross-partial derivative, 411, 437
 Cubic cost function, 195
 Current-valued Hamiltonian, 862–64
- Definite integral, 586, 593
 Definitely divergent sequence, 65
 Demand, theory, 473
 Demand function, 549
 Dependent variable, 45
 Derivative of a function, 134
 Determinant, geometric interpretation of,
 315–16
 Determinant of a 2×2 matrix, 305
 Determinant of a 3×3 matrix, 318–20
 Diagonal matrix, 271
 Difference equation, 634–35
 autonomous, 636–37
 linear, 637
 linear, autonomous, first-order, 643–44
 convergence of, 649
 general solution to, 647–48
 linear, first-order, form of general, 656–57
 stationary value, 649–51
 steady state of, 649
 steady-state equilibrium of, 654
 steady-state value of, 649–51
 linear, autonomous, second-order, 681–84
 characteristic equation of, 684
 characteristic roots of, 684
 eigenvalues of, 684
 homogeneous form of, 682–83
 linear, second-order with a variable term,
 708–11
 nonlinear, 637
 nonlinear, autonomous, first-order, 665
 global stability of, 670
 locally stable equilibrium of, 668
 order of, 639
 phase diagram of, 666
 solution to, 637–38
 stable limit cycle, 675–76
 steady-state equilibrium of, 666
 unstable equilibrium of, 668
- Differential, 135
 Differential equation, 635
 autonomous, 639
 linear, 639–40
 nonlinear, 639–40
 ordinary, 638
 order of, 639
 partial, 641
 solution, 640
 linear, autonomous, first-order 715–16
 general solution, 720
 general solution of homogeneous form, 717
 homogeneous form, 715–16
 particular solution, 719–20
 stable equilibrium of, 727
 steady-state value of, 719–20
 unstable equilibrium of, 727
 linear, autonomous, second-order, 753
 characteristic equation of, 755
 characteristic roots of, 755
 complete solution, 763
 convergence of, 768
 homogeneous form of, 754
 eigenvalues of, 755
 particular solution, 762
 linear, nonautonomous, first-order, 731
 general solution, 731
 linear, second-order with a variable term,
 772–77
 nonlinear, autonomous, first-order, 739
 initial-value problem, 740
 qualitative analysis of, 740
 stability analysis of, 742
 steady-state equilibrium point of, 742
 nonlinear, nonautonomous, first-order, 748
 separable equations, 749–50
 nonlinear system of two autonomous
 differential equations, 820
 global behavior of, 823
 phase diagram of, 823–24
- Differentiation rules, 147–48
 Dimension of a space, 359

- Dimensions of variables, 29
- Diminishing marginal productivity, 151
 - of an input, law of, 400
- Discontinuous function, 106
- Discount factor, 70
- Discount rate, 70
- Discounting, 860
- Discrete compounding, 76
- Discriminating monopoly, 473, 496–98
- Disjoint set, 16
- Disposable income, 245
- Distance between points, 34
- Distributive law, 26
- Divergent sequence, 65–66
- Domain, 41
- Dornbusch overshooting model, 816–19
- Dual consumer problem, 513
- Dynamic optimization problem, general form, 849

- ϵ -neighborhood, 35
- Economic dynamics, 633
- Eigenvalue problem, 363
- Eigenvalue of a matrix, 363
- Eigenvector, 364
- Elasticity, 127, 167–68
- Elasticity of substitution, 460–61
- Elements of a set, 11
- Empty set, 15
- Endogenous variables, 529
- Envelope theorem, 554–57
- Equality, 28
- Equilibrium
 - in n -markets, 342–43
 - in three markets, 258
 - in two markets, 241–43
- Equilibrium price, 214–15
- Euclidean distance, 34–35, 394
- Euclidean distance condition, 367
- Euclidean norm, 349
- Euler’s theorem, 459
- Excess demand, 541
- Exogenous variables, 529
- Expenditure function, 514, 529
- Exponent, 49
- Exponential function, 49, 165
 - derivative of, 148
- Exponential rule, 587
- Extreme values, 196

- Feasible set, 520
- Finite-dimensional space, 359

- First derivative function, 175
- First-order condition, 197
- First-order total differential, 417, 434
- Fixed-endpoint problem, 872–78, 899
- Flow variable, 591
- Free endpoint, 849
- Free-endpoint problem, 851
- Free-terminal-time problems, 909–15
- Free variable, 248
- Function, 41
- Functional, 846
- Fundamental theorem of integral calculus, 599
- Fundamental equation, 537

- Gauss-Jordan elimination, 235, 301
- General optimal control problem, 900
- Geometric series, 86–87
- Giffen good, 551
- Global maximum, 197, 486
- Gradient vector, 408

- Half-open interval, 33
- Hamiltonian function, 850
 - current-valued, 862–64
- Harmonic series, 86
- Hessian matrix, 410
- Homogeneous function, 456–57
- Homogeneous system of equations, 260
- Hotelling’s location model, 103

- Idempotent matrices, 294
- Identity matrix, 272
- Image, 41
- Image set, 42
- Implicit function theorem, 420, 422, 552
- Implicit differentiation, 418–23
- Implicit function, 45
- Improper integrals, 613–22
- Improper node, 811
- Income effect, 550
- Inconsistent system of equations, 254
- Indefinite integral, 585–86
- Independent variable, 45
- Indifference curves, 54, 430
- Indirect utility function, 529
- Industry of a good, 14
- Inequality-constrained endpoint problem, 878–81
- Infinite time horizon, 886
- Inner product, 348
- Input-output matrix, 269

- Input-output model, 269–70
- Input-requirements matrix, 269, 311
- Instantaneous rate of change, 139
- Integers, 24
- Integrable function, 598
- Integral of a constant multiple, 587
- Integral of a sum, 586
- Integrand, 586
- Integrating factor, 731–34
- Integration, 585
 - by parts, 624–26
- Intercept term, 45
- Interior point, 33, 39
- Interior solution, 225
- Intermediate-value theorem, 103
- Internal rate of return, 93
- Intersection of sets, 15
- Intervals, 33
- Inverse function rule, 162
- Inverse matrix, 302
- Inverse of a 2×2 matrix, 303–305
- Inverse of a 3×3 matrix, 320–22, 324–28
- Inverse of a function, derivative of, 148
- Inverse of an $n \times n$ matrix, 326
- Investment, theory of, 845
- Investment function, 244
- Irrational numbers, 25
- IS curve, 237, 246, 247, 544–46
- IS-LM model, 237, 243, 314–15, 332–35, 529, 543–46
 - linear, 237–40
- IS-LM-BP model, 235
- Isocline, 808
- Isoquant, 54, 426
- Isosectors, 808

- Keynesian consumption function, 183
- Keynesian model of income determination, 530
- Keynesian multiplier, 61
- Kronecker delta, 370
- Kuhn-Tucker (K-T) conditions, 568
- Kuhn-Tucker theorem, 569, 575

- Lagrange function, 505–508
- Lagrange method, 507–508
- Lagrange multiplier, 506–507, 515
- Lagrangian, 506–507
- Latent root, 364
- Latent vector, 364
- Leading principal minors, 380

- Leading principal submatrix, 380
- Left-hand derivative, 141
- Left-hand limit of a function, 104
- Leontief model, “closed,” 339–41
- Leontief model, “open,” 335–38
- Level curve, 423–24
- Level set, 423
- Level set of a function, 53
- Limit of a sequence, 62, 65–68
- Linear dependence, 253
- Linear differential equation system, 781
 - homogeneous form of, 782
 - steady-state solution to, 789
- Linear function, 44
 - derivative of, 147
- Linear independence, 352
- Linear indifference curves, 239
- Linear production technology, 311–12
- Linear programming, 567
- Linear system of equations, 236–37
- Linear system of n autonomous differential equations, 796
- Linear system of n -variables, 250–63
- Linearly dependent equations, 253
- Linearly dependent vectors, 352
- Linearly independent equations, 253
- Linearly independent vectors, 353
- LM curve, 237, 246, 247, 544–45
- Local maximum, 197, 485
- Logarithmic function, 50, 166
 - derivative of, 148
- Logarithmic rule, 587
- Long-run cost curves, 557
- Lower sum, 596

- Malthusian growth model, 665
- Mapping, 42
- Marginal analysis, 127
- Marginal-cost curve, 159
- Marginal cost of production, 137
- Marginal product, 116–17
 - curve, 151–52
 - function, 117, 393, 399–406, 590
- Marginal propensity to consume, 183, 244, 530
- Marginal rate of substitution, 432
- Marginal rate of technical substitution, 393, 426
- Marginal revenue
 - function, 149, 155
 - product of labor, 127
- Market equilibrium, 214
- Markov model, 781

- Matrix, 270
 addition, 273
 arrays, 254
 characteristic equation of, 364, 797
 characteristic polynomial, 364
 characteristic root of, 364
 characteristic vector of, 364
 eigenvalues, 797
 eigenvector, 797
 equality, 271
 equation, 275
 multiplication, 276
 of minors, 325
 operations, 273–77
 reduced row-echelon form, 255–57
 subtraction, 273
 Maximum of a function, 197
 Maximum principle, 848–58
 Mean-value theorem, 190
 Method of undetermined coefficients, 708
 Minor, 318, 325, 380
 Monotonic sequence, 82
 Monotonically decreasing sequence, 82
 Monotonically increasing sequence, 82
 Multimarket equilibrium, 241
 Multiproduct monopoly, 473, 479

 National debt accumulation, 715
 Natural logarithm, 50
 Natural numbers, 23
 Neoclassical model of economic growth, 745–47
 Nonnegative real numbers, 28
 Nonbinding constraint, 226
 “Nonsatiation” assumption, 572–73
 Nonsingular matrix, 302, 360
 Normalization, 367
 n th derivative test, 217
 Null matrix, 272
 Null set, 15

 Oligopoly, 118
 One-to-one correspondence, 42
 Open interval, 33
 Open set, 36
 Optimization, 473
 Ordered pair, 31
 Orthogonal matrix, 368
 Orthogonal vectors, 358
 Orthonormal basis, 358
 Overdetermined system of equations, 246, 254

 Partial derivative, 394–96
 Partial differentiation, 393–406
 Partition of a closed interval, 593–94
 Partition of the universal set, 19
 Partitioned matrices, 295
 Phase plane, 808
 Point elasticity of demand, 168
 Point sets, 32
 Points of inflection, 199, 401
 Points rationing, 503
 Pointwise continuity, 103
 Positive monotonic transformation, 431
 Power function, 49, 150
 derivative of, 150
 Power rule, 586
 Power set, 11, 21
 Present value, 69
 Present-value calculations, 87–93
 Price-adjustment model, 764–66
 Price competition, 118
 Price-regulated monopoly, 227–28
 Principal submatrix, 314–15
 Producer surplus, 609
 Product matrix, 280
 Production function, 113, 426–30
 Production possibility set, 14
 Production set, 14
 Profit function, 103, 119
 Profit maximization, 209, 481
 Profit-maximizing input choice, 490
 Proper subset, 13
 Proportional consumption function, 183
 Pure number, 30

 Quadratic form, 378
 indefinite, 380
 negative definite, 380
 negative semidefinite, 380
 positive definite, 380
 positive semidefinite, 380
 statistical distribution of, 347
 Quadratic function, 48
 Quasiconcave function, 54, 55
 Quasiconcavity, 51, 53–54, 451–56
 Quasiconvex function, 56
 Quasiconvexity, 51, 55, 451–56

 Range, 41
 Rank of a matrix, 360
 Rational numbers, 25
 Reaction function, 482

- Real line, 26
- Real numbers, 26
- Real-valued functions, 42
- Rectangular hyperbola, 48
- Reduced form, 243
- Reduced-form equation, 334
- Reflexivity, 28
- Relative difference, 18
- Remainder formula, 464–67
- Remainder term, 186
- Returns to scale, 139, 459
- Revenue function, 103, 119
- Riemann integral, 593
- Riemann sum, 594
- Right-hand derivative, 141
- Right-hand limit of a function, 105
- Row matrix, 270
- Row operations, 250
- Row vector, 270, 347

- Saddle path, 805
- Saddle point, 474, 568
 - conditions, 571
 - equilibrium, 805
- Savings, theory of, 845
- Scalar, 275
- Scalar multiplication, 275, 350–51
- Secant, 129
- Secant line, 129
- Second derivative function, 175
- Second-order conditions, 211
 - for constrained optimization, 516–19
- Second-order total differential, 437–38
- Sequence, 61–62
- Series, 84–85
- Set, 11
- Shadow price, 227, 498, 858
- Shadow wage rate, 559–61
- Short-run cost curves, 557
- Short-run production function, 179
- Singular matrix, 302
- Slater's condition, 569–71
- Slope, 45, 150
- Slope coefficient, 45
- Slutsky equation, 529, 549–51
- Solow model, 745–47
- Solow, Robert, 745
- Spectral decomposition, 366
- Square matrix, 271
- St. Petersburg Paradox, 61
- Stable focus, 814
- Stable node, 810
- State variable, 849
- Stationary value, 200, 474
- Step function, 112
- Stock variable, 591
- Strictly concave function, 51, 178
- Strictly convex function, 52, 177
- Strictly convex set, 39
- Strictly quasiconcave function, 55
- Strictly quasiconvex function, 56
- Structural equations, 242, 244
- Subset, 13
- Substitution effect, 550
- Substitution rule of integration, 623
- Supply, theory of, 473
- Supply function, 195
- Symmetric matrix, 289
- System of two linear difference equations,
 - general form of, 825

- Tangent, 127
- Tangent hyperplane, 469
- Tangent line, 130
- Taylor series (expansion) formula, 185–93, 217, 464–65
- Theory of demand, 473
- Theory of investment, 845
- Theory of savings, 845
- Theory of supply, 473
- Third derivative function, 175
- Time constraint, 576–78
- Time-derivative of a variable, 635
- Total differential, 135
- Total-product function, 151, 192
- Total revenue, 155
- Total revenue function, 155–57
- Trace of a matrix, 295–96
- Trajectory, 809
- Transitivity, 28
- Transpose matrix, 288
- Transpose of a matrix, 288
- Transversality condition, 851
- Triangular matrix, 310
- Twice differentiable function, 175

- Unbounded intervals, 33
- Unbounded sequence, 65
- Uncompensated-demand functions, 513
- Uncompensated-price effect, 514
- Unconstrained problem, 196
- Underdetermined system of equations, 247, 254

- Union of sets, 16
- Universal set, 14
- Unstable focus, 815
- Unstable node, 810
- Upper sum, 596–97
- Utility function, 430

- Value function, 555
- Vector, 271
 - addition, 349
 - diagonalization, 371–72
 - length, 348–49
 - space, 355
 - subtraction, 350
- Venn diagram, 15
- Vertical asymptote, 108

- Walrusian price adjustment model, 727–29, 753
- Walrusian price adjustment model with entry,
781
- Weierstrass's theorem, 520
- Worse set, 55

- Young's theorem, 411–12, 429