Mathematical Tools for Real-World Applications

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Solutions for Exercises

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Introduction

This document contains solutions to all the exercises in the book *Mathematical Tools for Real-World Applications* from MIT Press. Let me say two things before we plunge into the equations:

1. Many exercises allow for multiple ways to solve them. Please do not view the solutions below as canon.
2. Occasionally a solution may refer to an equation or a figure in the main text. Such references are marked with an asterisk (as in (3.78*)).

1 Units

1. Problem

Section A.31 provides an approximate formula for the payment rate on a mortgage. Given the initial loan amount D, interest rate r, and duration of the loan T, the payment rate is given by

$$p \approx \frac{rDe^{rT}}{e^{rT} - 1}. \tag{1.1}$$

Jane used this formula to estimate payments on a mortgage that she is planning to apply for. She used the loan amount D = \$230,000, interest rate 4.5 percent (r = 0.045/year), and loan duration of T = 30 years. Using these values and equation (1.1), she estimated her payments (principal and interest) to be p = \$13,972.14, which seemed very high. The loan officer at her bank said that the correct amount of a monthly payment was p_0 = \$1,164.34. Use dimensional analysis to find a mistake in Jane's calculations.

Solution

The rate r is in percent/year. The loan amount is in dollars. Therefore, formula (1.1) produces a result in dollars per year, that is, for the yearly payment. To obtain the monthly payment, p must be divided by 12. With this correction, the numbers from formula (1.1) match the amount from the loan officer.

2. Problem

Consider syrups with masses m_1, m_2, and m_3 and sugar concentrations p_1, p_2, and p_3. Sections A.16 and A.17 show that concentrations of sugar in the blend of two and three syrups are given respectively by

$$\begin{aligned} p_{12} &= \frac{p_1 m_1 + p_2 m_2}{m_1 + m_2}, \\ p_{123} &= \frac{p_1 m_1 + p_2 m_2 + p_3 m_3}{m_1 + m_2 + m_3}. \end{aligned} \tag{1.2}$$

Go through the derivations of these results and check all equations for units at each step.

Solution

A check shows that units are correct at each step.

3. Problem

A riverboat travels from town A to town B in time T_{AB} and from town B to town A in time T_{BA}. The time to go from town B to town A on a raft is given by

$$T_r = \frac{2T_{AB}T_{BA}}{T_{AB} - T_{BA}}. \tag{1.3}$$

Check the units for the solution of this problem in section A.2.

Solution A check shows that units are correct at each step.

4. Problem

Suggest a formula for the velocity of a satellite on a circular orbit as a function of the orbit radius and the gravity acceleration. Note that orbit radius is measured in meters (m) and satellite velocity in m/s, and that the gravity acceleration is in m/s^2.

Solution We denote the orbit radius as R, and the gravity acceleration as g. The only way to produce the correct units for velocity V is

$$V = C\sqrt{Rg}, \tag{1.4}$$

where C is a dimensionless constant.[1] (For reference, the exact formula is $V = \sqrt{Rg}$, which corresponds to $C = 1$.)

5. Problem

The law of sines and law of cosines link the lengths of the sides of a triangle and the measures of its angles (figure 1.1). Use dimensional analysis to determine which of the following formulations of these two laws are incorrect:

a) $\sin(a\beta) = \sin(b\alpha)$

b) $c^2 = a^2 + b^2 - 2\cos(ab\gamma)$

c) $a\sin\beta = b\sin\alpha$

d) $a\sin\alpha = b\sin\beta$

e) $c^2 = a^2 + b^2 - 2\sqrt{ab}\cos\gamma$

f) $a\sin(c+\beta) = b\sin(c+\alpha)$

g) $c^2 = a^2 + b^2 - 2ab\cos\gamma$

h) $a^2\sin\beta = b^2\sin\alpha$

i) $c^3 = a^3 + b^3 - 2(ab)^{\frac{3}{2}}\cos\gamma$

1. From equation (1.4) it may seem that satellite velocity is larger for larger orbit radii R. However, the gravity acceleration g quickly decreases with R. The net effect is a lower satellite velocity for higher orbits.

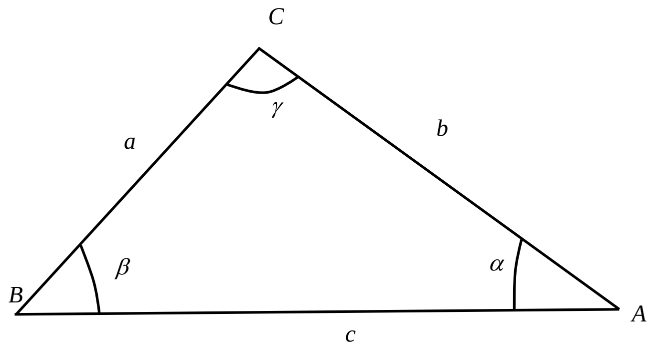

Figure 1.1
Law of sines and law of cosines: units

Solution

Dimensional analysis shows that formulas *a, b, e*, and *f* are incorrect. The correct formulas are *c* and *g*. (Formulas *d, h*, and *i* are also incorrect, but this cannot be detected by dimensional analysis.)

6. Problem

The waves that you may see on a beach are called gravity waves. As implied by the name, gravity acceleration $g \approx 9.8\ \text{m/s}^2$ is an important parameter in modeling these waves mathematically. Two other important parameters are measured in meters:

1. Wavelength λ, which is the distance between two consecutive crests
2. Ocean depth h in the wave propagation area

Using dimensional analysis, suggest two formulas for the speed of gravity waves in the ocean, one using g and λ and another one using g and h.

Solution

The two ways to get the correct units for the speed of gravity waves are

$$\begin{aligned} V_d &= \alpha\sqrt{\lambda g}, \\ V_s &= \beta\sqrt{hg}, \end{aligned} \tag{1.5}$$

where α, β are dimensionless constants. Waves with speeds V_d and V_s are respectively called deep and shallow water waves.

7. Problem

Old clocks used a pendulum to measure time. To design a clock, we need to know the period of pendulum oscillations. A formula for this period may contain some or all of the following parameters:

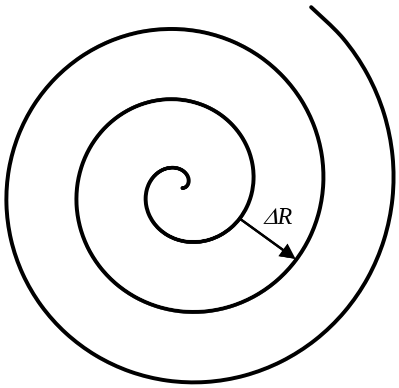

Figure 1.2
Archimedes's spiral: units

1. Gravity acceleration $g \approx 9.8$ m/s^2
2. Length of the pendulum l, measured in meters (m)
3. Mass of the pendulum bob M, measured in kilograms (kg)

Suggest a formula for the period of pendulum oscillations.

Solution

The only way to get the units of time is

$$T = C\sqrt{\frac{l}{g}}, \tag{1.6}$$

where C is a dimensionless constant. (The exact formula for small amplitude oscillations has $C = 2\pi$.)

8. Problem

The radius of Archimedes's spiral increases linearly with the turn angle (figure 1.2). Use dimensional analysis to flag the incorrect formulas for the arc length of the spiral:

a) $L = \frac{1}{4\pi}(\Delta R + 1)\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right)$

b) $L = \frac{1}{4\pi}\Delta R\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right)$

c) $L = \frac{1}{4\pi}\Delta R\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right) + \ln\Delta R\right)$

d) $L = \frac{1}{4\pi}\Delta R\left(\Delta R\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right)$

Here ΔR is the distance between the adjacent loops and θ is the total turn angle.

Solution

Formulas *a, c*, and *d* are incorrect.

9. Problem

For a heavy object falling from a small height, the air resistance is relatively small, and the motion is affected primarily by the gravity (free fall). Knowing that gravity acceleration is measured in m/s^2, suggest a formula for the velocity of a falling object as a function of height h.

Solution

The only way to get units of velocity (m/s^2) is

$$V = C\sqrt{gh}, \tag{1.7}$$

where C is a dimensionless constant. (The exact formula has $C = \sqrt{2}$.)

10. Problem

A pool must be drained for winter. There are three pumps that can be used separately or jointly. Individually, they can drain this pool in T_1, T_2, and T_3 hours. The time for draining the pool using jointly two or three pumps is given respectively by

$$\begin{aligned} T_{12} &= \frac{T_1 T_2}{T_1 + T_2}, \\ T_{123} &= \frac{T_1 T_2 T_3}{T_1 T_2 + T_1 T_3 + T_2 T_3}. \end{aligned} \tag{1.8}$$

Check the units in the derivations of these solutions in sections A.18 and A.19.

Solution

A check shows that units are correct at each step.

11. Problem

Section A.4 solves the problem of finding intersections between a circle and an ellipse (see figure 1.3). The circle and the ellipse are given by the following equations:

$$\begin{aligned} x^2 + y^2 &= R^2, \\ \frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} &= 1. \end{aligned} \tag{1.9}$$

Assume that all coordinates are measured in meters. Use dimensional analysis for this problem to flag the wrong solutions among the following options:

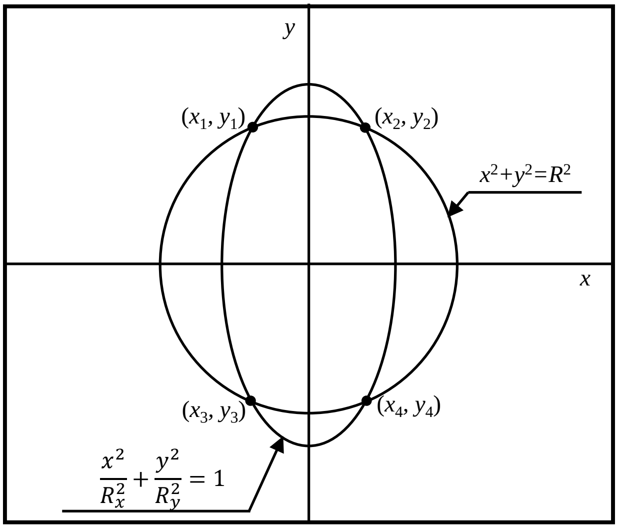

Figure 1.3
A circle and an ellipse: units

a) $x = \pm\sqrt{R_x^4 \dfrac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $\quad y = \pm\sqrt{R_y^4 \dfrac{R_x^2 - R^2}{R_x^2 - R_y^2}}$

b) $x = \dfrac{1}{4} \pm \sqrt{R_x^2 \dfrac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $\quad y = \dfrac{1}{4} \pm \sqrt{R_y^2 \dfrac{R_x^2 - R^2}{R_x^2 - R_y^2}}$

c) $x = \pm\sqrt{R_x^2 \dfrac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $\quad y = \pm\sqrt{R_y^2 \dfrac{R_x^2 - R^2}{R_x^2 - R_y^2}}$

d) $x = \pm\sqrt{R_x^2 \dfrac{R_y^2 - R^{-2}}{R_y^2 - R_x^2}}$; $\quad y = \pm\sqrt{R_y^2 \dfrac{R_x^2 - R^{-2}}{R_x^2 - R_y^2}}$

Solution

Formulas *a, b*, and *d* are incorrect.

12. Problem

Section A.11 deals with the sum of two scaled ratios:

$$\frac{p}{x-a} + \frac{q}{x-b} = d. \tag{1.10}$$

The solution of this equation for x is as follows:

$$x_{1,2} = \frac{(p+q) + d(a+b) \pm \sqrt{d^2(a-b)^2 + (p+q)^2 + 2d(a-b)(p-q)}}{2d}, \tag{1.11}$$

where subscripts 1, 2 correspond to the ± signs in the right-hand side. Assume that p and q are measured in meters (m), x, a, and b are measured in seconds (s), and d is measured in m/s. Go line by line through the solution of this problem in section A.11 and check the units in each equation.

Solution

A check shows that units are correct at each step.

13*. The maximum angle α_{max} between a pendulum and a vertical is called the amplitude. The formula for the period of pendulum oscillations that is the solution of problem 7 is approximately valid for small amplitudes ($\alpha_{max} \ll 1$). In a case of larger amplitudes, the period should also be a function of the amplitude. For a researcher who is trying to derive such a formula, having a clue about its general structure beforehand would be extremely helpful. Identify incorrect formulas for the period of pendulum oscillations among the following options, where $F(\alpha_{max})$ is a yet unspecified transcendental function of the amplitude (refer to problem 7 for notations):

a) $T = F(\alpha_{max})\sqrt{\frac{l}{g}}$

b) $T = F(\alpha_{max})\sqrt{\frac{lM}{g}}$

c) $T = F(\alpha_{max})\sqrt{\frac{g}{l}}$

d) $T = F(\alpha_{max})\sqrt{lgM}$

e) $T = \sqrt{\frac{l + F(\alpha_{max})}{g}}$

f) $T = \sqrt{\frac{l}{g}} + F(\alpha_{max})$

Solution

Formulas b through f are incorrect. (The exact solution is $T = 4\sqrt{l/g}F(\pi/2, \sin(\alpha_{\max}/2))$, where function $F(x, y)$ is called the *incomplete elliptic integral of the first kind*.)

14. Problem

A satellite on a circular orbit revolves around Earth every T seconds. A formula for T may contain some or all of the following parameters:

1. Gravity acceleration g at the altitude of the satellite, measured in m/s^2
2. Orbit radius R, measured in meters (m)
3. Mass of the satellite M, measured in kilograms (kg)

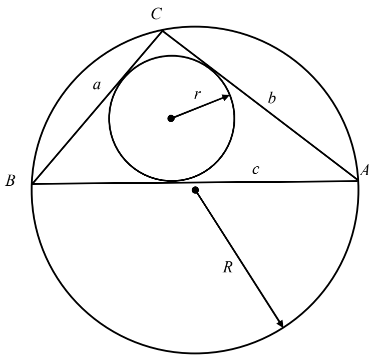

Figure 1.4
A triangle, an inscribed circle, and a circumscribed circle: units

Suggest a formula for the orbit period of the satellite.

Solution

The only way to get units of time is

$$T = C\sqrt{\frac{R}{g}}, \tag{1.12}$$

where C is a dimensionless constant. (The exact formula for small amplitude oscillations has $C = 2\pi$.)

15. Problem

Solutions of the depressed cubic equation

$$x^3 + px + q = 0 \tag{1.13}$$

can be expressed through trigonometric functions (see section A.29):

$$x_k = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right), \tag{1.14}$$

where $k = 0, 1, 2$. Assume that x is measured in meters. Deduce the proper measurement units for parameters p and q and check the units in equation (1.14).

Solution

In the depressed cubic equation, p must be measured in m^2 and q must be measured in m^3. Since the value of the cosine function is dimensionless, x_k is measured in the same units as $\sqrt{p}$, that is, in meters. We must also check that the argument of the $\cos^{-1}$ function is dimensionless. A check shows that it is.

16. Problem

Refer to figure 1.4. The radius of the inscribed circle r can be computed using one of the following two formulas:

$$
\begin{aligned}
r &= \frac{S}{p}, \\
r &= \frac{\sqrt{p(p-a)(p-b)(p-c)}}{p},
\end{aligned}
\tag{1.15}
$$

where S is the area of the triangle, $p = (a+b+c)/2$ is its half-perimeter, and a, b, c are sides. Check the units in these two formulas.

Solution

A check shows that units are correct.

17. Problem

We again refer to the drawing in figure 1.4. Using dimensional analysis, identify incorrect formulas for the radius of the circumscribed circle R:

a) $R = \dfrac{\sqrt[3]{a^2b^2c^2}}{\sqrt[4]{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}$

b) $R = \dfrac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}$

c) $R = \dfrac{abc}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$

d) $R = \dfrac{abc}{\sqrt[3]{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}$

Solution

From dimensional analysis we conclude that formulas c and d are incorrect. The correct formula is b. Formula a is incorrect, but this cannot be detected by dimensional analysis.

18*. The solution of the quartic equation is even more cumbersome than that of the cubic equation. The general quartic equation is given by

$$ax^4 + bx^3 + cx^2 + dx + f = 0. \tag{1.16}$$

The root behavior is governed by the discriminant. In particular, equation (1.16) has two distinct real roots and two complex roots if the following discriminant is negative:

$$
\begin{aligned}
D = {} & 256a^3f^3 - 192a^2bdf^2 - 128a^2c^2f^2 + 144a^2cd^2f - 27a^2d^4 \\
& + 144ab^2cf^2 - 6ab^2d^2f - 80abc^2d + 18abcd^3 + 16ac^4f \\
& - 4ac^3d^2 - 27b^4f^2 + 18b^3cdf - 4b^3d^3 - 4b^2c^3f + b^2c^2d^2 .
\end{aligned}
\tag{1.17}
$$

One term in the above expression for the discriminant contains an error. Assume that f is measured in meters, and x in seconds. Identify the erroneous term using the dimensional analysis.

Solution

For unit consistency we must assume that a is measured in m/s^4, b is measured in m/s^3, c is measured in m/s^2, and d is measured in m/s. A check shows that all terms except one have units of m^6/s^{12}. The incorrect term $-80abc^2d$ has units of m^5/s^{12}. (The correct expression for this term is $-80abc^2df$.)

19*. An explosion in the air creates a spherical shock wave. The following parameters affect the shock wave propagation:

1. The energy of the explosion E, measured in $\text{kg} \cdot \text{m}^2/\text{s}^2$
2. The density of the air ρ, measured in kg/m^3
3. Radius of the shock wave R, measured in meters (m)

Propagation of this wave away from the point of explosion is seen from the side as an expanding sphere. As the radius of the shock wave increases, the wave slows down. Using dimensional analysis, suggest a formula that computes the velocity of the shock wave (measured in m/s) as a function of the radius.

Solution

The only way to get the units of velocity is

$$V = C\sqrt{\frac{E}{\rho R^3}}, \tag{1.18}$$

where C is a dimensionless constant.

20. Problem

Section A.33 presents formulas for the linear regression algorithm. There are N data points (x_i, y_i) for variables x and y. We assume a linear model for the link between these two variables:

$$y = ax + b + R. \tag{1.19}$$

The data may not fit exactly a straight line because of measurement errors R. The linear regression algorithm states that the best estimate for parameters a and b from the data is given by the following equations:

$$\begin{aligned} a &= \frac{N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}, \\ b &= \frac{\sum_{i=1}^{N} y_i \cdot \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}. \end{aligned} \tag{1.20}$$

Table 1.1
Units for expressions in the linear regression equations

Sum	Units
$\sum_{i=1}^{N} x_i y_i$	$\mathrm{m} \cdot \mathrm{s}$
$\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i$	$\mathrm{m} \cdot \mathrm{s}$
$\sum_{i=1}^{N} x_i^2$	s^2
$\left(\sum_{i=1}^{N} x_i\right)^2$	s^2
$\sum_{i=1}^{N} y_i \cdot \sum_{i=1}^{N} x_i^2$	$\mathrm{m} \cdot \mathrm{s}^2$
$\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i$	$\mathrm{m} \cdot \mathrm{s}^2$

Determine units for a and b in these equations if x is measured in seconds (s) and y in meters (m).

Solution

From dimensional analysis of equation (1.19) we conclude that a is measured in m/s and b is measured in m. Next, we turn to equations (1.20). Units for different expressions in these equations are listed in table 1.1. Then the left-hand sides of equations (1.20) will produce m/s for a and m for b, which is consistent with equation (1.19).

2 Limiting Cases

1. Problem

Use limiting cases $a \to b$ and $a \to -b$ to select the two correct formulas from the four options below. (Hint: For each equation, select a limiting case that nulls its left-hand side.)

a) $a^3 - b^3 = (a + b)(a^2 + ab + b^2)$

b) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

c) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

d) $a^3 + b^3 = (a - b)(a^2 - ab + b^2)$

Solution

For $a \to b$ we must have $a^3 - b^3 \to 0$. This shows that formula a is incorrect. For $a \to -b$ we must have $a^3 + b^3 \to 0$. This shows that formula d is incorrect.

2. Problem

Check a limiting case $\alpha = 0$ for the following trigonometric identities. (Hint: Use $\sin(0) = 0$ and $\cos(0) = 1$.)

a) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

b) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

c) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

d) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Solution

We obtain:

a) $\sin\beta = 0 \cdot \cos\beta + 1 \cdot \sin\beta$

b) $\sin(-\beta) = 0 \cdot \cos\beta - 1 \cdot \sin\beta$

c) $\cos\beta = 1 \cdot \cos\beta - 0 \cdot \sin\beta$

d) $\cos(-\beta) = 1 \cdot \cos\beta + 0 \cdot \sin\beta$

3. Problem

Check a limiting case $\alpha = 0$ for the following trigonometric identities. (Hint: Use $\tan(0) = 0$ and $\cot(\alpha) \to \infty$ when $\alpha \to 0$.)

a) $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

b) $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

c) $\cot(\alpha + \beta) = \dfrac{\cot\alpha \cot\beta - 1}{\cot\beta + \cot\alpha}$

d) $\cot(\alpha - \beta) = \dfrac{\cot\alpha \cot\beta + 1}{\cot\beta - \cot\alpha}$

Solution

We obtain:

a) $\tan\beta = \dfrac{0 + \tan\beta}{1 - 0 \cdot \tan\beta}$

b) $\tan(-\beta) = \dfrac{0 - \tan\beta}{1 + 0 \cdot \tan\beta}$

c) $\cot\beta \to \dfrac{\cot\alpha \cot\beta}{\cot\alpha} = \cot\beta$

d) $\cot(-\beta) \to \dfrac{\cot\alpha \cot\beta}{-\cot\alpha} = -\cot\beta$

4. Problem

Check limiting cases for $\alpha \to 0$ and $\alpha \to \pi$ for the following trigonometric identities:

a) $\sin 2\alpha = 2\cos\alpha \sin\alpha$

b) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

c) $\sin^2 \dfrac{\alpha}{2} = \dfrac{1 - \cos\alpha}{2}$

d) $\cos^2 \dfrac{\alpha}{2} = \dfrac{1 + \cos\alpha}{2}$

Solution

We use $\sin 0 = \sin\pi = \sin 2\pi = 0$, $\cos 0 = \cos 2\pi = 1$, $\cos\pi = -1$, $\sin\pi/2 = 1$, and $\cos\pi/2 = 0$.

For $\alpha = 0$ we obtain:

a) $\sin 0 = 2\sin 0 = 0$

b) $\cos 0 = \cos^2 0 - \sin^2 0 = 1$

c) $\sin^2 0 = \dfrac{1 - \cos 0}{2} = 0$

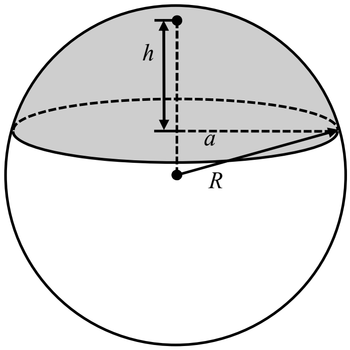

Figure 2.1
A spherical cap: limiting cases

d) $\cos^2 0 = \dfrac{1+\cos 0}{2} = 1$

For $\alpha = \pi$ we obtain:

a) $\sin 2\pi = -2\sin\pi = 0$

b) $\cos 2\pi = \cos^2\pi - \sin^2\pi = 1$

c) $\sin^2 \dfrac{\pi}{2} = \dfrac{1-\cos\pi}{2} = 1$

d) $\cos^2 \dfrac{\pi}{2} = \dfrac{1+\cos\pi}{2} = 0$

5. Problem

Here we revisit the problems for the sum of two ratios and the sum of two scaled ratios. They are solved in sections A.10 and A.11 and compared in section 2.7. Check that each equation in the first column of table 2.1 in section 2.7 can be obtained from the corresponding equation in the second column in the limiting case $p \to 1, q \to 1$.

Solution

A check shows that limiting case $p \to 1, q \to 1$ holds.

6. Problem

A spherical cap is the part of a sphere that lies above a plane that crosses this sphere (see figure 2.1 and section A.30). The volume V and the surface area S of a spherical cap are given by

$$\begin{aligned} V &= \frac{1}{3}\pi h^2(3R-h), \\ S &= 2\pi R h. \end{aligned} \tag{2.1}$$

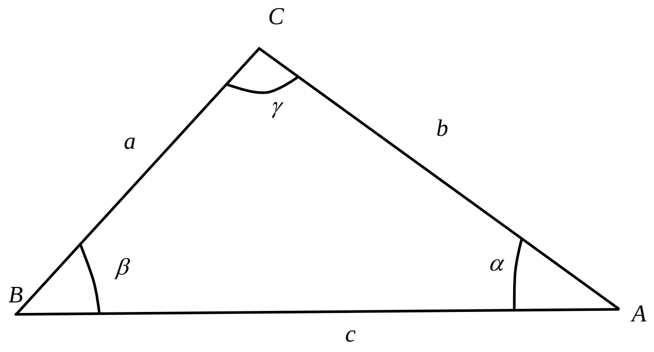

Figure 2.2
Law of cosines: limiting cases

Explore limiting cases $h \to 0, h = R$, and $h \to 2R$. Corresponding formulas for a sphere are

$$V_{\text{sphere}} = \frac{4}{3}\pi R^3, \quad S_{\text{sphere}} = 4\pi R^2. \tag{2.2}$$

Solution

For $h = 0$ we get $V = 0, S = 0$, as expected. For $h = R$ we get

$$V = \frac{1}{3}\pi R^2(3R - R) = \frac{2}{3}\pi R^3 = \frac{V_{\text{sphere}}}{2}, \quad S = 2\pi R^2 = \frac{S_{\text{sphere}}}{2}. \tag{2.3}$$

For $h = 2R$ we get

$$V = \frac{1}{3}\pi(2R)^2(3R - 2R) = \frac{4}{3}\pi R^3 = V_{\text{sphere}}, \quad S = 2\pi R(2R) = 4\pi R^2 = S_{\text{sphere}}. \tag{2.4}$$

7. Problem

The law of cosines computes the length c of a side of a triangle using lengths a, b of two other sides and the angle γ between these two sides (figure 2.2):

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \tag{2.5}$$

Check the following limiting cases and provide an interpretation for each:

a) $a \to 0$

b) $a \to \infty$ and c remains finite

c) $\gamma \to 0$

d) $\gamma \to \dfrac{\pi}{2}$

e) $\gamma \to \pi$

Solution

Various limiting cases correspond to different geometries:

a) From $a \to 0$ and the law of cosines we obtain $c^2 \to b^2$. This is a degenerate triangle, where sides c and b coincide.

b) From $a \to \infty$, c finite, and the law of cosines we obtain:

$$a^2 + b^2 - 2ab\cos\gamma \ll a^2. \tag{2.6}$$

This is equivalent to

$$(a^2 + b^2 - 2ab) + 2ab(1 - \cos\gamma) \ll a^2. \tag{2.7}$$

Expressions $a^2 + b^2 - 2ab$ and $2ab(1 - \cos\gamma)$ are nonnegative and cannot cancel or nearly cancel each other. We conclude that each of them is much smaller than a^2:

$$\begin{aligned} a^2 + b^2 - 2ab = (a - b)^2 \ll a^2 \\ 2ab(1 - \cos\gamma) \ll a^2. \end{aligned} \tag{2.8}$$

Therefore, we have $a \to b$ and $\gamma \to 0$. This is a degenerate triangle, where sides a and b coincide.

c) For $\gamma = 0$ we get

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \\ &= (a - b)^2. \end{aligned} \tag{2.9}$$

This means that either $c = a - b$ or $c = b - a$. In both cases the triangle degenerates into a line, and either c and b add up to a or c and a add up to b.

d) For $\gamma = \pi/2$ we have $\cos\gamma = 0$. Then $c^2 = a^2 + b^2$. This is a right triangle, and the law of cosines degenerates into the Pythagorean theorem.

e) For $\gamma = \pi$ we get $\cos\gamma = -1$ and $c^2 = a^2 + b^2 + 2ab = (a + b)^2$. Then $c = a + b$. The triangle degenerates into a line, and sides a and b add up to c.

8. Problem

Section A.12 solves the following equation:

$$\frac{1}{x-a} - \frac{1}{x-b} = d. \tag{2.10}$$

The solution is as follows:

$$x_{1,2} = \frac{d(a+b) \pm \sqrt{d^2(a-b)^2 + 4d(a-b)}}{2d}, \tag{2.11}$$

where subscripts 1, 2 correspond to the ± signs in the right-hand side. Formulate and check limiting cases for equation (2.10) and its solution (2.11). How many limiting cases can you come up with? (Hint: Use the analysis in section 2.6 as a template.)

Solution

We explore the following limiting cases:

a) From $d \to \infty$ and from equation (2.10) we must have $x \to a$ or $x \to b$. In solution (2.11) we observe that $d^2(a-b)^2 \gg 4d(a-b)$. Then we get

$$x_{1,2} \approx \frac{d(a+b) \pm d(a-b)}{2d}, \tag{2.12}$$

which does produce $x_1 = a$ and $x_2 = b$ in the limit.

b) From $d \to 0$ and from equation (2.10) we must have

$$\frac{1}{x-a} \to \frac{1}{x-b}. \tag{2.13}$$

Then $x \to \infty$ or $a \to b$ (or both).
In solution (2.11) we observe that $d^2(a-b)^2 \ll 4d(a-b)$. Then

$$x_{1,2} \approx \frac{d(a+b) \pm \sqrt{4d(a-b)}}{2d}. \tag{2.14}$$

Since for small d we have $\sqrt{d} \gg d$, we can neglect the first term in the numerator:

$$x_{1,2} \approx \frac{\pm\sqrt{4d(a-b)}}{2d} = \pm\frac{\sqrt{a-b}}{\sqrt{d}}. \tag{2.15}$$

For $d \to 0$, this expression goes to infinity, unless $a = b$. This limiting case holds.

c) Finally, we may consider the case when d is expressed through some other parameter c in a particular way:

$$d = \frac{1}{c-a} - \frac{1}{c-b}. \tag{2.16}$$

Similar to the reasoning in section 2.6, one of the roots of the equation for x must be equal to c. This means that the square root in solution (2.11) should contain an expression that is a complete square. Checking this condition is cumbersome, but still less difficult than the similar check in section 2.6. First, we simplify the expression for d:

$$d = \frac{a-b}{(c-a)(c-b)}. \tag{2.17}$$

Next, we substitute this into the expression in the square root:

$$d^2(a-b)^2 + 4d(a-b) = \frac{(a-b)^4}{(c-a)^2(c-b)^2} + 4\frac{(a-b)^2}{(c-a)(c-b)}. \tag{2.18}$$

We combine the two terms in the right-hand side to get

$$d^2(a-b)^2 + 4d(a-b) = (a-b)^2 \frac{(a-b)^2 + 4(c-a)(c-b)}{(c-a)^2(c-b)^2}. \tag{2.19}$$

The expression in the numerator is as follows:

$$(a-b)^2 + 4(c-a)(c-b) = ((a-b)^2 + 4ab) - 4c(a+b) + 4c^2. \tag{2.20}$$

We observe that $(a-b)^2 + 4ab = (a+b)^2$. Then the right-hand side in equation (2.20) is a complete square

$$\begin{aligned} (a-b)^2 + 4(c-a)(c-b) &= (a+b)^2 - 4c(a+b) + 4c^2 \\ &= (2c - (a+b))^2. \end{aligned} \tag{2.21}$$

Then the expression under the square root in the solution for x is also a complete square:

$$d^2(a-b)^2 + 4d(a-b) = (a-b)^2 \frac{(2c-(a+b))^2}{(c-a)^2(c-b)^2}. \tag{2.22}$$

9. Problem

A radar measures range (distance) to the object it is tracking. Assume that there are two radars that detect a sea vessel at ranges R_1 and R_2 (see figure 2.3). The respective coordinates of the radars are $x_1 = 0, y_1 = 0$ and $x_2 = D, y_2 = 0$. Section A.25 provides a solution for the coordinates of the vessel as

$$\begin{aligned} x &= \frac{D^2 + R_1^2 - R_2^2}{2D}, \\ y &= \pm\sqrt{R_1^2 - x^2}. \end{aligned} \tag{2.23}$$

Explore the effect of the following limiting cases on x and y, and explain the meaning of each:

a) $R_2^2 = D^2 + R_1^2$

b) $R_1^2 = D^2 + R_2^2$

c) $D = R_1 + R_2$

d) $R_1 = R_2$

e) $R_1 = D + R_2$

f) $R_2 = D + R_1$

Solution

We explore the following limiting cases:

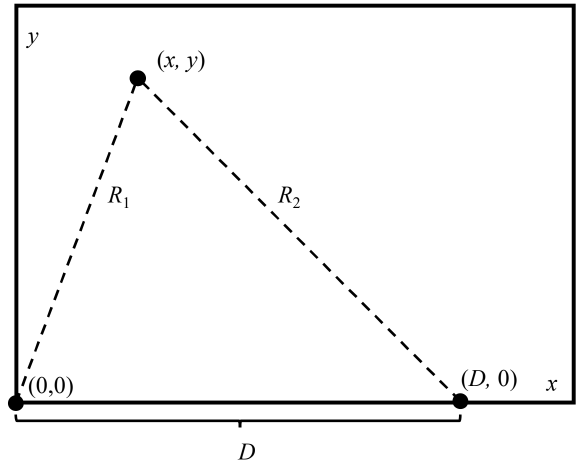

Figure 2.3
Detecting a vessel by two radars: limiting cases

a) $R_2^2 = D^2 + R_1^2$. In this case R_2, R_1 and D satisfy the Pythagorean theorem, which means that they are the sides of a right triangle with the target positioned right above or below the point (0,0). Then the horizontal coordinate of the target must be the same as that of one of the radars: $x = 0$. The vertical coordinate of the target will be equal to $\pm R_1$. Let's check this result in the solution. We substitute $R_2^2 = D^2 + R_1^2$ in equations (2.23):

$$\begin{aligned} x &= \frac{D^2 + R_1^2 - (D^2 + R_1^2)}{2D} = 0, \\ y &= \pm\sqrt{R_1^2 - x^2} = \pm R_1. \end{aligned} \tag{2.24}$$

b) $R_1^2 = D^2 + R_2^2$. Similarly, this is also a right triangle, but the target now is at $(D, \pm R_2)$. We substitute the limiting case condition in the solution to get

$$\begin{aligned} x &= \frac{D^2 + (D^2 + R_2^2) - R_2^2}{2D} = D, \\ y &= \pm\sqrt{R_1^2 - x^2} = \pm\sqrt{R_1^2 - D^2} = \pm R_2. \end{aligned} \tag{2.25}$$

c) $D = R_1 + R_2$. In this case the target is on the line that connects the radars and is located between them. We expect to have $x = R_1, y = 0$. Checking this limiting case in the solution for x produces:

$$\begin{aligned} x &= \frac{D^2 + R_1^2 - R_2^2}{2D} \\ &= \frac{(R_1 + R_2)^2 + R_1^2 - R_2^2}{2(R_1 + R_2)} \\ &= \frac{R_1^2 + R_2^2 + 2R_1R_2 + R_1^2 - R_2^2}{2(R_1 + R_2)} \\ &= \frac{2R_1(R_1 + R_2)}{2(R_1 + R_2)} \\ &= R_1. \end{aligned} \tag{2.26}$$

For the y coordinate we get $y = \pm\sqrt{R_1^2 - x^2} = \pm\sqrt{R_1^2 - R_1^2} = 0$.

d) $R_1 = R_2$. In this case we have an isosceles triangle, which means that we should have the horizontal coordinate of the target at the midpoint between the radars: $x = D/2$. Indeed,

$$\begin{aligned} x &= \frac{D^2 + R_1^2 - R_2^2}{2D} \\ &= \frac{D^2}{2D} \\ &= \frac{D}{2}. \end{aligned} \tag{2.27}$$

e) $R_1 = D + R_2$. Like in case c, the triangle degenerates into a straight line, but now the target is located to the right of both radars. We expect to have $x = R_1, y = 0$. Indeed,

$$\begin{aligned} x &= \frac{D^2 + R_1^2 - R_2^2}{2D} \\ &= \frac{D^2 + (D^2 + 2DR_2 + R_2^2) - R_2^2}{2D} \\ &= \frac{2D(D + R_2)}{2D} \\ &= D + R_2 \\ &= R_1. \end{aligned} \tag{2.28}$$

For the vertical coordinate we get $y = \pm\sqrt{R_1^2 - R_1^2} = 0$.

f) Finally, we consider $R_2 = D + R_1$. The triangle again degenerates into a straight line, but now the target is located to the left of both radars. We expect to get $x = -R_1, y = 0$. Indeed,

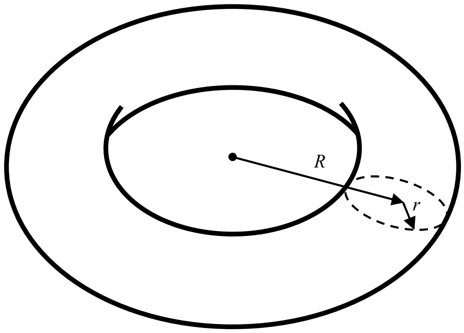

Figure 2.4
A torus: limiting cases

$$
\begin{aligned}
x &= \frac{D^2 + R_1^2 - R_2^2}{2D} \\
&= \frac{D^2 + R_1^2 - (D^2 + 2DR_1 + R_1^2)}{2D} \\
&= \frac{-2DR_1}{2D} \\
&= -R_1.
\end{aligned} \tag{2.29}
$$

For the vertical coordinate we get $y = \pm\sqrt{R_1^2 - R_1^2} = 0$.

10. Problem

A torus is a donut-shaped body (see figure 2.4). Use limiting cases to flag the incorrect formulas for the volume of a torus. Assume that $R > r$.

a) $V = 2\pi^2\left(R^3 + r^3\right)$

b) $V = 2\pi^2 R r^2$

c) $V = 2\pi r^3 e^{-\frac{R}{r}}$

d) $V = 2\pi R^3 e^{-\frac{r}{R}}$

Solution

We consider these four formulas in turn and apply limiting cases to check them:

a) $V = 2\pi^2(R^3 + r^3)$. If $r \to 0$ for a fixed R, the torus becomes very thin, and its volume should go to zero. We see that this formula is incorrect.

b) $V = 2\pi^2 R r^2$. Checking $R \to 0, R \to \infty$ and $r \to 0$ does not produce any red flags. This formula passes limiting case checks. (In fact, it is correct.)

c) $V = 2\pi r^3 e^{-R/r}$. For $R \to \infty$ and a fixed value of r we should expect $V \to \infty$, but the formula gives $V \to 0$. We see that it is incorrect.

d) $V = 2\pi R^3 e^{-r/R}$. For $r \to 0$ and a fixed value of R we should expect $V \to 0$, but the formula gives $V \to 2\pi R^3$. We see that it is incorrect.

11. Problem

Section A.15 solves an equation for the ratio of two cosine functions:

$$\frac{\cos(\alpha + x)}{\cos(\alpha - x)} = \frac{p}{q}. \tag{2.30}$$

The solution is given by

$$x = \tan^{-1}\left(\cot\alpha \cdot \frac{q-p}{q+p}\right) + n\pi, \tag{2.31}$$

where n is an integer.

a) Check the limiting case $p \to q$ for both the formulation of the problem and the solution.

b) Note that the cosine function is periodic with the period of 2π, which implies that for any solution x, the value $x + 2\pi m$ is also a solution. Yet, equation (2.31) has a term $n\pi$, where n may be an odd number. Would odd values of n make the equation not valid if you substitute x in the limiting case $p \to q$ from equation (2.31) in equation (2.30)?

c) Use limiting cases to flag the wrong solutions among the following options and explain how each limiting case works:

i. $x = \tan^{-1}\left(\cot\alpha \cdot \frac{q-p}{p+q}\right) + n\pi$

ii. $x = \tan^{-1}\left(\cot\alpha \cdot \frac{p+q}{p-q}\right) + n\pi$

iii. $x = \cos^{-1}\left(\sin\alpha \cdot \frac{q-p}{p+q}\right) + n\pi$

iv. $x = \sin^{-1}\left(\cos\alpha \cdot \frac{q-p}{p+q}\right) + n\pi$

v. $x = \cos^{-1}\left(\sin\alpha \cdot \frac{q+p}{p-q}\right) + n\pi$

vi. $x = \sin^{-1}\left(\cos\alpha \cdot \frac{q+p}{p-q}\right) + n\pi$

Solution

a) First, we refer to the original equation (2.30) and use $p = q$ there. We see that we must get $\cos(\alpha + x) = \cos(\alpha - x)$. Then one of the following two conditions must be satisfied:

$$\alpha + x = \alpha - x + 2\pi n \tag{2.32}$$

or

$$\alpha + x = -(\alpha - x) + 2\pi n, \tag{2.33}$$

where n is an integer. Condition (2.33) imposes a constraint on α, and not on x. For $\alpha \neq \pi n$, it does not hold, and therefore condition (2.32) must be true. This produces $x = \pi n$.

Let's check this limiting case in the solution. We use

$$\begin{aligned} x &= \tan^{-1}\left(\cot\alpha \cdot \frac{q-p}{q+p}\right) + \pi n \\ &= \tan^{-1}(\cot\alpha \cdot 0) + \pi n \\ &= \pi n. \end{aligned} \tag{2.34}$$

This limiting case holds. (Note that if $\alpha \to \pi n$, then $\cot\alpha \to \infty$ and therefore $p \to q$ no longer implies $x \to \pi n$.)

b) In the limiting case $p \to q$, adding πn to x would change the signs of both numerator and denominator in the ratio of cosines, leaving the ratio intact. Therefore, if x is a solution, $x + \pi n$ is also a solution.

c) We consider these formulas in turn.

i. $x = \tan^{-1}\left(\cot\alpha \cdot \frac{q-p}{q+p}\right) + \pi n.$

We have already considered limiting case $p \to q$ for this formula and concluded that it holds. To be thorough, we may also consider the case $p \to -q$. Then from equation (2.30) we get $\cos(\alpha + x) \to -\cos(\alpha - x)$. This is equivalent to

$$\alpha + x \to \alpha - x + (2n+1)\pi \tag{2.35}$$

or

$$\alpha + x \to -\alpha + x + (2n+1)\pi, \tag{2.36}$$

where n is an integer. The first condition produces $x \to \pi/2 + \pi n$. In the solution (2.31) the argument of $\tan^{-1}$ goes to infinity, which produces $\pi/2$ for the value of $\tan^{-1}$.

The second condition produces $\alpha \to \pi n + \pi/2$, but does not impose any condition on x. In the solution (2.31) we get $\cot\alpha \to 0$, which in the combination with $q + p \to 0$ in the denominator makes the value of x indeterminate.

This limiting case also holds.

ii. $x = \tan^{-1}\left(\cot\alpha \cdot \frac{q+p}{q-p}\right) + \pi n.$

In this solution checking limiting case $p \to q$ is similar to checking limiting case $p \to -q$ for the previous formula and the other way around. We conclude that both limiting cases do not hold.

iii. $x = \cos^{-1}\left(\sin\alpha \cdot \frac{q-p}{q+p}\right) + \pi n.$

We consider two limiting cases. If $q \to p$, the argument of $\cos^{-1}$ approaches zero, and we get $x \to \pi/2 + \pi n$. We already know that this does not hold for equation (2.30), so this solution must be incorrect. In another confirmation, we use $\alpha = 0$. For the solution we again get $x \to \pi/2 + \pi n$. However, this does not satisfy the original equation (2.30), unless $p = -q$.

iv. $x = \sin^{-1}\left(\cos\alpha \cdot \dfrac{q-p}{q+p}\right) + \pi n.$

We consider $\alpha \to \pi/2$ and $p \neq -q$. Then the argument of $\sin^{-1}$ approaches zero, and $x \to \pi n$. However, this value does not satisfy the original equation (2.30):

$$\frac{\cos\left(\frac{\pi}{2} + x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\sin(-x)}{\sin x} = -1 \neq \frac{p}{q}. \tag{2.37}$$

v. $x = \cos^{-1}\left(\sin\alpha \cdot \dfrac{q+p}{q-p}\right) + \pi n.$

We use $\alpha = 0$ and follow the reasoning for formula in task *iii* above to conclude that this solution is incorrect.

vi. $x = \sin^{-1}\left(\cos\alpha \cdot \dfrac{q+p}{q-p}\right) + \pi n.$

We use $\alpha \to \pi/2$ and follow the reasoning for formula in task *iv* above to conclude that this solution is incorrect.

12*. In section A.14 we seek a solution for an equation that contains a sum of two sine functions:

$$p\sin(x+\phi) + q\sin x = c. \tag{2.38}$$

The solution is given as

$$x = -\tan^{-1}\frac{p\sin\phi}{p\cos\phi + q} + (-1)^n \sin^{-1}\left(\frac{c}{\sqrt{p^2 + 2pq\cos\phi + q^2}}\right) + n\pi. \tag{2.39}$$

a) Check the validity of the solution using the following limiting cases:

i. $\phi \to 0$

ii. $q \to 0$

iii. $p \to 0$

b) Use a limiting case $\phi = -\pi/2$. Show that this problem is reduced to one that is solved in section A.13. Use the solution in section A.13 to check this limiting case for equation (2.39).

Solution

a) We check limiting cases for the original equation (2.38) and the solution:

i.) Case $\phi \to 0$. For equation (2.38) we have

$$(p+q)\sin x = c. \tag{2.40}$$

The solutions of this equation are given as

$$x = (-1)^n \sin^{-1}\frac{c}{p+q} + \pi n. \tag{2.41}$$

Now we check this limiting case for solution (2.39). For $\phi = 0$ the argument of $\tan^{-1}$ is zero, as is the value of $\tan^{-1}$. Under the square root we get $p^2 + 2pq\cos\phi + q^2 = p^2 + 2pq + q^2 = (p+q)^2$. Then the square root produces $|p+q|$. We see that this limiting case holds if $|p+q| = p+q$. This constraint is explained by a footnote in section A.13 in the textbook, which says that depending on the values of parameters, we may have to add π to the final result.

ii.) Case $q \to 0$. Equation (2.38) is transformed to

$$p\sin(x+\phi) = c. \tag{2.42}$$

Its solutions are given by

$$x = (-1)^n \sin^{-1}\frac{c}{p} - \phi + \pi n. \tag{2.43}$$

For $q = 0$ we get for solution (2.39)

$$x = -\tan^{-1}(\tan\phi) + (-1)^n \sin^{-1}\frac{c}{\sqrt{p^2}} + \pi n. \tag{2.44}$$

This produces

$$x = (-1)^n \sin^{-1}\frac{c}{|p|} - \phi + \pi n. \tag{2.45}$$

Similar to the previous case, this limiting case holds if $|p| = p$. This constraint is explained by a footnote in section A.13 in the textbook, which says that depending on the values of parameters, we may have to add π to the final result.

iii.) Case $p \to 0$. Equation (2.38) is transformed to

$$q\sin x = c. \tag{2.46}$$

Its solutions are given by

$$x = (-1)^n \sin^{-1}\frac{c}{q} + \pi n. \tag{2.47}$$

For $p = 0$ we get for solution (2.39)

$$x = -\tan^{-1} 0 + (-1)^n \sin^{-1}\frac{c}{\sqrt{q^2}} + \pi n. \tag{2.48}$$

This limiting case holds with the same caveat as above.

b) Case $\phi = -\pi/2$. The original equation (2.38) takes the form

$$p \sin\left(x - \frac{\pi}{2}\right) + q \sin x = c. \tag{2.49}$$

This is equivalent to

$$-p \cos x + q \sin x = c. \tag{2.50}$$

This equation is indeed reduced to one solved in section A.13 if we denote $a = q, b = -p$. Its solution is

$$x = -\tan^{-1}\frac{b}{a} + (-1)^n \sin^{-1}\frac{c}{\sqrt{a^2 + b^2}} + \pi n. \tag{2.51}$$

In equation (2.39) for $\phi = -\pi/2$ we get $\sin\phi = -1, \cos\phi = 0$, which yields

$$x = \tan^{-1}\frac{p}{q} + (-1)^n \sin^{-1}\frac{c}{\sqrt{p^2 + q^2}} + \pi n. \tag{2.52}$$

For $a = q, b = -p$ we get

$$x = -\tan^{-1}\frac{b}{a} + (-1)^n \sin^{-1}\frac{c}{\sqrt{a^2 + b^2}} + \pi n. \tag{2.53}$$

We see that this limiting case holds.

13. Problem

The radius of Archimedes's spiral increases linearly with the turn angle (figure 2.5). Use limiting cases to flag the incorrect formulas for the arc length of the spiral:

a) $L = \frac{1}{4\pi}(\Delta R + 1)\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right)$

b) $L = \frac{1}{4\pi}\Delta R\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right)$

c) $L = \frac{1}{4\pi}\Delta R\left(\theta\sqrt{1+\theta^2} + \ln(\theta)\right)$

d) $L = \frac{1}{4\pi}\Delta R\left(\frac{1+\theta^2}{\theta} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right)$

Here ΔR is the distance between the adjacent loops and θ is the total turn angle.

Solution

For formulas *a* through *d* we consider the following limiting cases:

a) For $\Delta R \to 0$ and a fixed turn angle, we expect the total arc length of the spiral to be zero. We conclude that this formula is incorrect.

b) We consider four limiting cases: $\Delta R \to 0, \theta \to 0, \Delta R \to \infty$, and $\theta \to \infty$. For small $\Delta R, \theta$ we expect the arc angle to be small, and for large $\Delta R, \theta$ we expect the arc angle to be large. These conditions hold for formula *b*.

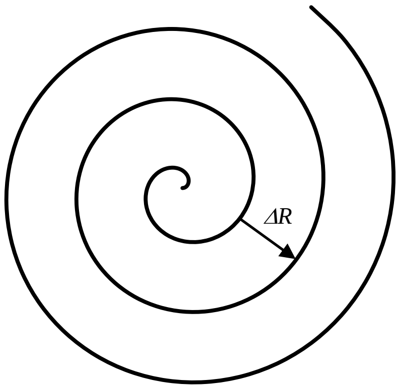

Figure 2.5
Archimedes's spiral: limiting cases

c) We consider $\theta \to 0$. The formula predicts a negative (and large by the absolute value) arc length, which is obviously incorrect.

d) We again consider $\theta \to 0$. The formula predicts a large value of the arc length, which is obviously incorrect.

14. Problem

Chapter 3 shows how to solve the following trigonometric equation:

$$\sin(x-a) + \sin(x-b) = c. \tag{2.54}$$

Use a limiting case $a \to b$ to select the correct solution among the following options:

a) $x = \dfrac{a+b}{2} + (-1)^n \sin^{-1}\left(\dfrac{c}{2\cos\frac{a-b}{2}}\right) + \pi n$

b) $x = \dfrac{a-b}{2} + (-1)^n \sin^{-1}\left(\dfrac{c}{2\cos\frac{a+b}{2}}\right) + \pi n$

c) $x = \dfrac{a-b}{2} + (-1)^n \sin^{-1}\left(\dfrac{c}{2\cos\frac{a-b}{2}}\right) + \pi n$

d) $x = \dfrac{a+b}{2} + (-1)^n \sin^{-1}\left(\dfrac{c}{2\cos\frac{a+b}{2}}\right) + \pi n$

Solution

For equation (2.54), limiting case $a \to b$ produces

$$2 \sin (x - a) = c. \tag{2.55}$$

The solution of this equation is

$$x = a + (-1)^n \sin^{-1} \frac{c}{2} + \pi n. \tag{2.56}$$

The limiting case $a \to b$ in application to the four options is as follows:

a) $x = a + (-1)^n \sin^{-1} \frac{c}{2} + \pi n$

b) $x = (-1)^n \sin^{-1} \frac{c}{2 \cos a} + \pi n$

c) $x = (-1)^n \sin^{-1} \frac{c}{2} + \pi n$

d) $x = a + (-1)^n \sin^{-1} \frac{c}{2 \cos a} + \pi n$

We conclude that option *a* is correct.

15. Problem

There are syrups with masses m_1, m_2, and m_3 with sugar concentrations p_1, p_2, and p_3. Sections A.16 and A.17 show that blends of two or three of these syrups will have sugar concentrations respectively given by

$$\begin{aligned} p_{12} &= \frac{p_1 m_1 + p_2 m_2}{m_1 + m_2}, \\ p_{123} &= \frac{p_1 m_1 + p_2 m_2 + p_3 m_3}{m_1 + m_2 + m_3}. \end{aligned} \tag{2.57}$$

a) Investigate the following limiting cases for blending two syrups and explain the results:

- **i.** The amount of syrup 2 is very small ($m_2 \ll m_1$).
- **ii.** Both syrups have the same concentration of sugar ($p_1 = p_2$).
- **iii.** Show that the sugar concentration of the mix is always in between the concentrations of the mixing ingredients ($p_1 \leq p_{12} \leq p_2$ or $p_1 \geq p_{12} \geq p_2$). (Hint: Try to compute $(p_{12} - p_1)(p_{12} - p_2)$ and show that the result is nonpositive.)

b) Show that if the amount of the third syrup is small ($m_3 \ll m_1; m_3 \ll m_2$), the equation for the problem for three syrups reduces to that for the problem for two syrups. Explain this result.

Solution

a) We consider three limiting cases

i.) For $m_2 \ll m_1$ we get

$$p_{12} \approx \frac{p_1 m_1}{m_1} = p_1. \tag{2.58}$$

This equation reduces to p_1, which is the concentration of the first syrup before mixing. Indeed, adding a small amount of syrup 2 should not alter the concentration of syrup 1 much.

ii.) For $p_2 = p_1$ we get $p_{12} = p_1 = p_2$. This is expected: mixing two equally concentrated syrups does not alter the concentration.

iii.) We compute

$$\begin{aligned}(p_{12} - p_1)(p_{12} - p_2) &= \left(\frac{p_1 m_1 + p_2 m_2}{m_1 + m_2} - p_1\right)\left(\frac{p_1 m_1 + p_2 m_2}{m_1 + m_2} - p_2\right)\\ &= \frac{p_2 m_2 - p_1 m_2}{m_1 + m_2} \cdot \frac{p_1 m_1 - p_2 m_1}{m_1 + m_2}\\ &= \frac{(p_2 - p_1) m_2}{m_1 + m_2} \cdot \frac{(p_1 - p_2) m_1}{m_1 + m_2}\\ &= -\frac{(p_2 - p_1)^2 m_2 m_1}{(m_1 + m_2)^2}\\ &\leq 0.\end{aligned} \tag{2.59}$$

b) We set $m_3 \ll m_1; m_3 \ll m_2$ in the equation for the concentration of a mix of three syrups to get

$$p_{123} = \frac{p_1 m_1 + p_2 m_2 + p_3 m_3}{m_1 + m_2 + m_3} \approx \frac{p_1 m_1 + p_2 m_2}{m_1 + m_2}, \tag{2.60}$$

which is indeed the equation for the concentration of a mix of two syrups. This is expected, because adding the third syrup does not make any difference in this scenario.

16. Problem

Section A.4 solves the problem of finding intersections between a circle and an ellipse (see figure 2.6). The circle and the ellipse are given by the following equations:

$$\begin{aligned}&x^2 + y^2 = R^2,\\ &\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} = 1.\end{aligned} \tag{2.61}$$

Formulate one or more limiting cases for this problem and use them to flag the wrong solutions among the following options:

a) $x = \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$; $\quad y = \pm\sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$

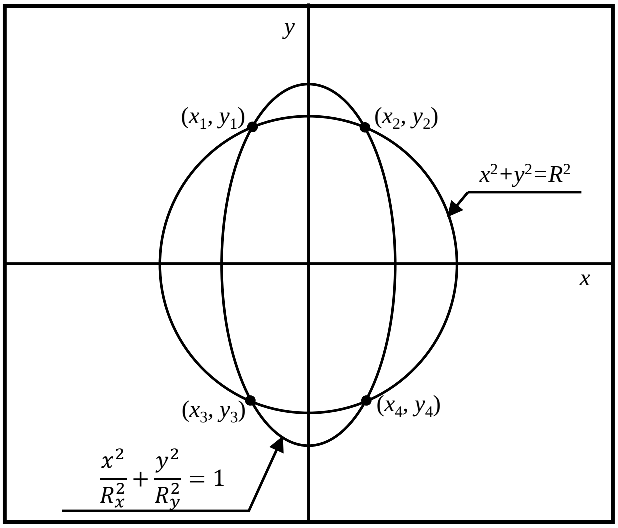

Figure 2.6
A circle and an ellipse: limiting cases

b) $x = \pm\sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $\qquad y = \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$

c) $x = \pm\sqrt{R_x^2 \frac{R_y^2 + R^2}{R_y^2 - R_x^2}}$; $\qquad y = \pm\sqrt{R_y^2 \frac{R_x^2 + R^2}{R_x^2 - R_y^2}}$

(Hint: Note that the top and the bottom points on the ellipse are given by $y_{t,b} = \pm R_y$, and the rightmost and leftmost points are given by $x_{r,l} = \pm R_x$.)

Solution

We consider $R_x \to 0$. The ellipse degenerates into a vertical line segment. The points of intersection (if any) must have $x = 0$. The circle will intersect this line segment if $R_y \geq R$, and the vertical coordinates of the intersection will be given by $y = \pm R$. We consider the three options in turn.

a) A check for the solution shows that for $R_x = 0$

$$x = \pm\sqrt{R_y^2 \frac{-R^2}{-R_y^2}} = \pm R, \tag{2.62}$$

which is incorrect.

b) For $R_x = 0$ we get

$$
\begin{aligned}
x &= \pm\sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}} = 0, \\
y &= \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}} = \pm\sqrt{R_y^2 \frac{-R^2}{-R_y^2}} = \pm R.
\end{aligned}
\tag{2.63}
$$

Note that for small but nonzero values of R_x and for $R > R_y$ we get imaginary values of x, which means that the circle and the ellipse do not intersect. This limiting case holds.

c) For $R_x = 0$ we get imaginary values for y. This formula is incorrect.

Application of limiting case $R_y = 0$ is analogous.

17. Problem

We again refer to section A.4 and figure 2.6. The circle and the ellipse are given by the following equations:

$$
\begin{aligned}
x^2 + y^2 &= R^2, \\
\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} &= 1.
\end{aligned}
\tag{2.64}
$$

The coordinates of the intersection points (if any) for these two curves are given by

$$
\begin{aligned}
x &= \pm\sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}, \\
y &= \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}.
\end{aligned}
\tag{2.65}
$$

Show that solutions exist if one of the following conditions holds:

$$
\begin{aligned}
R_x \le R \le R_y, \\
R_y \le R \le R_x.
\end{aligned}
\tag{2.66}
$$

Explain why this is true from figure 2.6. (Hint: Note that the top and the bottom points on the ellipse are given by $y_{t,b} = \pm R_y$, and the rightmost and leftmost points are given by $x_{r,l} = \pm R_x$.)

Solution

There are four options for the geometry of the problem

a) If $R < R_x, R < R_y$, the circle lies inside the ellipse and there are no intersections between the two curves. In this case we have $R_y^2 - R^2 > 0, R_x^2 - R^2 > 0$. Both numerators in the square roots in solution (2.65) for x, y are positive. However, denominators have different signs. Therefore, one of the expressions under the square roots will be negative, producing an imaginary value for the solution. This indicates that there are no intersection points.

b) If $R > R_x, R > R_y$, the ellipse lies entirely inside the circle and there are no intersections between the two curves. In this case we have $R_y^2 - R^2 < 0, R_x^2 - R^2 < 0$. In solution (2.65) one of the expressions under the square roots is negative, which indicates that there are no solutions for x, y.

c) If $R_x \geq R \geq R_y$ (but $R_x \neq R_y$), the ellipse extends beyond the circle in the x direction, but is more narrow than the circle in the y direction. We expect to have four intersections between the two curves. Expressions under the square roots in (2.65) are positive, yielding valid solutions for x, y.

d) The case $R_x \leq R \leq R_y$ is analogous to case c.

18*. Consider the problem for the difference between an unknown and its reciprocal:

$$x - \frac{1}{x} = d. \tag{2.67}$$

Its solution is presented in section A.9:

$$x_{1,2} = \frac{d \pm \sqrt{d^2 + 4}}{2}. \tag{2.68}$$

Following the examples in section 2.11, explore the following limiting cases in this problem:

a) A large value in the right-hand side: $d \to \infty$

b) A special case when d can be represented as $d = q - 1/q$, where q is a parameter

c) What happens if we square equation (2.67)? Show that the roots for the squared equation form two pairs and are reciprocal in each pair. Prove the reciprocity of x_1^2 and x_2^2 from equation (2.68).

Solution

We consider the three limiting cases in turn.

a) For $d \to \infty$, equation (2.67) shows that we should have either $x \to \infty$ or $x \to 0$. For the plus sign in the solution, we get

$$x_1 = \frac{d + \sqrt{d^2 + 4}}{2} \to \infty. \tag{2.69}$$

To prove that $x_2 \to 0$, we note that $x_1 x_2 = -1$. Indeed,

$$\begin{aligned} x_1 x_2 &= \frac{d + \sqrt{d^2 + 4}}{2} \cdot \frac{d - \sqrt{d^2 + 4}}{2} \\ &= \frac{d^2 - (d^4 + 4)}{4} \\ &= -1. \end{aligned} \tag{2.70}$$

Then, if $x_1 \to \infty$, we have $x_2 \to 0$.

b) For $d = q - 1/q$ the equation takes the form

$$x - \frac{1}{x} = q - \frac{1}{q}, \tag{2.71}$$

and we expect to have one of the roots $x = q$. Note that $x = -1/q$ also satisfies this equation, so it must be the second root. Let's check this in solution (2.68). We substitute $d = q - 1/q$ in the solution. First, it is helpful to simplify the expression for the discriminant:

$$d^2 + 4 = q^2 + \frac{1}{q^2} - 2 + 4 = \left(q + \frac{1}{q}\right)^2. \tag{2.72}$$

Then

$$x_{1,2} = \frac{\left(q - \frac{1}{q}\right) \pm \left(q + \frac{1}{q}\right)}{2}, \tag{2.73}$$

which does produce $x_1 = q, x_2 = -1/q$, as expected.

c) Squaring equation (2.67) produces the following

$$x^2 + \frac{1}{x^2} - 2 = d^2. \tag{2.74}$$

We multiply this equation by x^2 to get a quadratic equation for x^2 (such equations are called biquadratic):

$$(x^2)^2 - (d^2 + 2)x^2 + 1 = 0. \tag{2.75}$$

Its four roots are given by

$$x_{1,2,3,4} = \mp\sqrt{\frac{d^2 + 2 \pm \sqrt{d^4 + 4d^2}}{2}}, \tag{2.76}$$

where we can have any combination of the signs in the right-hand side. Pairs of these roots are reciprocal. Indeed,

$$\begin{aligned} &\sqrt{\frac{d^2 + 2 + \sqrt{d^4 + 4d^2}}{2}} \cdot \sqrt{\frac{d^2 + 2 - \sqrt{d^4 + 4d^2}}{2}} \\ =&\sqrt{\frac{(d^2 + 2)^2 - (d^4 + 4d^2)}{4}} \\ =&1. \end{aligned} \tag{2.77}$$

We can also prove reciprocity of x_1^2 and x_2^2 from equation (2.68). We already proved that $x_1 = -1/x_2$, so x_1^2 and x_2^2 must be reciprocal.

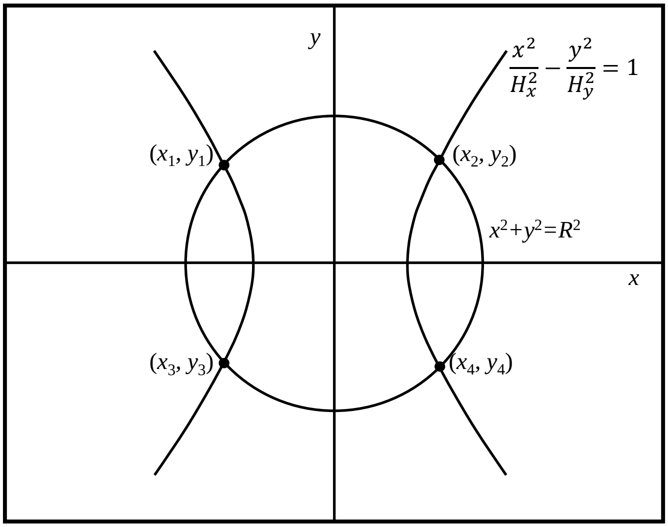

Figure 2.7
A circle and a hyperbola: limiting cases

19. Problem

Section A.5 solves the problem of finding the intersections between a circle and a hyperbola (see figure 2.7). The circle and the hyperbola are given by the following equations:

$$\begin{aligned} x^2 + y^2 &= R^2, \\ \frac{x^2}{H_x^2} - \frac{y^2}{H_y^2} &= 1. \end{aligned} \tag{2.78}$$

a) What is the value of x that satisfies the hyperbola equation for the limiting case $y = 0$?

b) Use the answer to the previous question to identify the wrong solutions for the intersections between the circle and the hyperbola:

i. $y = \pm\sqrt{H_y^2 \frac{R^2 - H_x^2}{H_x^2 + H_y^2}}$; $\quad x = \pm\sqrt{H_x^2 \frac{R^2 - H_y^2}{H_x^2 + H_y^2}}$

ii. $y = \pm\sqrt{H_y^2 \frac{R^2 + H_x^2}{H_x^2 + H_y^2}}$; $\quad x = \pm\sqrt{H_x^2 \frac{R^2 + H_y^2}{H_x^2 + H_y^2}}$

iii. $y = \pm\sqrt{H_y^2 \frac{R^2 - H_x^2}{H_x^2 + H_y^2}}$; $\quad x = \pm\sqrt{H_x^2 \frac{R^2 + H_y^2}{H_x^2 + H_y^2}}$

iv. $y = \pm\sqrt{H_y^2 \dfrac{R^2 + H_x^2}{H_x^2 + H_y^2}}$; $x = \pm\sqrt{H_x^2 \dfrac{R^2 - H_y^2}{H_x^2 + H_y^2}}$

Solution

a) For $y = 0$ we obtain from the equation for the hyperbola $x = \pm H_x$.

b) From the previous question and from the figure we can see that if $R = H_x$, then the intersections of the hyperbola and the circle will have coordinates $x = \pm H_x, y = 0$. We use this observation to consider the four solution options in turn.

i.) Substitution of $R = H_x$ does not produce $x = \pm H_x$. This solution is incorrect.

ii.) Substitution of $R = H_x$ does not produce $y = 0$. This solution is incorrect.

iii.) Substitution of $R = H_x$ produces $x = \pm H_x, y = 0$. This solution may be correct.

iv.) Substitution of $R = H_x$ does not produce $x = \pm H_x$ or $y = 0$. This solution is incorrect.

20. Problem

The law of sines links the angles and lengths of adjacent sides in a triangle (figure 2.2):

$$a \sin\beta = b \sin\alpha. \tag{2.79}$$

Check the following limiting cases and provide an interpretation for each:

a) $a \to 0$

b) $\alpha \to \dfrac{\pi}{2}$

c) $\alpha \to 0$

d) $\alpha \to \beta$

e) $\alpha \to \dfrac{\pi}{2} - \beta$

Solution

We consider the limiting cases in turn.

a) For $a \to 0$ we must have $b \to 0$ or $\alpha \to 0$ (or both). The first case simply corresponds to a very small triangle. The second case corresponds to a triangle that nearly degenerates into a straight line, with sides b and c nearly overlapping.

b) For $\alpha \to 0$ we have $\sin\alpha \to 0$. From equation (2.79) we obtain either $a \to 0$ or $\beta \to 0$. The first case ($\alpha \to 0$ and $a \to 0$) was already analyzed: sides b and c nearly overlap. The second case ($\alpha \to 0$ and $\beta \to 0$) corresponds to a triangle that nearly degenerates into a straight line, where sides b and a add up to side c.

c) For $\alpha = \pi/2$ we have a right triangle. We use $\sin\alpha = 1$. From equation (2.79) we obtain $a \sin\beta = b$, which directly follows from the definition of the sine function.

d) For $\alpha = \beta$ we see from equation (2.79) that $a = b$. This means that if a triangle has two equal angles, it is isosceles.

e) For $\alpha = \pi/2-\beta$ we compute the third angle $\gamma = \pi-\alpha-\beta = \pi/2$. This is also a right triangle. We use $\sin\alpha = \sin(\pi/2 - \beta) = \cos\beta$. From equation (2.79) we obtain $a\sin\beta = b\cos\beta$, which is equivalent to $\tan\beta = b/a$. This directly follows from the definition of the tangent function for angle β.

21. Problem

The law of tangents is another way to link the angles and lengths of adjacent sides in a triangle (figure 2.2):

$$\frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}. \tag{2.80}$$

Check the following limiting cases and provide an interpretation for each:

a) $\beta \to 0$ and α remains finite

b) $a = b$

c) $\alpha \to \pi/2-\beta$ (Hint: You may need to use a formula for the tangent of the difference of two angles from exercise 3.)

Solution

We consider the limiting cases in turn.

a) For $\beta \to 0$ and a fixed nonzero value of α we obtain from equation (2.80)

$$\frac{a-b}{a+b} \to 1, \tag{2.81}$$

which implies $b \to 0$. This is a triangle with a very small side b. Since b is the opposing side to angle β, a small value of β is expected.

b) For $a = b$ we get from equation (2.80) $\alpha = \beta$. This means that in an isosceles triangle two angles are equal.

c) We use $\alpha = \pi/2 - \beta$ and $\alpha + \beta + \gamma = \pi$ to get $\gamma = \pi/2$. Therefore, this is a right triangle. From equation (2.80) we obtain

$$\begin{aligned}\frac{a-b}{a+b} &= \frac{\tan\left(\frac{\pi}{4}-\beta\right)}{\tan\frac{\pi}{4}} \\ &= \tan\left(\frac{\pi}{4}-\beta\right),\end{aligned} \tag{2.82}$$

where we used $\tan\pi/4 = 1$. In the right-hand side we use the formula for the tangent of a difference:

$$\begin{aligned}\frac{a-b}{a+b} &= \tan\left(\frac{\pi}{4}-\beta\right)\\ &= \frac{\tan\frac{\pi}{4}-\tan\beta}{1+\tan\frac{\pi}{4}\tan\beta}\\ &= \frac{1-\tan\beta}{1-\tan\beta}.\end{aligned} \tag{2.83}$$

For the right triangle we have $\tan\beta = b/a$. We substitute this in the right-hand side of the last equation to get

$$\begin{aligned}\frac{a-b}{a+b} &= \frac{1-\tan\beta}{1-\tan\beta}\\ &= \frac{1-\frac{b}{a}}{1+\frac{b}{a}}\\ &= \frac{a-b}{a+b}.\end{aligned} \tag{2.84}$$

We arrived at an identity. This limiting case checks.

22[*]. The law of tangents above is closely related to Mollweide's formula (see figure 2.2 for notations):

$$\begin{aligned}\frac{a+b}{c} &= \frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}},\\ \frac{a-b}{c} &= \frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}.\end{aligned} \tag{2.85}$$

Explore the following limiting cases:

a) $c \to 0$, and a, b remain finite.

b) $a \to b$ (two sides are equal).

c) For the first Mollweide's formula, $c \to a+b$ and a, b, c remain finite.

d) For the second Mollweide's formula, $c \to a-b$ and a, b, c remain finite.

Solution

We consider these limiting cases in turn.

a) For $c \to 0$ and the first Mollweide's formula we have the left-hand side going to infinity. This means that in the right-hand side we must have $\gamma \to 0$. This corresponds to a very narrow triangle, with a small side opposing angle γ. For such a triangle, the two remaining sides have nearly equal lengths. In the second Mollweide's formula the right-hand side is not going to

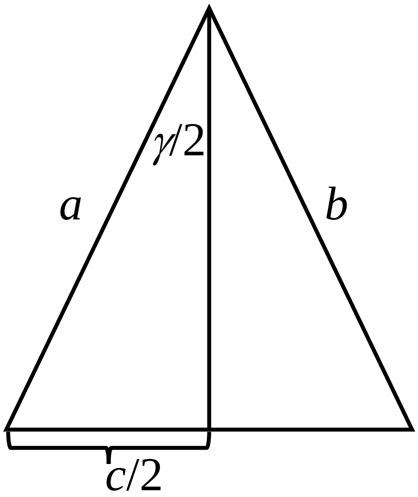

Figure 2.8
A limiting case for the first Mollweide's formula

infinity. Therefore, the left-hand side $(a-b)/c$ remains finite, even though $c \to 0$. This is made possible by $a \to b$.

b) For $a = b$ and c being finite, we have the left-hand side of the second Mollweide's formula equal to zero. This produces $\alpha = \beta$. Indeed, in an isosceles triangle two angles must be equal.

Next, we use $a = b$ and $\alpha = \beta$ (the latter equation we proved above). Then from the first Mollweide's formula we get

$$\sin\frac{\gamma}{2} = \frac{c}{2a}. \tag{2.86}$$

This case is illustrated in figure 2.8. For the half-angle $\gamma/2$, equation (2.86) follows from the definition of the sine function.

c) We consider $c = a + b$. If one side is equal to the sum of two other sides, such a "triangle" degenerates into a straight line. Let's check if this also follows from the equations. The left-hand side of the first Mollweide's formula is equal to 1. Therefore

$$\cos\frac{\alpha-\beta}{2} = \sin\frac{\gamma}{2}. \tag{2.87}$$

This is equivalent to

$$\cos\frac{\alpha-\beta}{2} = \cos\left(\frac{\pi}{2} - \frac{\gamma}{2}\right). \tag{2.88}$$

Then

$$\frac{\alpha-\beta}{2}=\frac{\pi}{2}-\frac{\gamma}{2}, \tag{2.89}$$

which yields

$$\pi=\alpha+\gamma-\beta. \tag{2.90}$$

But we also know that in a triangle

$$\pi=\alpha+\gamma+\beta. \tag{2.91}$$

The last two equations are possible to satisfy only if $\beta = 0$. This shows that the triangle degenerates into a straight line, as expected.

d) We consider $c = a - b$. This is equivalent to $a = c + b$, which is also a case of a triangle that degenerates into a straight line. For the second Mollweide's formula we follow a derivation analogous to the one for the previous limiting case $c = a + b$, arriving at the same conclusion.

23*. Heron's formula links the area S of a triangle, its half-perimeter $p = (a + b + c)/2$, and the lengths of its sides (see figure 2.2):

$$S = \sqrt{p(p-a)(p-b)(p-c)}. \tag{2.92}$$

Explore the following limiting cases:

a) $a \to 0, \quad b \to c$

b) $a \to b + c$

c) $a^2 + b^2 = c^2$

d) $a^2 + c^2 = b^2$

Solution

We consider these limiting cases in turn.

a) If $a \to 0; b \to c$, the triangle is very narrow, with two sides b and c nearly coinciding. The area of such a triangle should be going to zero. Indeed, we have

$$p-b=\frac{a+b+c}{2}-b \to 0, \tag{2.93}$$

which yields $S \to 0$.

b) For $a = b + c$ the triangle degenerates into a straight line, and its area should be zero. In Heron's formula one of the factors is computed as

$$\begin{aligned} p-a &= \frac{a+b+c}{2}-a \\ &= \frac{b+c-a}{2} \\ &= 0, \end{aligned} \tag{2.94}$$

which yields $S = 0$.

c) We consider $a^2 + b^2 = c^2$. This is a right triangle, and its area is $S = ab/2$. Let's check this limiting case in Heron's formula. This formula contains four factors under the square root:

$$\begin{aligned} p &= \frac{a+b+c}{2}, \\ p-a &= \frac{b+c-a}{2}, \\ p-b &= \frac{a+c-b}{2}, \\ p-c &= \frac{a+b-c}{2}. \end{aligned} \tag{2.95}$$

We compute products of pairs of these factors:

$$\begin{aligned} p(p-c) &= \frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \\ &= \frac{(a+b)^2 - c^2}{4} \\ &= \frac{a^2 + b^2 - c^2 + 2ab}{4} \\ &= \frac{ab}{2} \end{aligned} \tag{2.96}$$

and

$$\begin{aligned} (p-a)(p-b) &= \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \\ &= \frac{c^2 - (a-b)^2}{4} \\ &= \frac{c^2 - (a^2 + b^2) + 2ab}{4} \\ &= \frac{ab}{2}. \end{aligned} \tag{2.97}$$

Then Heron's formula yields

$$\begin{aligned} S &= \sqrt{p(p-a)(p-b)(p-c)} \\ &= \sqrt{\left(\frac{ab}{2}\right)^2} \\ &= \frac{ab}{2}. \end{aligned} \tag{2.98}$$

This limiting case holds.

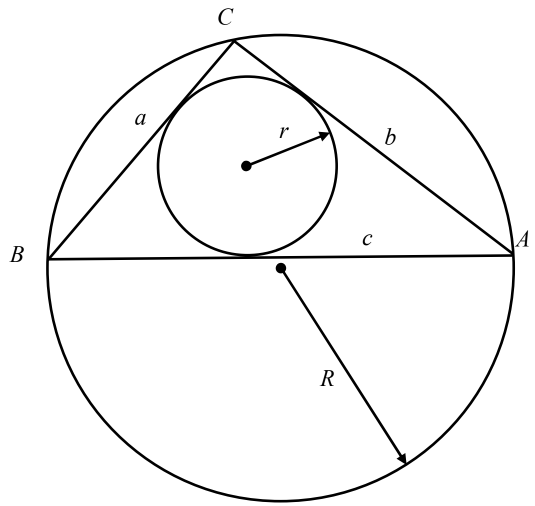

Figure 2.9
A triangle, an inscribed circle, and a circumscribed circle: limiting cases

d) For $a^2 + c^2 = b^2$ the derivation is analogous, but the factors under the square root should be coupled into pairs in a different way.

24. Problem

It is known that the radius of the inscribed circle r is as follows (see figure 2.9):

$$r = \frac{S}{p}, \tag{2.99}$$

where S is the area of the triangle and p is its half-perimeter: $p = (a + b + c)/2$. Explore the limiting case $S \to 0$ and p remains finite.

Solution

If the area of a triangle is small, but its perimeter remains finite, such a triangle is nearly flattened into a straight line. Then the radius of the inscribed circle must be small, which is consistent with formula (2.99).

25. Problem

Another formula for the radius of the inscribed circle r is given by

$$r = \frac{\sqrt{p(p-a)(p-b)(p-c)}}{p}, \tag{2.100}$$

where p is its half-perimeter: $p = (a + b + c)/2$. Explore what happens if $p \to c$. Show that r becomes small in this case and why that follows from figure 2.9.

Solution

For $p = c$, formula (2.100) predicts $r = 0$. We use the explicit expression for p to get

$$\frac{a + b + c}{2} - c = 0. \tag{2.101}$$

This yields $a + b = c$, which means that this "triangle" degenerates into a straight line. For such a triangle, the radius of the inscribed circle must be zero.

26. Problem

We again refer to the drawing in figure 2.9. The radius of the circumscribed circle R is given by

$$R = \frac{abc}{\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}}. \tag{2.102}$$

Consider the following limiting cases:

a) Three separate limiting cases that lead to a small value in the denominator even if lengths of all sides remain finite:

i. $a \to b + c$

ii. $a \to b - c$

iii. $a \to c - b$

Why does the radius of the circumscribed circle become large in all these cases? Show this from the equation for R and from figure 2.9.

b) If $a = b$ and $c \ll a; c \ll b$, the radius of the circumscribed circle is equal to $R \approx a/2 = b/2$. Prove this property from the above equation for R and explain it from figure 2.9.

Solution

We consider the limiting cases in turn.

a) For cases $a \to b+c, a \to b-c$, or $a \to c-b$ we always have one side nearly equal to the sum of the two other sides. Such a triangle is nearly flattened into a straight line (see figure 2.10A). The circumscribed circle must have a very small curvature to go through all three vertices and therefore would have a large radius.

b) For $a = b$ and $c \ll a; c \ll b$ we have a narrow isosceles triangle (see figure 2.10B). A circumscribed circle will have a diameter nearly equal to a or b, and the radius nearly equal to $a/2$. From formula (2.102) we get

$$\begin{aligned} R &= \frac{abc}{\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}} \\ &\approx \frac{a^2 c}{\sqrt{(2a)cc(2a)}} = \frac{a}{2}. \end{aligned} \tag{2.103}$$

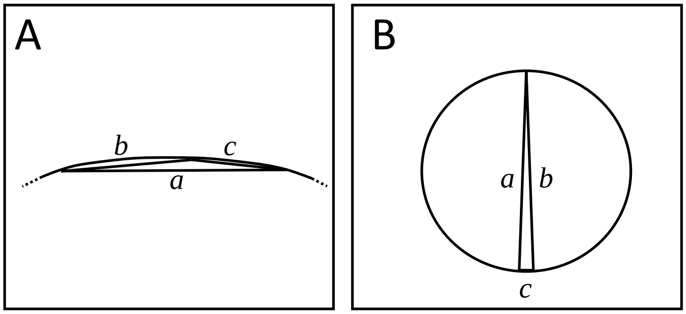

Figure 2.10
A circumscribed circle in two limiting cases

This limiting case holds.

27*. The depressed cubic equation is given by

$$x^3 + px + q = 0. \tag{2.104}$$

Its three solutions can be expressed through trigonometric functions (see section A.29):

$$x_k = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right), \tag{2.105}$$

where $k = 0, 1, 2$. For the limiting case $q \to 0$, equation (2.104) factorizes to

$$x(x^2 + p) = 0, \tag{2.106}$$

and we can see that one of its roots must be equal to zero and the other ones are equal to $\pm\sqrt{-p}$. Show that these limiting cases also apply to solution (2.105).

Solution

For $q = 0$ the argument of $\cos^{-1}$ in equation (2.105) is zero, and we get

$$\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) = \frac{\pi}{2}. \tag{2.107}$$

We consider cases $k = 0, 1, 2$ separately using this value for $\cos^{-1}$.

a) For $k = 0$ and $q = 0$ we get

$$\begin{aligned} x_0 &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\pi}{6}\right) \\ &= 2\sqrt{-\frac{p}{3}}\frac{\sqrt{3}}{2} \\ &= \sqrt{-p}. \end{aligned} \tag{2.108}$$

b) For $k = 1$ we get

$$\begin{aligned} x_1 &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) \\ &= 2\sqrt{-\frac{p}{3}}\cos\left(-\frac{\pi}{2}\right) \\ &= 0. \end{aligned} \tag{2.109}$$

c) For $k = 2$ we get

$$\begin{aligned} x_2 &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\pi}{6} - \frac{4\pi}{3}\right) \\ &= 2\sqrt{-\frac{p}{3}}\cos\left(-\pi - \frac{\pi}{6}\right) \\ &= -2\sqrt{-\frac{p}{3}}\cos\left(-\frac{\pi}{6}\right) \\ &= -2\sqrt{-\frac{p}{3}}\frac{\sqrt{3}}{2} \\ &= -\sqrt{-p}. \end{aligned} \tag{2.110}$$

We do get $x_1 = 0$, $x_{0,2} = \pm\sqrt{-p}$, as expected.

28*. Section A.32 gives a basic explanation of the so-called Kalman filter. It is an algorithm that computes an estimate X for a quantity that is measured by two (possibly different) instruments. Suppose that the first measurement produced a value X_1 with a variance of the measurement error σ_1^2, and the second measurement produced a value X_2 with a variance of the measurement error σ_2^2. Then the best estimate for the X from these two measurements is given by the following equation:

$$X = \frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}. \tag{2.111}$$

The accuracy of X is characterized by its own variance:

$$\sigma^2 = \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}. \tag{2.112}$$

Read section A.32, explore the following limiting cases, and explain the results:

a) Instrument 1 is much more accurate than instrument 2: $\sigma_1^2 \ll \sigma_2^2$.

b) Both instruments produced the same value: $X_1 = X_2$.

c) Show that for nonzero variances σ_1^2, σ_2^2, the value of X is always between the values of X_1 and X_2. Why do you expect this to be true?

d) Show that $\sigma^2 < \sigma_1^2,\ \sigma^2 < \sigma_2^2$. Why do you expect this to be true?

Solution

We consider the four limiting cases in turn.

a) If instrument 1 is much more accurate than instrument 2, a sensible algorithm should "trust" this instrument more, and the final result should be close to the output of that more accurate instrument. In other words, adding data from an inferior instrument should not alter the estimate in a substantial way. Indeed, if $\sigma_1^2 \ll \sigma_2^2$, we have

$$\begin{aligned} X &= \frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} \\ &\approx \frac{X_1\sigma_2^2}{\sigma_2^2} \\ &= X_1, \\ \sigma^2 &= \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} \\ &\approx \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2} \\ &= \sigma_1^2. \end{aligned} \tag{2.113}$$

b) If both instruments produce the same value, a combined estimate should be equal to that value. Indeed, for $X_1 = X_2$ we have

$$\begin{aligned} X &= \frac{X_1\sigma_2^2 + X_1\sigma_1^2}{\sigma_2^2 + \sigma_1^2} \\ &= X_1. \end{aligned} \tag{2.114}$$

c) We expect the best estimate to be in between the two measurements. If X is in between X_1 and X_2, then $(X-X_1)(X-X_2) < 0$. This is a convenient way to check if the Kalman filter estimate is in between X_1 and X_2:

$$
\begin{aligned}
(X - X_1)(X - X_2) &= \left(\frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} - X_1\right)\left(\frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} - X_2\right) \\
&= \frac{X_1\sigma_2^2 + X_2\sigma_1^2 - X_1\sigma_2^2 - X_1\sigma_1^2}{\sigma_2^2 + \sigma_1^2} \cdot \frac{X_1\sigma_2^2 + X_2\sigma_1^2 - X_2\sigma_2^2 - X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} \\
&= \frac{(X_2 - X_1)\sigma_1^2}{\sigma_2^2 + \sigma_1^2} \cdot \frac{(X_1 - X_2)\sigma_2^2}{\sigma_2^2 + \sigma_1^2} \\
&= \frac{-(X_2 - X_1)^2\sigma_1^2\sigma_2^2}{(\sigma_2^2 + \sigma_1^2)^2} \\
&< 0.
\end{aligned} \tag{2.115}
$$

d) Variance is a measure of uncertainty in a measurement. Adding more data should reduce uncertainty. Therefore, we expect the variance of the Kalman filter estimate to be smaller than the variance of either measurement. Indeed, we compute the difference between σ^2 and σ_1^2 as follows:

$$
\begin{aligned}
\sigma^2 - \sigma_1^2 &= \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} - \sigma_1^2 \\
&= \frac{\sigma_2^2\sigma_1^2 - \sigma_2^2\sigma_1^2 - \sigma_1^4}{\sigma_2^2 + \sigma_1^2} \\
&= \frac{-\sigma_1^4}{\sigma_2^2 + \sigma_1^2} \\
&< 0.
\end{aligned} \tag{2.116}
$$

A comparison between σ^2 and σ_2^2 is analogous.

29*. Section A.31 provides a formula for the payment rate on a mortgage. Given the initial loan amount D, interest rate r, and duration of the loan T, the annual payment rate is

$$
p \approx \frac{rDe^{rT}}{e^{rT} - 1}. \tag{2.117}
$$

Explore the following limiting cases and explain the results:

a) A very short loan duration or a low interest rate: $rT \ll 1$

b) A very long loan duration or a high interest rate: $rT \gg 1$

(Hint: Use the following properties of the exponent: $e^x \approx 1 + x$ for $|x| \ll 1$ and $e^x \gg 1$ for $x \gg 1$.)

Solution

We consider two limiting cases in turn.

a) For a very short loan duration or a low interest rate, the effect of interest is negligible. The payment of the loan is then computed as paying the original loan amount D over time T.

Therefore, the payment formula should reduce to $p \approx D/T$. We use $e^x \approx 1 + x$ for small x in equation (2.117) to get

$$\begin{aligned} p &\approx \frac{rDe^{rT}}{e^{rT} - 1} \\ &\approx \frac{rD(1 + rT)}{rT} \\ &= \frac{D}{T} + rD \\ &\approx \frac{D}{T}. \end{aligned} \tag{2.118}$$

b) If the interest is large, it takes the bulk of each payment. Similarly, for a long duration of a loan, the principal is paid off slowly, and payment toward the principal must be small compared with the payment of interest. In both cases, the total payment is approximately equal to the payment of interest and can be computed as $p \approx rD$. We use $e^x \gg 1$ for large x in equation (2.117) to get

$$\begin{aligned} p &\approx \frac{rDe^{rT}}{e^{rT} - 1} \\ &\approx \frac{rDe^{rT}}{e^{rT}} \\ &= rD. \end{aligned} \tag{2.119}$$

3 Symmetry

1. Problem

By swapping a and b, select three correct formulas from the six options below:

a) $a^3 - b^3 = (a + b)(a^2 + ab + b^2)$

b) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

c) $a^2 - b^2 = (a - b)(a + b)$

d) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

e) $a^3 + b^3 = (a - b)(a^2 - ab + b^2)$

f) $a^2 + b^2 = (a - b)(a + b)$

Solution

A swap of a and b affects the left-hand and the right-hand sides as follows:

a) The left-hand side changes sign, and the right-hand side stays invariant. This option is incorrect.

b) Both sides change sign. This option may be correct.

c) Both sides change sign. This option may be correct.

d) Both sides stay invariant. This option may be correct.

e) The left-hand side stays invariant, and the right-hand side changes sign. This option is incorrect.

f) The left-hand side stays invariant, and the right-hand side changes sign. This option is incorrect.

2. Problem

Check symmetry with respect to swapping α and β in the following trigonometric identities. (Hint: Use $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.)

a) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

b) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

c) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

d) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Solution

Swapping α and β produces the following equations:

a) $\sin(\beta + \alpha) = \sin\beta\cos\alpha + \sin\alpha\cos\beta$

b) $\sin(\beta - \alpha) = -\sin(\alpha - \beta) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$

c) $\cos(\beta + \alpha) = \cos\beta\cos\alpha - \sin\beta\sin\alpha$

d) $\cos(\beta - \alpha) = \cos(\alpha - \beta) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$

In all cases the modified equation is equivalent to the original one.

3. Problem

Check symmetry with respect to replacing α with $\pi/2 - \alpha$ in the following trigonometric identities. (Hint: Use $\sin(\pi/2 - \alpha) = \cos\alpha; \cos(\pi/2 - \alpha) = \sin\alpha; \sin(\pi - \alpha) = \sin\alpha; \cos(\pi - \alpha) = -\cos\alpha$.)

a) $\sin 2\alpha = 2\cos\alpha\sin\alpha$

b) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

Solution

Replacement α with $\pi/2 - \alpha$ produces the following:

a)

$$\sin(\pi - 2\alpha) = 2\cos\left(\frac{\pi}{2} - \alpha\right)\sin\left(\frac{\pi}{2} - \alpha\right). \tag{3.1}$$

This is equivalent to

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \tag{3.2}$$

leaving the formula invariant.

b)

$$\cos(\pi - 2\alpha) = \cos^2\left(\frac{\pi}{2} - \alpha\right) - \sin^2\left(\frac{\pi}{2} - \alpha\right). \tag{3.3}$$

This is equivalent to

$$-\cos 2\alpha = \sin^2\alpha - \cos^2\alpha. \tag{3.4}$$

This formula is obtained by multiplying the original one by -1.

4. Problem

Check symmetry with respect to replacing α with $\pi-\alpha$ in the following trigonometric identities. (Hint: Use $\sin(\pi/2-\alpha)=\cos\alpha$; $\cos(\pi/2-\alpha)=\sin\alpha$; $\sin(\pi-\alpha)=\sin\alpha$; $\cos(\pi-\alpha)=-\cos\alpha$.)

a) $\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}$

b) $\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$

Solution

Replacement α with $\pi-\alpha$ produces the following:

a)

$$\sin^2\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)=\frac{1-\cos(\pi-\alpha)}{2}. \tag{3.5}$$

This is equivalent to

$$\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}. \tag{3.6}$$

b)

$$\cos^2\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)=\frac{1+\cos(\pi-\alpha)}{2}. \tag{3.7}$$

This is equivalent to

$$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}. \tag{3.8}$$

As a result of replacement α with $\pi-\alpha$, formulas for options a and b switch places.

5. Problem

Section A.22 considers the following equation:

$$\frac{x-a}{x-b}+\frac{x-b}{x-a}=c. \tag{3.9}$$

Separately, section A.23 considers a similar equation:

$$\frac{x-a}{x-b}-\frac{x-b}{x-a}=c. \tag{3.10}$$

One of these equations is invariant with respect to swapping $a\leftrightarrow b$, and another is invariant with respect to swapping $a\leftrightarrow b$ and $c\leftrightarrow -c$ simultaneously. Using these properties, determine which of the following solutions applies to which equation:

$$\begin{aligned} x_{1,2} &= \frac{(2-c)(a+b)\pm(a-b)\sqrt{c^2-4}}{2(2-c)}, \\ x'_{1,2} &= \frac{c(a+b)-2(a-b)\pm(a-b)\sqrt{c^2+4}}{2c}, \end{aligned} \tag{3.11}$$

where subscripts 1, 2 correspond to the $\pm$ signs in the right-hand side.

Solution

Equation (3.9) is invariant with respect to swapping a and b, and equation (3.10) is invariant with respect to swapping a and b and simultaneous change of sign $c \leftrightarrow -c$.

Formula for $x_{1,2}$ is not invariant with respect to any operation that includes a change of sign $c \leftrightarrow -c$, because it includes expression $2 - c$. However, it is invariant with respect to change $a \leftrightarrow b$ (with an immaterial change $1 \leftrightarrow 2$ in the indexing of the two solutions).

Formula for $x'_{1,2}$ is invariant with respect to the simultaneous change of sign $c \leftrightarrow -c$ and swapping $a \leftrightarrow b$.

We conclude $x_{1,2}$ is the solution of equation (3.9) and $x'_{1,2}$ is the solution of equation (3.10).

6. Problem

Section A.15 solves the following equation for a ratio of two cosine functions:

$$\frac{\cos(\alpha + x)}{\cos(\alpha - x)} = \frac{p}{q}. \tag{3.12}$$

This problem is invariant with respect to swapping $p \leftrightarrow q$ and $x \leftrightarrow -x$. In addition, it is invariant with respect to swapping $p \leftrightarrow q$ and $\alpha \leftrightarrow -\alpha$. Use symmetry analysis to flag the wrong solutions among the following options:

a) $x = \cot^{-1}\left(\tan\alpha \cdot \frac{q-p}{p+q}\right) + n\pi$

b) $x = \cot^{-1}\left(\tan\alpha \cdot \frac{(q-p)^2}{p^2+q^2}\right) + n\pi$

c) $x = \cot^{-1}\left(\sin\alpha \cdot \frac{q-p}{p+q}\right) + n\pi$

d) $x = \cot^{-1}\left(\cos\alpha \cdot \frac{q-p}{p+q}\right) + n\pi$

Solution

We use the fact that sine, tangent, and cotangent are odd functions and cosine is an even function. Checks for the two invariances show that options *b* and *d* are incorrect.

7. Problem

A riverboat travels from town A to town B in time T_{AB} and from town B to town A in time T_{BA}. Section A.2 shows that the amount of time it takes to travel by a raft from town B to town A is given by

$$T_r = \frac{2T_{AB}T_{BA}}{T_{AB} - T_{BA}}. \tag{3.13}$$

A swap of towns A and B changes the sign for the time that is required to go from town B to town A on a raft. Why is the result not invariant for this swap, and what may the change of sign mean?

Solution

There is no invariance with respect to a swap $A \leftrightarrow B$ because the symmetry is violated by the river flowing in the direction from B to A. If we replace $A \leftrightarrow B$ in the solution, that would correspond to a case when the river flows from A to B. The formula for travel time is obtained by multiplying the speed of the raft by the distance between the towns. Since in the modified scenario the river flows in the opposite direction, the raft speed changes the sign, which in turn changes the sign of T_r.

Another way to interpret this result is as follows. Consider the raft travel time as the difference in the time moment $t(A)$ when the raft is at town A and the time moment $t(B)$ when it is at town B: $T_r = t(A) - t(B)$. If town A is downstream from town B, then $t(A) > t(B)$, and the travel time is positive. However, if A is upstream from town B, then $t(A) < t(B)$, and the travel time is negative.

8. Problem

Section A.11 solves the following equation for x:

$$\frac{p}{x-a} + \frac{q}{x-b} = d. \tag{3.14}$$

The solution is as follows:

$$x_{1,2} = \frac{(p+q) + d(a+b) \pm \sqrt{d^2(a-b)^2 + (p+q)^2 + 2d(a-b)(p-q)}}{2d}, \tag{3.15}$$

where subscripts 1, 2 correspond to the ± signs in the right-hand side. Which of the following swap symmetries are valid for this problem?

a) $a \leftrightarrow b$

b) $p \leftrightarrow q$

c) $x \leftrightarrow -x$ and $d \leftrightarrow -d$

d) $a \leftrightarrow b$ and $p \leftrightarrow q$

Check each condition in both the original equation and the solution.

Solution

a) Equation (3.14) is not invariant with respect to $a \leftrightarrow b$ because

$$\frac{p}{x-a} + \frac{q}{x-b} \neq \frac{p}{x-b} + \frac{q}{x-a}. \tag{3.16}$$

Solution (3.15) is not invariant either, because term $2d(a-b)(p-q)$ changes sign.

b) Similarly, equation (3.14) and its solution are not invariant with respect to $p \leftrightarrow q$.

c) Equation (3.14) is not invariant with respect to $x \leftrightarrow -x$ and $d \leftrightarrow -d$ because

$$\frac{p}{x-a} + \frac{q}{x-b} \neq -\frac{p}{-x-a} - \frac{q}{-x-b}. \tag{3.17}$$

Solution (3.15) is not invariant either, because

$$
\begin{aligned}
(p+q)+d(a+b) &\neq (p+q)-d(a+b) \\
(p+q)+d(a+b) &\neq -((p+q)-d(a+b)).
\end{aligned}
\tag{3.18}
$$

d) Both equation (3.14) and its solution are invariant with respect to $a \leftrightarrow b$ and $p \leftrightarrow q$.

9. Problem

A spherical cap is the part of a sphere that lies on one side of a plane that crosses this sphere (see section A.30 and figure 3.1). The volume V and the surface area S are given by

$$
\begin{aligned}
V &= \frac{1}{3}\pi h^2(3R-h), \\
S &= 2\pi R h.
\end{aligned}
\tag{3.19}
$$

Note that a plane crossing a sphere creates not one but two spherical caps that are located on both sides of the plane. The sum of the volumes of these two spherical caps must equal the volume of the sphere. Similarly, the sum of the surface areas of the two spherical caps must equal the surface area of the sphere. Prove these properties by using the formulas for the volume and the surface area of a sphere:

$$
\begin{aligned}
V_{\text{sphere}} &= \frac{4}{3}\pi R^3, \\
S_{\text{sphere}} &= 4\pi R^2.
\end{aligned}
\tag{3.20}
$$

Solution

The volume and surface area of the complementary spherical cap are obtained from equations (3.19) by replacing h with $2R-h$ there:

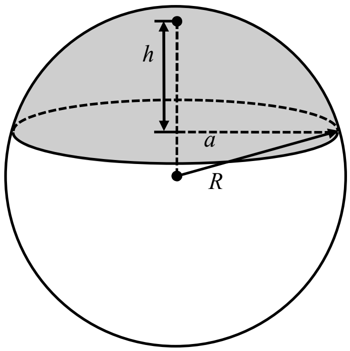

Figure 3.1
A spherical cap: symmetry

$$\begin{aligned}\tilde{V} &= \frac{1}{3}\pi(2R-h)^2(3R-(2R-h)),\\ \tilde{S} &= 2\pi R(2R-h).\end{aligned} \tag{3.21}$$

We compute the sum $\tilde{V} + V$:

$$\begin{aligned}\tilde{V} + V &= \frac{1}{3}\pi(2R-h)^2(3R-(2R-h)) + \frac{1}{3}\pi h^2(3R-h)\\ &= \frac{1}{3}\pi(4R^2-4Rh+h^2)(R+h) + \frac{1}{3}\pi h^2(3R-h)\\ &= \frac{1}{3}\pi(4R^3-4R^2h+h^2R+4R^2h-4Rh^2+h^3) + \frac{1}{3}\pi(3Rh^2-h^3)\\ &= \frac{4}{3}\pi R^3,\end{aligned} \tag{3.22}$$

which is equal to the volume of the sphere. Next, we compute sum $\tilde{S} + S$:

$$\begin{aligned}\tilde{S} + S &= 2\pi R(2R-h) + 2\pi Rh\\ &= 4\pi R^2,\end{aligned} \tag{3.23}$$

which is equal to the surface area of the sphere.

10. Problem

The radius of a circle that is inscribed in a right triangle (figure 3.2) is given by

$$R = \frac{a+b-c}{2}. \tag{3.24}$$

This formula is invariant with respect to only one of the following swaps of variables a, b, and c:

a) $a \leftrightarrow b$

b) $a \leftrightarrow c$

c) $b \leftrightarrow c$

Which invariance holds? Explain why.

Solution

The hypotenuse is qualitatively different from the two legs of a right triangle, because it is opposite of an angle with a unique property ($\gamma = \pi/2$). Therefore, we should expect equations to be invariant with respect to swapping the two legs, but not a leg and the hypotenuse. (This would be true for any formula that specifically applies to right triangles. A formula that applies to any triangle should be invariant with respect to a swap of any two sides.)

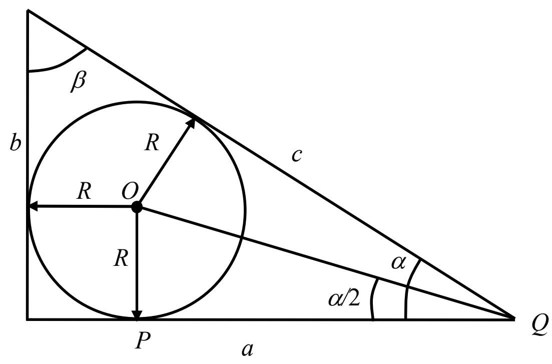

Figure 3.2
A circle inscribed in a right triangle: symmetry

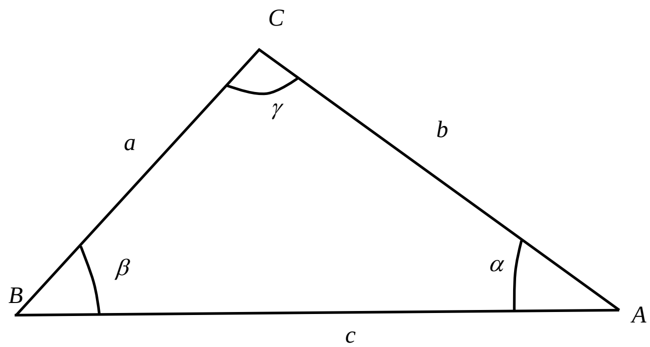

Figure 3.3
Heron's formula: symmetry

11. Problem

Heron's formula links area S of a triangle, its half-perimeter $p = (a + b + c)/2$, and the lengths of its sides (figure 3.3):

$$S = \sqrt{p(p - a)(p - b)(p - c)}. \tag{3.25}$$

Exercise 23 in chapter 2 explored a limiting case for $a \to b + c$, which leads to $S \to 0$. Use symmetry to show that a case $a \to b - c$ also results in $S \to 0$. Explain this result.

Solution

A limiting case $a \to b - c$ is equivalent to $b \to a + c$, which can be obtained from limiting case $a \to b + c$ by swapping $a \leftrightarrow b$. We can see that Heron's formula is invariant with respect to a swap of any two sides of the triangle. Therefore, if limiting case $a \to b + c$ leads to $S \to 0$, so should limiting case $a \to b - c$ do.

12. Problem

The number of ways to select k objects from a set of n objects is given by

$$_nC_k = \frac{n!}{k!(n-k)!}, \tag{3.26}$$

where a factorial for an integer is the product of all positive integers less than or equal to that integer: $n! = 1 \times 2 \times \cdots \times n$. Note that if we select k items from a set of n items, $n-k$ items are left out. Therefore, each particular selection of k objects is equivalent to a complementary selection of $n-k$ objects. This creates a symmetry: the number of ways to select k objects should be equal to the number of ways to select $n-k$ objects from the same set. Prove that $_nC_k = {}_nC_{n-k}$ using the formula above.

Solution

The complementary selection has $k' = n - k$ objects. Substitution of k' in place of k leaves equation (3.26) intact:

$$_nC'_k = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{k!(n-k)!}, \tag{3.27}$$

13. Problem

Section 3.5 showed that concentrations of water in a two-syrup mix obey the same equations as concentrations of sugar. Following the same logic, show that this is also true for a three-syrup mix.

Solution

The equation for the concentration of sugar in a three-syrup mix is as follows:

$$p_{123} = \frac{p_1 m_1 + p_2 m_2 + p_3 m_3}{m_1 + m_2 + m_3}. \tag{3.28}$$

Concentration of water in the syrups is given by

$$\begin{aligned} q_1 &= 1 - p_1, \\ q_2 &= 1 - p_2, \\ q_3 &= 1 - p_3, \\ q_{123} &= 1 - p_{123}. \end{aligned} \tag{3.29}$$

We expect these values to satisfy the following equation

$$q_{123} = \frac{q_1 m_1 + q_2 m_2 + q_3 m_3}{m_1 + m_2 + m_3}. \tag{3.30}$$

If we substitute into this equation values from equations (3.29), we will get

$$1 - p_{123} = \frac{(m_1 + m_2 + m_3) - (p_1 m_1 + p_2 m_2 + p_3 m_3)}{m_1 + m_2 + m_3}. \tag{3.31}$$

This does reduce to equation (3.28), as expected.

14. Problem

Explore a quadratic equation:

$$ax^2 + bx + c = 0. \tag{3.32}$$

a) What happens to the roots if we flip the sign of the second coefficient ($b \leftrightarrow -b$)? Identify this symmetry both in the original equation and in its solution

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.33}$$

b) Refer to figure 2.2 in section 2.4 and sketch a plot of $x_{1,2}$ as a function of a for $b = -20; c = 10$.

Solution

a) The quadratic equation remains invariant if $b \leftrightarrow -b$ and $x \leftrightarrow -x$. In the solution, this corresponds to a change $b \leftrightarrow -b$ and flipping the sign for the square root.

b) The plot will be flipped with respect to the vertical axis. It is sketched in figure 3.4.

15. Problem

Equations (3.78*) in section 3.11 must have the same symmetry $x \leftrightarrow y$ as the original equations (3.65*) in that section. Prove that this symmetry holds for equations (3.78*) and then solve them for x and y and show that the symmetry holds for the solution as well. (Hint: Use the analysis in section 3.7 as a template.)

Solution

Equations (3.78*) are reproduced below:

$$\begin{aligned} x &= \frac{c}{y}, \\ y^2 - ay + c &= 0. \end{aligned} \tag{3.34}$$

The first of these equations is obviously symmetric with respect to $x \leftrightarrow y$. To prove the same symmetry for the second equation, we substitute $y = c/x$ into it:

$$\left(\frac{c}{x}\right)^2 - a\frac{c}{x} + c = 0. \tag{3.35}$$

We multiply this equation by x^2/c to get

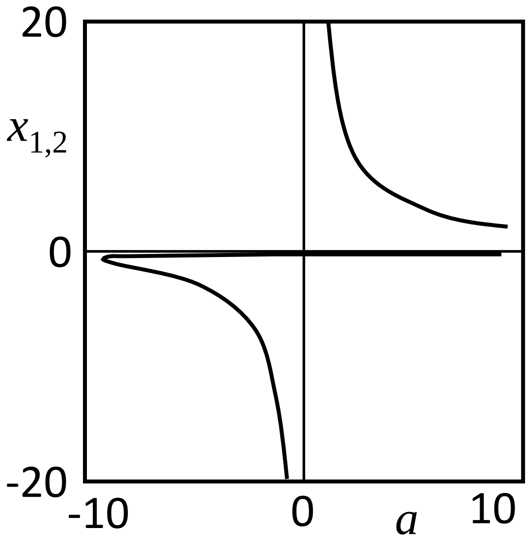

Figure 3.4
Solutions of a quadratic equation as a function of *a*

$$c - ax + x^2 = 0. \tag{3.36}$$

This proves the symmetry of the second equation in (3.34).

Next, we solve equations (3.34) and prove that the symmetry holds for the solution. We get:

$$\begin{aligned} y_{1,2} &= \frac{a \pm \sqrt{a^2 - 4c}}{2}, \\ x_{1,2} &= \frac{c}{y_{1,2}}. \end{aligned} \tag{3.37}$$

We know that original equations (3.65*) have symmetry $x \leftrightarrow y$. In the solution, this symmetry is expressed as $x_1 = y_2, x_2 = y_1$. Indeed, let's prove $y_1 = x_2$. We start from

$$\frac{a + \sqrt{a^2 - 4c}}{2} = \frac{2c}{a - \sqrt{a^2 - 4c}}. \tag{3.38}$$

We multiply both sides by $2(a - \sqrt{a^2 - 4c})$ to get

$$(a + \sqrt{a^2 - 4c})(a - \sqrt{a^2 - 4c}) = 4c. \tag{3.39}$$

We expand the parentheses in the right-hand side to get $4c = 4c$, which proves the symmetry.

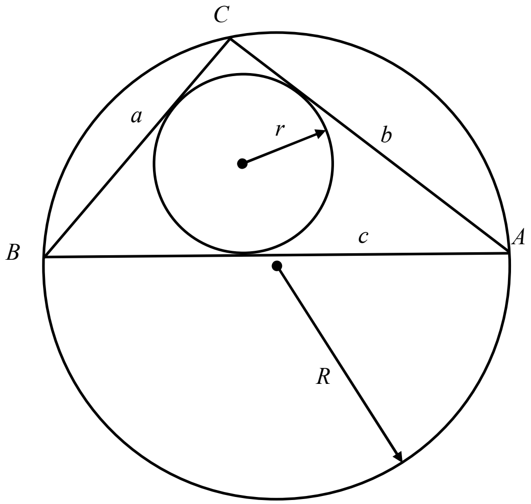

Figure 3.5
A triangle, an inscribed circle, and a circumscribed circle: symmetry

16. Problem

Consider the drawing in figure 3.5. The radii of the inscribed circle r and of the circumscribed circle R are given by

$$\begin{aligned} r &= \frac{\sqrt{p(p-a)(p-b)(p-c)}}{p}, \\ R &= \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}, \end{aligned} \tag{3.40}$$

where p is the half-perimeter of the triangle: $p = (a+b+c)/2$. Show that both formulas are symmetric with respect to swapping any pair of sides of the triangle.

Solution

Formula for r is obviously symmetric for a swap $a \leftrightarrow b$ or for any other pair. Next, we replace $\tilde{a} = b, \tilde{b} = a$ in the formula for R. We observe for the four factors under the square root that

$$\begin{aligned} \tilde{a} + \tilde{b} + c &= a + b + c, \\ -\tilde{a} + \tilde{b} + c &= a - b + c, \\ \tilde{a} - \tilde{b} + c &= -a + b + c, \\ \tilde{a} + \tilde{b} - c &= a + b - c. \end{aligned} \tag{3.41}$$

The product of these four factors remains invariant. This proves the symmetry.

17. Problem

Section A.9 solves the following equation for x:

$$x - \frac{1}{x} = d. \tag{3.42}$$

The solution is given by

$$x_{1,2} = \frac{d \pm \sqrt{d^2 + 4}}{2}. \tag{3.43}$$

Following the examples in section 3.7, explore the following symmetries in this problem:

a) This equation is invariant with respect to swapping $x \leftrightarrow -\frac{1}{x}$.

b) What happens with the roots of equation (3.42) when we flip the sign of d?

Solution

a) Equation (3.42) is obviously invariant for $x \leftrightarrow -\frac{1}{x}$. This means that $x_1 x_2 = -1$. Indeed, for the solutions (3.43) we get

$$\frac{d + \sqrt{d^2 + 4}}{2} \cdot \frac{d - \sqrt{d^2 + 4}}{2} = \frac{d^2 - (d^2 + 4)}{4} = -1. \tag{3.44}$$

b) Flipping the sign of d in equation (3.42) is the same as flipping the sign of x. Equivalently, it is the same as swapping $x \leftrightarrow 1/x$. Indeed, flipping the sign of d in the solution (3.43) and the sign in front of the square root (which is immaterial for the purposes of proving symmetry) flips the sign of x. We have already established that flipping the sign of the square root is equivalent to swapping $x \leftrightarrow -1/x$.

18. Problem

Exercise 13 in chapter 1 deals with pendulum oscillations. The maximum angle $\alpha_{\max}$ between the pendulum and the vertical is called the amplitude. The formula for the period of pendulum oscillations is in the form

$$T = F(\alpha_{\max})\sqrt{\frac{l}{g}}, \tag{3.45}$$

where $F(\alpha_{\max})$ is a yet unspecified function of the amplitude, l is the length of the pendulum, and g is the gravity acceleration.

a) Do you expect the period of the pendulum oscillations to be dependent on whether the initial angle between the pendulum and the vertical was positive or negative?

b) Based on the previous question, should function $F(\alpha_{\max})$ be even, odd, or neither?

Solution

a) The period of the pendulum oscillations must be invariant with respect to the sign of the initial angle. Indeed, a positive angle becomes negative if we observe this pendulum from behind, and the point of observation must not change the period.

b) Based on the previous argument, $F(\alpha_{\max})$ must be an even function. (In fact, the exact solution is $T = 4\sqrt{l/g}F(\pi/2, \sin(\alpha_{\max}/2))$, where function $F(x, y)$ is called the *incomplete elliptic integral of the first kind*. This function is indeed even with respect to the second argument.)

19. Problem

Section A.5 solves the problem of finding intersections between a circle and a hyperbola (see figure 3.6). The circle and the hyperbola are given by the following equations:

$$\begin{aligned} x^2 + y^2 &= R^2, \\ \frac{x^2}{H_x^2} - \frac{y^2}{H_y^2} &= 1. \end{aligned} \tag{3.46}$$

Use symmetry analysis to determine which solutions below for the intersections between the circle and the hyperbola are wrong:

a) $y = \pm\sqrt{H_y^2 \frac{R^2 - H_x^2}{H_x^2 + H_y^2}}; \quad x_{1,2} = \sqrt{H_x^2 \frac{R^2 - H_y^2}{H_x^2 + H_y^2}}; \quad x_{3,4} = -\sqrt{H_x^2 \frac{R^2 + H_y^2}{H_x^2 + H_y^2}}.$

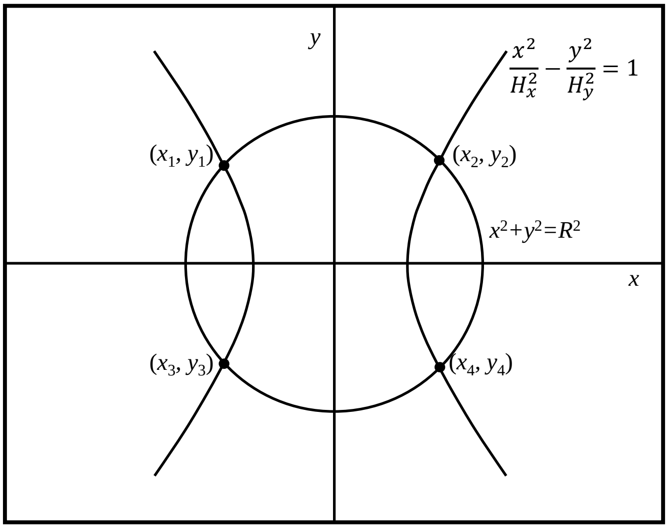

Figure 3.6
A circle and a hyperbola: symmetry

b) $y = \pm\sqrt{H_y^2 \frac{R^2 + H_x^2}{H_x^2 + H_y^2}}; \qquad x = -\frac{R}{4} \pm \sqrt{H_x^2 \frac{R^2 + H_y^2}{H_x^2 + H_y^2}}.$

c) $y = \pm\sqrt{H_y^2 \frac{R^2 - H_x^2}{H_x^2 + H_y^2}}; \qquad x = \pm\sqrt{H_x^2 \frac{R^2 + H_y^2}{H_x^2 + H_y^2}}.$

d) $y = \frac{R}{4} \pm \sqrt{H_y^2 \frac{R^2 + H_x^2}{H_x^2 + H_y^2}}; \; x = \pm\sqrt{H_x^2 \frac{R^2 - H_y^2}{H_x^2 + H_y^2}}.$

Solution

From the figure and from equations (3.46) we expect invariances $x \leftrightarrow -x$ and $y \leftrightarrow -y$ to hold. Checking these invariances in the four options above shows that only option *c* may be correct.

20*. Problem

The Kalman filter computes an estimate for a quantity X that is measured by two (possibly different) instruments (see section A.32). Suppose that the first measurement produced a value X_1 with a variance of the measurement error σ_1^2, and the second measurement produced a value X_2 with a variance of the measurement error σ_2^2. Then the best estimate for X from these two measurements is given by the following equation:

$$X = \frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}. \tag{3.47}$$

Its accuracy is characterized by its own variance:

$$\sigma^2 = \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}. \tag{3.48}$$

Read section A.32 and explore the following symmetries in the Kalman filter algorithm:

a) Show that the Kalman filter algorithm is symmetric with respect to swapping measurements 1 and 2.

b) Suppose we have three measurements for the same quantity, X_1, X_2, and X_3, and variances σ_1^2, σ_2^2, and σ_3^2, respectively. The algorithm given by equations (3.47) and (3.48) tells us how to combine any two of them. We can generalize this algorithm for combining three measurements. First, we combine measurements 1 and 2. We can view the result of this computation as a virtual "measurement" with the value X and the variance σ^2, which we can then combine with measurement 3, again using the same Kalman filter algorithm. We expect that the final result of this computation should not depend on our choice of the order in which we combined measurements.[1] For example, we should get the same result if we combine measurements 2 and 3 first and then add measurement 1. Prove that this is the case.

Solution

1. Note that this line of argument is similar to those in sections 3.5 and 3.6.

a) Swapping subscripts 1 and 2 in the Kalman filter equations leaves them intact. This symmetry holds.

b) We use solutions (3.47) for combining measurements X and X_3:

$$X_{123} = \frac{X\sigma_3^2 + X_3\sigma^2}{\sigma_3^2 + \sigma^2}, \tag{3.49}$$

where in turn

$$\begin{aligned} X &= \frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}, \\ \sigma^2 &= \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}. \end{aligned} \tag{3.50}$$

We substitute X, σ^2 into equation (3.49) to get:

$$X_{123} = \left(\frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} + \sigma_3^2\right)^{-1} \left(\frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}\sigma_3^2 + X_3\frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}\right). \tag{3.51}$$

Bringing the sums in parentheses to common denominators produces

$$\begin{aligned} X_{123} &= \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2} \cdot \frac{X_1\sigma_2^2\sigma_3^2 + X_2\sigma_1^2\sigma_3^2 + X_3\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \\ &= \frac{X_1\sigma_2^2\sigma_3^2 + X_2\sigma_1^2\sigma_3^2 + X_3\sigma_1^2\sigma_2^2}{\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2}. \end{aligned} \tag{3.52}$$

For the variance we get

$$\sigma_{123}^2 = \frac{\sigma^2\sigma_3^2}{\sigma^2 + \sigma_3^2}. \tag{3.53}$$

We substitute the expression for σ^2 to get

$$\sigma_{123}^2 = \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}\sigma_3^2\left(\frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2} + \sigma_3^2\right)^{-1}. \tag{3.54}$$

This transforms to

$$\begin{aligned} \sigma_{123}^2 &= \frac{\sigma_2^2\sigma_1^2\sigma_3^2}{\sigma_2^2 + \sigma_1^2} \cdot \frac{\sigma_2^2 + \sigma_1^2}{\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2} \\ &= \frac{\sigma_2^2\sigma_1^2\sigma_3^2}{\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2}. \end{aligned} \tag{3.55}$$

Equations (3.52) and (3.55) are fully symmetric with respect to any swap of subscripts 1, 2, and 3. Therefore, if we combine measurements 2 and 3 first and then add measurement 1, we would get the same result.

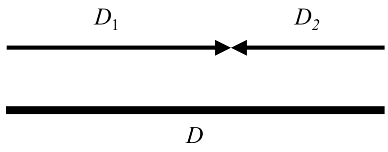

Figure 3.7
Two hikers on a trail: symmetry

21. Problem

Two hikers are starting to walk toward each other from the opposite ends of a trail (see figure 3.7). One hiker maintains speed V_1, and another maintains speed V_2. The total length of the trail is D. Section A.1 shows that the hikers will cover the following distances:

$$\begin{aligned} D_1 &= \frac{DV_1}{V_1 + V_2}, \\ D_2 &= \frac{DV_2}{V_1 + V_2}. \end{aligned} \tag{3.56}$$

The derivation of the final result in section A.1 uses the positions $x_1(t)$, $x_2(t)$ of the two hikers as functions of time. Follow this derivation and show that the final result is invariant with respect to the following transformations:[2]

a) space shift: $x'_1 = x_1 + \Delta x$ and $x'_2 = x_2 + \Delta x$

b) time shift: $t' = t + \Delta t$

Solution

We apply time (Δt) and space (Δx) shifts to the variables in this problem. This means that the first hiker starts from the point Δx, the second hiker starts from the point $D + \Delta x$, and they both start moving at time Δt. The hikers' positions as functions of time are (compare these equations to those in section A.1):

2. Time- and space-shift invariances are universal: all natural laws must be time- and space-shift invariant. Emmy Noether, a genius German mathematician, proved a theorem from which it follows that some invariances are intimately linked with conservation laws. Specifically, the time-shift invariance is linked with the law of conservation of energy, and the space-shift invariance is linked with the conservation of momentum.

$$x_1 = V_1(t - \Delta t) + \Delta x,$$
$$x_2 = D - V_2(t - \Delta t) + \Delta x. \tag{3.57}$$

The two hikers meet when $x_1 = x_2$. The time when they meet t_m is then given by

$$V_1(t_m - \Delta t) + \Delta x = D - V_2(t_m - \Delta t) + \Delta x, \tag{3.58}$$

Solving this equation for t_m produces

$$t_m = \frac{D}{V_1 + V_2} + \Delta t. \tag{3.59}$$

The hikers will be in motion for the period of time from Δt to t_m. The time elapsed will be computed as

$$t_m - \Delta t = \frac{D}{V_1 + V_2}. \tag{3.60}$$

This is the same value as in section A.1 and it will produce the same results for the distances traveled by the hikers. This shows that the solution is invariant with respect to time and space shifts.

22. Problem

We are already familiar with Archimedes's spiral. Its radius increases linearly with the turn angle (figure 3.8). The arc length of the spiral is given by the following equation:

$$L = \frac{\Delta R}{4\pi}\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right), \tag{3.61}$$

where ΔR is the distance between the adjacent loops and θ is the total turn angle. In the figure, the spiral rolls out counterclockwise. The formula for the length should also be applicable to a spiral that rolls out clockwise. We expect that a swap $\theta \leftrightarrow -\theta$ would result either in the same value of L or in the change for the sign of L. Mathematically, this is expressed as $L(\theta) = L(-\theta)$ or $L(\theta) = -L(-\theta)$. The choice between these two options would depend on the convention for the directions for arc length computation. Prove that the second option holds: $L(\theta) + L(-\theta) = 0$.

Solution

We compute $L(\theta) + L(-\theta)$ as follows:

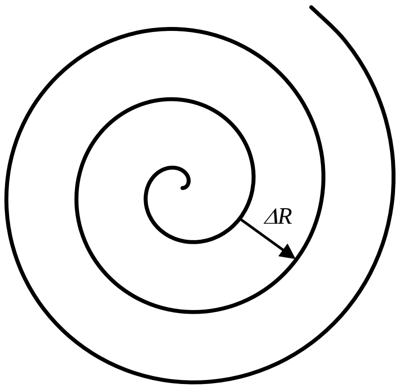

Figure 3.8
Archimedes's spiral: symmetry

$$
\begin{aligned}
L(\theta) + L(-\theta) &= \frac{\Delta R}{4\pi}\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right) + \frac{\Delta R}{4\pi}\left(-\theta\sqrt{1+\theta^2} + \ln\left(-\theta + \sqrt{1+\theta^2}\right)\right) \\
&= \frac{\Delta R}{4\pi}\ln\left[\left(\theta + \sqrt{1+\theta^2}\right)\left(-\theta + \sqrt{1+\theta^2}\right)\right] \\
&= \frac{\Delta R}{4\pi}\ln\left[\left(1+\theta^2\right) - \theta^2\right] \\
&= \frac{\Delta R}{4\pi}\ln 1 \\
&= 0.
\end{aligned}
\tag{3.62}
$$

23*. Problem

In section 3.1 we introduced a new variable to solve equation (3.3*). Using that example as a template, solve the following equation for x by introducing a new variable:

$$(x-a)(x-a+1)(x-a+3)(x-a+4) = b. \tag{3.63}$$

Solution

Zeroes of individual factors in the right-hand side are located at $a, a-1, a-3$, and $a-4$. We introduce a new variable that has a zero in the middle of these four points: $y = x - a + 2$. Then equation (3.63) is in the form:

$$(y-2)(y-1)(y+1)(y+2) = b. \tag{3.64}$$

We expand symmetric pairs of parentheses:

$$\begin{aligned}(y-1)(y+1) &= y^2 - 1,\\ (y-2)(y+2) &= y^2 - 4.\end{aligned} \tag{3.65}$$

This yields

$$(y^2 - 1)(y^2 - 4) = b, \tag{3.66}$$

which produces a quadratic equation for y^2:

$$(y^2)^2 - 5y^2 + 4 - b = 0. \tag{3.67}$$

Its solutions are

$$y = \pm\sqrt{\frac{5 \mp \sqrt{9+4b}}{2}}, \tag{3.68}$$

where we allow all four combinations of signs. For x we get:

$$x = \pm\sqrt{\frac{5 \mp \sqrt{9+4b}}{2}} + a - 2. \tag{3.69}$$

24. Problem

Solve the following system of equations for x and y:

$$\begin{aligned}x - y &= a,\\ x^4 + y^4 &= b^4.\end{aligned} \tag{3.70}$$

Solution

We make these equations symmetric with respect to $x \leftrightarrow y$ by introducing $z = -y$:

$$\begin{aligned}x + z &= a,\\ x^4 + z^4 &= b^4.\end{aligned} \tag{3.71}$$

Now the problem is equivalent to one solved in section 3.11. It is reduced to solving the following equations:

$$x = \frac{c}{z},$$
$$z^2 - az + c = 0, \tag{3.72}$$

where

$$c = \frac{2a^2 \pm \sqrt{2(a^4 + b^4)}}{2}. \tag{3.73}$$

25*. Problem

Solve the following equation for z:

$$\sqrt[4]{a-z} + \sqrt[4]{z-b} = c. \tag{3.74}$$

Use both of the following methods:

a) Introduce new variables

$$x = \sqrt[4]{a-z},$$
$$y = \sqrt[4]{z-b} \tag{3.75}$$

to obtain

$$x + y = c,$$
$$x^4 + y^4 = a - b. \tag{3.76}$$

Then use the technique that is presented in section 3.11.

b) Shift the original variable z by introducing $v = z - z_0$. Select the value of z_0 to make the equation symmetric with respect to $v \leftrightarrow -v$:

$$\sqrt[4]{d-v} + \sqrt[4]{d+v} = c, \tag{3.77}$$

where d is a function of a, b, and z_0. Then square and regroup the terms as necessary to produce a quadratic equation for $\sqrt[4]{d^2 - v^2}$. After that, solve for v and then for z.

Solution

a) Use of variables $x = \sqrt[4]{a-z}$ and $y = \sqrt[4]{z-b}$ produces equations (3.76) that have been solved in section 3.11 although with somewhat different notations. Following solution in section 3.11, we reduce the problem to solving the following equations for x, y:

$$x = \frac{g}{y},$$
$$y^2 - cy + g = 0, \tag{3.78}$$

where

$$g = \frac{2c^2 \pm \sqrt{2(c^4 + a - b)}}{2}. \tag{3.79}$$

It is sufficient to solve one of these equations to obtain the value of z; the other equation is redundant. We get

$$y = \frac{c \pm \sqrt{c^2 - 4g}}{2}. \tag{3.80}$$

The discriminant can be a bit simplified:

$$\begin{aligned} c^2 - 4g &= c^2 - 2(2c^2 \pm \sqrt{2(c^4 + a - b)}) \\ &= -3c^2 \mp 2\sqrt{2(c^4 + a - b)}. \end{aligned} \tag{3.81}$$

Then

$$y = \sqrt[4]{z - b} = \frac{c \pm \sqrt{-3c^2 \mp 2\sqrt{2(c^4 + a - b)}}}{2}. \tag{3.82}$$

This yields

$$z = \left(\frac{c \pm \sqrt{-c^2 \mp \sqrt{c^4 + a - b}}}{2} \right)^4 + b. \tag{3.83}$$

Since the derivation in section 3.11 included squaring, some of the roots may be superfluous. It is possible that values given by some combinations of plus and minus signs will not satisfy the original equation (3.77). In any application we would need to substitute all four values in the original equation and determine which roots are valid.

b) We introduce a variable that makes the two square roots in the original equation symmetric: $v = z - (a + b)/2$. Then

$$\begin{aligned} a - z &= d - v, \\ z - b &= d + v, \end{aligned} \tag{3.84}$$

where $d = (a - b)/2$. Then equation (3.77) takes the form:

$$\sqrt[4]{d - v} + \sqrt[4]{d + v} = c. \tag{3.85}$$

We square this equation and regroup the terms to get

$$\sqrt{d - v} + \sqrt{d + v} = c^2 - 2\sqrt[4]{d^2 - v^2}. \tag{3.86}$$

We square this equation again

$$d - v + d + v + 2\sqrt{d^2 - v^2} = c^4 - 4c^2\sqrt[4]{d^2 - v^2} + 4\sqrt{d^2 - v^2}. \tag{3.87}$$

we denote $u = \sqrt[4]{d^2 - v^2}$ to get a quadratic equation for u:

$$2u^2 - 4c^2u + c^4 - 2d = 0. \tag{3.88}$$

The solution for u is given by the quadratic formula

$$\begin{aligned} u_{1,2} &= \frac{4c^2 \pm \sqrt{16c^4 - 8(c^4 - 2d)}}{4} \\ &= \frac{2c^2 \pm \sqrt{2c^4 + 4d}}{2}. \end{aligned} \tag{3.89}$$

From that we solve for v:

$$\sqrt[4]{d^2 - v^2} = \frac{2c^2 \pm \sqrt{2c^4 + 4d}}{2}, \tag{3.90}$$

which yields

$$v = \mp \sqrt{d^2 - \left(\frac{2c^2 \pm \sqrt{2c^4 + 4d}}{2}\right)^4}, \tag{3.91}$$

Finally, we obtain for z

$$\begin{aligned} z &= v + \frac{a+b}{2} \\ &= \mp \sqrt{d^2 - \left(\frac{2c^2 \pm \sqrt{2c^4 + 4d}}{2}\right)^4} + \frac{a+b}{2}, \end{aligned} \tag{3.92}$$

where we allow all combinations of signs.

Similarly to the previous method, it is possible that values given by some combinations of plus and minus signs will not satisfy the original equation (3.77). We would need to substitute all four values in the original equation and determine which roots are valid.

26*. Problem

For what values of parameter a does the following equation have only one real solution for x (count any value only once, ignoring root multiplicity)?

$$x^{10} + e^{-a^2}x^2 - \frac{a^2 - 16}{2a} = 0. \tag{3.93}$$

Solution

This is a polynomial equation for x, and it has 10 real and complex roots. From the equation we see that all real roots are symmetric with respect to a change of sign: $x \leftrightarrow -x$ leaves the equation invariant. Therefore, all nonzero roots would come in pairs. The only case when this equation has only one solution is to have $x = 0$ as the only real root. Setting $x = 0$ in the equation yields $a^2 - 16 = 0$, or $a = \pm 4$.

In that case, we have

$$x^{10} + e^{-16}x^2 = 0. \tag{3.94}$$

Other than $x = 0$, roots are given by

$$x^8 + e^{-16} = 0. \tag{3.95}$$

All roots of this last equation are complex. Therefore, $x = 0$ is the only real root of equation (3.93).

27*. Problem

A solution for a cubic equation

$$ax^3 + bx^2 + cx + d = 0 \tag{3.96}$$

is given in section A.29. We denote

$$\begin{aligned}\Delta_0 &= b^2 - 3ac,\\ \Delta_1 &= 2b^3 - 9abc + 27a^2d,\\ C &= \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}},\end{aligned} \tag{3.97}$$

where the ± sign in this solution can be chosen arbitrarily, unless $C = 0$ for one of the signs, in which case we must choose the sign that yields a nonzero value of C. Then one of the roots is given by

$$x_1 = -\frac{1}{3a}\left(b + C + \frac{\Delta_0}{C}\right). \tag{3.98}$$

The solution says that the ± sign in this solution can be chosen arbitrarily (unless $C = 0$ for one of the signs). In section 3.3 we stated that an arbitrary choice in equations is associated with the symmetry with respect to that choice. Prove that cubic formula (3.98) is symmetric with respect to swapping $C_+ \leftrightarrow C_-$, where C_+ and C_- are given by the corresponding signs in the expression for C in equations (3.97). (Hint: Simplify the expression for C_+C_- and use the result to prove the desired symmetry.)

Solution

We compute C_+C_- to get

$$\begin{aligned}C_+C_- &= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2} \cdot \frac{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}\\ &= \sqrt[3]{\frac{\Delta_1^2 - (\Delta_1^2 - 4\Delta_0^3)}{4}}\\ &= \Delta_0.\end{aligned} \tag{3.99}$$

Therefore, we can replace C_+ with Δ_0/C_- in the solution for x:

$$\begin{aligned} x_1 &= -\frac{1}{3a}\left(b + C_+ + \frac{\Delta_0}{C_+}\right) \\ &= -\frac{1}{3a}\left(b + \frac{\Delta_0}{C_-} + \frac{\Delta_0}{\frac{\Delta_0}{C_-}}\right) \\ &= -\frac{1}{3a}\left(b + C_- + \frac{\Delta_0}{C_-}\right), \end{aligned} \tag{3.100}$$

which proves the invariance.

28. Problem

Section A.4 solves the problem of finding intersections between a circle and an ellipse (see figure 3.9). The circle and the ellipse are given by the following equations:

$$\begin{aligned} &x^2 + y^2 = R^2, \\ &\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} = 1, \end{aligned} \tag{3.101}$$

where R_x, R_y are the half-axes of the ellipse. Use symmetry for this problem to flag the wrong solutions among the following options:

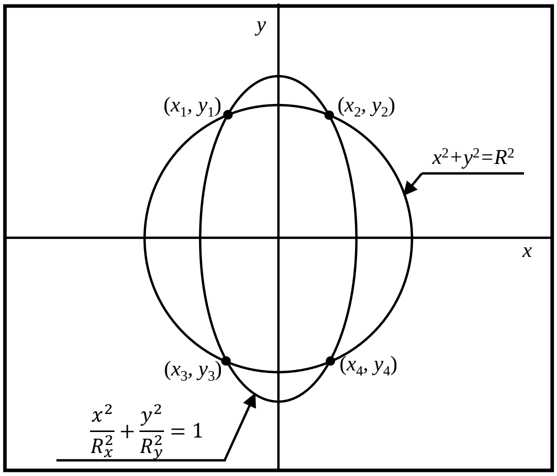

Figure 3.9
A circle and an ellipse: symmetry

a) $x = \pm\sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $y = \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$.

b) $x = \pm\sqrt{R_y^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $y = \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$.

c) $x = \frac{R_x}{2} \pm \sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $y = \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$.

d) $x = \pm\sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $y = \frac{R_y}{2} \pm \sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$.

Solution

From the figure and from the equations for x, y we see that a solution must be invariant with respect to a change in the signs of x, y or both. In addition, it must be symmetric with respect to change $x \leftrightarrow y$ and $R_x \leftrightarrow R_y$. Then

a) Symmetry holds. This solution may be correct.
b) Symmetry does not hold for $x \leftrightarrow y$ and $R_x \leftrightarrow R_y$.
c) Symmetry does not hold for $x \leftrightarrow -x$.
d) Symmetry does not hold for $y \leftrightarrow -y$.

29*. Problem

The depressed cubic equation is given by

$$x^3 + px + q = 0. \tag{3.102}$$

Its three solutions can be expressed through trigonometric functions (see section A.29):

$$x_k = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right), \tag{3.103}$$

where $k = 0, 1, 2$. Note that equation (3.102) is invariant with respect to a change in the signs of both x and q: $x \leftrightarrow -x; q \leftrightarrow -q$. Prove that this invariance also holds for solution (3.103).

Solution

This invariance implies that changing the sign of q in the solution would change the sign of x_k:

$$x_k(-q) = -x_m(q), \tag{3.104}$$

that is, there should be a one-to-one correspondence between values $x_k(-q)$ and $-x_m(q)$, even if they are numbered differently. Let's consider the effect of changing the sign of q on the right-hand side of equation (3.103). If the argument of $\cos^{-1}$ changes sign, then

$$\cos^{-1}\left(\frac{3(-q)}{2p}\sqrt{-\frac{3}{p}}\right) = \pi - \cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}.\right) \tag{3.105}$$

Next, we consider values of $x_k(-q)$:

$$\begin{aligned} x_k(-q) &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3(-q)}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right) \\ &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\pi}{3} - \frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right) \\ &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{\pi}{3} + \frac{2\pi k}{3}\right), \end{aligned} \tag{3.106}$$

where we flipped the sign of the argument of the cosine function, using the fact that this function is even. This last equation is similar to solution (3.103) for $x_k(q)$, except $-2\pi k/3$ is replaced there by $-\pi/3 + 2\pi k/3$. Values of these two expressions for different values of k are shown in table 3.1.

We observe that these values can be partitioned in pairs in such a way, that a value in the second column is equal to a value in the third column with a π subtracted from it. Indeed:

$$\begin{aligned} 0 &= \pi - \pi, \\ -\frac{2\pi}{3} &= \frac{\pi}{3} - \pi, \\ -\frac{4\pi}{3} &= -\frac{\pi}{3} - \pi. \end{aligned} \tag{3.107}$$

Therefore, the set of values for $x_k(-q)$ can be obtained from the set of values for $x_k(q)$ by subtracting a π from the argument of the cosine function in formula (3.103). Since $\cos(y - \pi) = -\cos y$, the set of values for $x_k(-q)$ is a reshuffled set of values for $-x_k(q)$.

30. Problem

A radar measures range (distance) to the object it is tracking. Assume that there are two radars that detect a sea vessel at ranges R_1 and R_2 (see figure 3.10). Given coordinates of the radars

Table 3.1
Values of expressions in the formulas for $x_k(q)$ and $x_k(-q)$

k	$-2\pi k/3$	$-\pi/3 + 2\pi k/3$
0	0	$-\pi/3$
1	$-2\pi/3$	$\pi/3$
2	$-4\pi/3$	π

$x_1 = 0; y_1 = 0$ and $x_2 = D; y_2 = 0$, the coordinates of the detected vessel are (see section A.25)

$$x = \frac{D^2 + R_1^2 - R_2^2}{2D}, \quad y = \pm\sqrt{R_1^2 - x^2}. \tag{3.108}$$

Show that this solution possesses the following two symmetries:

a) The solution is invariant with respect to $y \leftrightarrow -y$.

b) The solution is invariant with respect to $R_1 \leftrightarrow R_2$ and $x \leftrightarrow D - x$. (Compare this with limiting cases in exercises 9a and 9b in chapter 2.)

Explain the results.

Solution

a) Solution 3.108 is obviously symmetric with respect to $y \leftrightarrow -y$. This means that the target can be on either side of the line connecting the two radars (above or below the horizontal axis in figure 3.10).

b) If we swap $R_1 \leftrightarrow R_2$, then the horizontal position of the target will be a reflection of that in figure 3.10 with respect to a vertical line located at the midpoint between the two radars. This should produce $\tilde{x} = D - x$ and $\tilde{y} = y$. Let's check these conditions in equations (3.108).

Condition $\tilde{x} = D - x$ is equivalent to $\tilde{x} + x = D$. This condition holds:

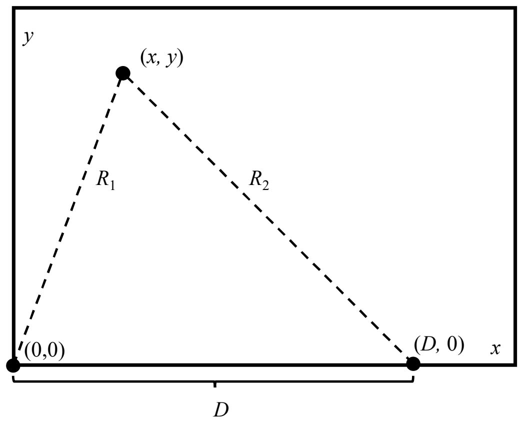

Figure 3.10
Detecting a vessel by two radars: symmetry

$$
\begin{aligned}
\tilde{x} + x &= \frac{D^2 + R_1^2 - R_2^2}{2D} + \frac{D^2 + R_2^2 - R_1^2}{2D} \\
&= \frac{2D^2}{2D} \\
&= D.
\end{aligned}
\tag{3.109}
$$

For $\tilde{y}$ we use solution (3.108), where we replace R_1 with $\tilde{R}_1 = R_2$ and x with $\tilde{x} = D - x$.

$$
\begin{aligned}
\tilde{y} &= \pm\sqrt{\tilde{R}_1^2 - \tilde{x}^2} \\
&= \pm\sqrt{R_2^2 - (D - x)^2} \\
&= \pm\sqrt{R_2^2 - D^2 - x^2 + 2Dx}.
\end{aligned}
\tag{3.110}
$$

In the linear term $2Dx$ we substitute x from the first equation in (3.108), but leave the quadratic term $-x^2$ intact:

$$
\begin{aligned}
\tilde{y} &= \pm\sqrt{R_2^2 - D^2 - x^2 + 2Dx} \\
&= \pm\sqrt{R_2^2 - D^2 - x^2 + D^2 + R_1^2 - R_2^2} \\
&= \pm\sqrt{R_1^2 - x^2} \\
&= y.
\end{aligned}
\tag{3.111}
$$

31*. Problem

Given data points x_i and y_i for two variables x and y, the linear regression algorithm estimates parameters a and b of the best-fit model $y = ax + b$ to the data:

$$
\begin{aligned}
a &= \frac{N\sum_{i=1}^N x_i y_i - \sum_{i=1}^N x_i \cdot \sum_{i=1}^N y_i}{N\sum_{i=1}^N x_i^2 - \left(\sum_{i=1}^N x_i\right)^2}, \\
b &= \frac{\sum_{i=1}^N y_i \cdot \sum_{i=1}^N x_i^2 - \sum_{i=1}^N x_i \cdot \sum_{i=1}^N x_i y_i}{N\sum_{i=1}^N x_i^2 - \left(\sum_{i=1}^N x_i\right)^2}.
\end{aligned}
\tag{3.112}
$$

In section 3.13 above we already considered several symmetries for the linear regression algorithm. In addition, this algorithm has two shift symmetries that are the subject of this exercise:

a) Substitute shifted values $x'_i = x_i + c, y'_i = y_i$ in equations (3.112) and prove that they will produce new parameter estimates $a' = a$ and $b' = b - ac$.

b) Similarly, prove that using $x''_i = x_i, y''_i = y_i + d$ will produce new parameter estimates $a'' = a$ and $b'' = b + d$.

c) Explain both symmetries by considering the effects of shifts on figure 3.9*.

Solution

a) We consider the effect of a shift in x_i:

$$x_i' = x_i + c. \tag{3.113}$$

We consider three expressions in equations (3.112) separately.

i. For the denominator in equations (3.112) we get:

$$\begin{aligned} N\sum_{i=1}^{N} x_i'^2 - \left(\sum_{i=1}^{N} x_i'\right)^2 &= N\sum_{i=1}^{N}(x_i + c)^2 - \left(\sum_{i=1}^{N}(x_i + c)\right)^2 \\ &= N\sum_{i=1}^{N} x_i^2 + N\sum_{i=1}^{N} 2x_i c + N\sum_{i=1}^{N} c^2 - \left(\sum_{i=1}^{N} x_i + Nc\right)^2 \\ &= N\sum_{i=1}^{N} x_i^2 + 2Nc\sum_{i=1}^{N} x_i + N^2c^2 - \left(\left(\sum_{i=1}^{N} x_i\right)^2 + 2Nc\sum_{i=1}^{N} x_i + N^2c^2\right) \\ &= N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2 . \end{aligned} \tag{3.114}$$

We see that the denominator is invariant with respect to the shift $x_i' = x_i + c$.

ii. Numerator for a'.

$$\begin{aligned} N\sum_{i=1}^{N} x_i' y_i - \sum_{i=1}^{N} x_i' \cdot \sum_{i=1}^{N} y_i &= N\sum_{i=1}^{N}(x_i + c)y_i - \sum_{i=1}^{N}(x_i + c) \cdot \sum_{i=1}^{N} y_i \\ &= N\sum_{i=1}^{N} x_i y_i + Nc\sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i - Nc\sum_{i=1}^{N} y_i \\ &= N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i. \end{aligned} \tag{3.115}$$

The numerator for a remains invariant. Since the denominator also remains invariant, we have $a' = a$.

iii. Numerator for b'.

$$\begin{aligned}
\sum_{i=1}^{N} y_i \cdot \sum_{i=1}^{N} x_i'^2 - \sum_{i=1}^{N} x_i' \cdot \sum_{i=1}^{N} x_i' y_i &= \sum_{i=1}^{N} y_i \cdot \sum_{i=1}^{N} (x_i + c)^2 - \sum_{i=1}^{N} (x_i + c) \cdot \sum_{i=1}^{N} (x_i + c) y_i \\
&= \sum_{i=1}^{N} y_i \sum_{i=1}^{N} (x_i^2 + 2x_i c + c^2) - \left(\sum_{i=1}^{N} x_i + Nc \right) \left(\sum_{i=1}^{N} x_i y_i + c \sum_{i=1}^{N} y_i \right) \\
&= \sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i^2 + 2c \sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i + Nc^2 \sum_{i=1}^{N} y_i \\
&\quad - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i - c \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i - Nc \sum_{i=1}^{N} x_i y_i - Nc^2 \sum_{i=1}^{N} y_i \\
&= \left(\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i \right) + c \left(\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i - N \sum_{i=1}^{N} x_i y_i \right).
\end{aligned} \tag{3.116}$$

This numerator for formula for b' should be divided by the denominator, which we know to remain invariant with respect to the shift of $x' = x + c$. By comparing the final expression in equation (3.116) with numerators in formulas (3.112) and keeping in mind that the denominator remains invariant, we see that $b' = b - ac$.

b) We consider the effect of a shift in y:

$$y'' = y + d. \tag{3.117}$$

We consider three expressions in equations (3.112) separately.

i. Since the denominators in formulas (3.112) do not include y_i, they are invariant with respect to a shift of y_i.

ii. For the numerator in the expression for a'' we get

$$\begin{aligned}
N \sum_{i=1}^{N} x_i y_i'' - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i'' &= N \sum_{i=1}^{N} x_i (y_i + d) - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} (y_i + d) \\
&= N \sum_{i=1}^{N} x_i y_i + Nd \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i - Nd \sum_{i=1}^{N} x_i \\
&= N \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i.
\end{aligned} \tag{3.118}$$

We see that this expression for the numerator remains invariant. Since the denominator also is invariant, we conclude that $a'' = a$.

iii. For the numerator in the expression for b'' we get

$$\begin{aligned}\sum_{i=1}^{N} y_i'' \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i'' &= \sum_{i=1}^{N} (y_i + d) \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i (y_i + d) \\ &= \left(\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i \right) + d \left(N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i \right)^2 \right).\end{aligned} \tag{3.119}$$

When divided by the denominator in (3.112), the first pair of parentheses will produce b, and the second will produce 1. We obtain:

$$b'' = b + d. \tag{3.120}$$

c) The explanation of these symmetries is as follows. For a uniform shift in all data points we should see an identical shift in the straight line that is produced by linear regression. For a horizontal shift $x_i' = x_i + c$, an equivalent horizontal shift in a line $y = ax + b$ would not change the value of y. Since $x = x' - c$, we get

$$\begin{aligned} y' &= y \\ &= ax + b \\ &= a(x' - c) + b \\ &= ax' + b - ac. \end{aligned} \tag{3.121}$$

We obtain an equation for a straight line, where the slope remains invariant and the intercept is reduced by ac:

$$\begin{aligned} a' &= a, \\ b' &= b - ac. \end{aligned} \tag{3.122}$$

For a uniform vertical shift of all data points, we should observe an equal uniform vertical shift of the line that is produced by regression: $y' = y + d$. This produces

$$\begin{aligned} y'' &= y + d \\ &= ax + b + d. \end{aligned} \tag{3.123}$$

Then

$$\begin{aligned} a'' &= a, \\ b'' &= b + d. \end{aligned} \tag{3.124}$$

4 Scaling

1. Problem

There are syrups with masses m_1, m_2, and m_3 with sugar concentrations p_1, p_2, and p_3. Sections A.16 and A.17 show that blends of two or three syrups will respectively have the following concentrations of sugar:

$$
\begin{aligned}
p_{12} &= \frac{p_1 m_1 + p_2 m_2}{m_1 + m_2}, \\
p_{123} &= \frac{p_1 m_1 + p_2 m_2 + p_3 m_3}{m_1 + m_2 + m_3}.
\end{aligned}
\tag{4.1}
$$

a) Is there a scaling property for the masses of syrups? What happens if we replace $m_1 \leftrightarrow am_1; m_2 \leftrightarrow am_2; m_3 \leftrightarrow am_3$?

b) Does this scaling remain valid for extremely small values of scaling multiplier a, for example, if $a = 10^{-30}$? (Hint: This is not a purely mathematical question. Think about the molecular structure of a syrup.)

c) Is there a scaling behavior for concentrations p_1, p_2, p_3?

d) Does the scaling for concentrations break down for large values of the scaling multiplier?

Solution

a) The solution is invariant with respect to $m'_j = am_j$ scaling. If we take proportionally more of each syrup, the concentration of the blend will remain the same.

b) This scaling is maintained for arbitrarily large values of a, but breaks down for extremely small values, when the mass of each syrup becomes on the order of or smaller than mass of molecules forming this syrup. If am_2 is the mass of a sugar molecule, and $m_1 < am_2$, such a blend cannot be made.

c) Concentrations exhibit linear scaling: multiplying the concentration of each syrup and the blend by the same factor leaves equations valid.

d) The scaling for concentration breaks down when one of the concentrations exceeds 1.

2. Problem

A torus is a donut-shaped body (see figure 4.1). Use scaling to flag the incorrect formulas for the volume of a torus.

a) $V = 2\pi^2 R^2 r$

b) $V = 2\pi^2 \left(R^3 + 3R^2 r + 3Rr^2 + r^3\right)$

c) $V = 2\pi^2 R r^2$

Solution

For small r, the cross section of the torus scales as r^2. The volume of the torus should scale as r^2 as well. Only option c has this property.

3. Problem

A riverboat travels from town A to town B in time T_{AB} and from town B to town A in time T_{BA}. Section A.2 shows that the amount of time required to travel by a raft from town B to town A is given by

$$T_r = \frac{2T_{AB}T_{BA}}{T_{AB} - T_{BA}}. \tag{4.2}$$

a) Is there a scaling property for travel times T_r, T_{AB}, and T_{BA}?

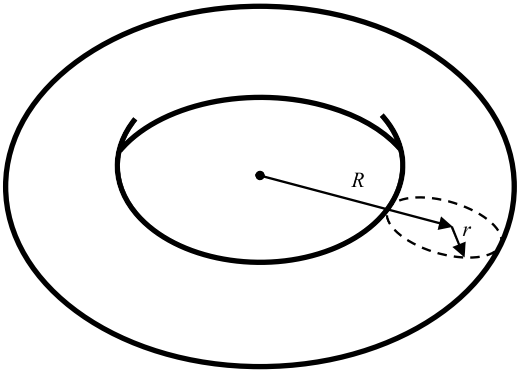

Figure 4.1
A torus: scaling

b) Is there a more general scaling for travel times, velocities, and the distance? Can you select such parameters α, β, and γ that replacing $T_r \leftrightarrow \alpha T_r, T_{AB} \leftrightarrow \alpha T_{AB}, T_{BA} \leftrightarrow \alpha T_{BA}, V_r \leftrightarrow \beta V_r, V_b \leftrightarrow \beta V_b, D \leftrightarrow \gamma D$ will leave the equations valid? (See section A.2 for notations.)

c) How is this scaling related to the units and dimensionality of this problem?

Solution

a) Scaling $T'_i = \alpha T_i$, where T_i includes T_{AB}, T_{BA}, and T_r, leaves equation (4.2) valid.

b) Scaling $T'_r = \alpha T_r, T'_{AB} = \alpha T_{AB}, T'_{BA} = \alpha T_{BA}, V'_r = \beta V_r, V'_b = \beta V_b, D' = \gamma D$ holds as long as

$$\beta = \frac{\gamma}{\alpha}. \tag{4.3}$$

c) Let's consider a change in the units for distance and time. If in the new units for all times are scaled by α and all distances are scaled by γ, then all velocities will be scaled by γ/α. This produces scaling (4.3).

4. Problem

A spherical cap is the part of a sphere that lies above a plane that crosses this sphere (see section A.30 and figure 4.2). The volume V and the surface area S are given by

$$\begin{aligned} V &= \frac{1}{3}\pi h^2(3R - h), \\ S &= 2\pi R h. \end{aligned} \tag{4.4}$$

a) What is the scaling of S with respect to h?

b) What is the scaling of V with respect to h for small values of h?

c) Are both scalings maintained for all possible values of h?

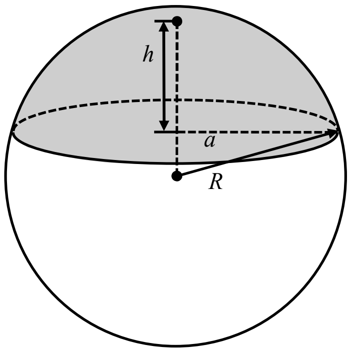

Figure 4.2
A spherical cap: scaling

Solution

a) The scaling of S with respect to h is linear.

b) If we expand the parentheses in the formula for V, there will be a cubic and a quadratic term for h. For small h the quadratic term dominates the scaling.

c) In the formula for S, the linear scaling with respect to h holds for all valid values of h, that is, for $0 \le h \le 2R$. The quadratic scaling of V with respect to h only holds for $h \ll R$.

5. Problem

A radar sends a powerful electromagnetic signal, which then bounces off the object that is being tracked. The radar detects the reflected signal, measures its travel delay, and estimates the distance to the object. The minimum detectable power of the received signal is an important parameter that drives the design of several radar subsystems. How does the power of the received signal scale with the following parameters of the problem:

a) The transmitted power P_t

b) The distance to the object R

c) The area of the object A_o

d) The area of the receiving antenna at the radar A_a

(Hint: See figure 4.2* in section 4.5 and the discussion there.)

Solution

We assume that the total energy is not a function of the distance (that is, attenuation of radio waves in the air is negligible). The radar beam diverges as the signal propagates farther from the radar. At large distances, an object obstructs a fraction of the beam, and therefore only a fraction of the total power hits the object. The power of the signal impinging on the object is equal to

$$P_o = P_t \frac{A_o}{A_b}, \tag{4.5}$$

where P_t is the total transmitted power of the signal in the beam, A_o is the cross section of the object, and A_b is the cross-section of the radar beam at the distance to the object. From section 4.5 we know that A_b scales as R^2.

After that, the radio wave scatters or reflects from the object and propagates away from it. The power of the reflected wave P'_o is proportional to (and is a fraction of) the power of the wave that has hit the object:

$$P'_o = \alpha P_o, \tag{4.6}$$

where $0 < \alpha < 1$. That radiation also diverges as the reflected signal propagates farther from the object. Some fraction of this power hits the receiving antenna of the radar. For the received power P_r we get an equation that is similar to equation (4.5), but now we have to use the power P'_o of the radiation that is reflected by the object, the area A_a of the radar antenna on the receiving end, and a cross-section A'_b of the radiation cone emitted by the object:

$$P_r = P'_o \frac{A_a}{A'_b}. \tag{4.7}$$

Similarly, A'_b scales as R^2.

Combining equations (4.5), (4.6), and (4.7) produces the following scaling for the received power of the signal:

a) Linear with respect to the transmitted power P_t

b) As R^{-4} with respect to the distance to the object R

c) Linear with respect to the area of the object A_o

d) Linear with respect to the area of receive antenna A_a. (Depending on the design, a larger antenna may be able to produce a more narrow beam. In such cases, the scaling with respect to the antenna size may be nonlinear.)

6*. Problem

The Kalman filter estimates quantity X that is measured by two (possibly different) instruments. Suppose that the first measurement produced a value X_1 with a variance of the measurement error σ_1^2, and the second measurement produced a value X_2 with a variance of the measurement error σ_2^2. Then the best estimate for X from these two measurements is given by the following equation:

$$X = \frac{X_1\sigma_2^2 + X_2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}. \tag{4.8}$$

The accuracy of X is characterized by its own variance:

$$\sigma^2 = \frac{\sigma_2^2\sigma_1^2}{\sigma_2^2 + \sigma_1^2}. \tag{4.9}$$

Read section A.32 and explore the following scaling behaviors in the Kalman filter algorithm:

a) Values X_1, X_2 are scaled by the same factor: $\tilde{X}_1 = \alpha X_1; \quad \tilde{X}_2 = \alpha X_2$.

b) Values σ_1^2, σ_2^2 are scaled by the same factor: $\tilde{\sigma}_1^2 = \beta\sigma_1^2; \quad \tilde{\sigma}_2^2 = \beta\sigma_2^2$.

In what way are the implications of these two scalings for the value of X in equation (4.8) different?

Solution

a) X scales linearly with respect to X_1, X_2, and σ^2 remains invariant.

b) σ^2 scales linearly with respect to σ_1^2, σ_2^2, and X remains invariant.

If two measurements of X are scaled by the same factor, the best estimate for X should be scaled by the same factor. This does not affect the uncertainty of the estimate.

Scaling the uncertainties in these two measurements does not change the estimate for X, but linearly scales its uncertainty.

7. Problem

The maximum radius of the spot on Earth's surface covered by a beam from a low-orbit satellite is given by (see section 2.12)

$$L \approx \sqrt{2RH}, \tag{4.10}$$

where R is Earth's radius and H is the orbit altitude.

a) How does L scale with the satellite orbit height H?

b) Will this scaling break down for large values of H? Why?

Solution

a) L scales as a square root of H.

b) The maximum coverage area cannot exceed half of the sphere. Therefore, the scaling of L with respect to H has to break down for large values of H. In fact, this particular formula is valid only for low-orbit satellites, that is, when $H \ll R$.

8. Problem

A circle is given by the following equation:

$$x^2 + y^2 = R^2. \tag{4.11}$$

Compare this with the equation for an ellipse,

$$\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} = 1, \tag{4.12}$$

and show that an ellipse is a scaled version of a circle.

Solution

We introduce $\tilde{x}, \tilde{y}$:

$$\begin{aligned} x &= \alpha\tilde{x}, \\ y &= \beta\tilde{y}. \end{aligned} \tag{4.13}$$

In these new variables the equation for the circle is as follows:

$$\alpha^2\tilde{x}^2 + \beta^2\tilde{y}^2 = R^2. \tag{4.14}$$

We divide this equation by R^2 and denote

$$\begin{aligned} R_x^2 &= \frac{R^2}{\alpha^2}, \\ R_y^2 &= \frac{R^2}{\beta^2} \end{aligned} \tag{4.15}$$

to get the equation for an ellipse:

$$\frac{\tilde{x}^2}{R_x^2} + \frac{\tilde{y}^2}{R_y^2} = 1. \tag{4.16}$$

9. Problem

A circle and a line are given by the following equations (see section A.3):

$$\begin{aligned} x^2 + y^2 &= R^2, \\ y &= px + q. \end{aligned} \tag{4.17}$$

The coordinates $x_{1,2}$ of their intersections (if any) are given by

$$x_{1,2} = \frac{-pq \pm \sqrt{(1+p^2)R^2 - q^2}}{1+p^2}. \tag{4.18}$$

Use the known solution of this problem (equation (4.18)) and the results from problem 8 above to find the horizontal coordinates of the intersections between an ellipse and a straight line. Use the following equations for the ellipse and the straight line:

$$\begin{aligned} \frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} &= 1, \\ y &= px + q. \end{aligned} \tag{4.19}$$

(Do not solve this problem from scratch: leveraging scaling laws here is more economical!)

Solution

We start from equations for the ellipse and the straight line

$$\begin{aligned} \frac{\tilde{x}^2}{R_x^2} + \frac{\tilde{y}^2}{R_y^2} &= 1, \\ \tilde{y} &= \tilde{p}\tilde{x} + \tilde{q}, \end{aligned} \tag{4.20}$$

where we use tilde notations to differentiate variables from the analogous variables (which are not marked with tildes) in the circle and line problem.

We need to map the ellipse and line problem into a scaled circle and line problem. To do that, we select some arbitrary radius of a circle R. Then, following the solution of problem 8, we observe that there is a scaling relationship between these two problems that is given by

$$\begin{aligned} x &= \alpha\tilde{x}, \\ y &= \beta\tilde{y}, \\ R &= \alpha R_x, \\ R &= \beta R_y. \end{aligned} \tag{4.21}$$

We see that equation $\tilde{y} = \tilde{p}\tilde{x} + \tilde{q}$ scales to $y = px + q$ if

$$\begin{aligned} p &= \frac{\beta}{\alpha}\tilde{p}, \\ q &= \beta\tilde{q}. \end{aligned} \tag{4.22}$$

The solution of the circle and line problem is given by equation (4.18), where the right-hand side is a function of p, q, and R:

$$x_{1,2} = f(p, q, R). \tag{4.23}$$

Using the scaling relationships (4.21) and (4.22) this can be written as

$$\tilde{x}_{1,2} = \alpha^{-1} f\left(\frac{\beta}{\alpha}\tilde{p}, \beta\tilde{q}, R\right), \tag{4.24}$$

where parameters α and β are given by

$$\begin{aligned} \alpha &= \frac{R}{R_x}, \\ \beta &= \frac{R}{R_y}, \end{aligned} \tag{4.25}$$

and R is a preselected arbitrary radius of the circle. Equations (4.24) and (4.25), where function f is given by the right-hand side of equation (4.18), jointly define the solution for intersections of an ellipse and a straight line:

$$\begin{aligned} \tilde{x}_{1,2} &= \alpha^{-1}\left(-\frac{\beta^2}{\alpha}\tilde{p}\tilde{q} \pm \sqrt{\left(1 + \frac{\beta^2}{\alpha^2}\tilde{p}^2\right)R^2 - \beta^2\tilde{q}^2}\right)\left(1 + \frac{\beta^2}{\alpha^2}\tilde{p}^2\right)^{-1} \\ &= \frac{R_x}{R}\left(-\frac{RR_x}{R_y^2}\tilde{p}\tilde{q} \pm \sqrt{\left(1 + \frac{R_x^2}{R_y^2}\tilde{p}^2\right)R^2 - \frac{R^2}{R_y^2}\tilde{q}^2}\right)\left(1 + \frac{R_x^2}{R_y^2}\tilde{p}^2\right)^{-1} \\ &= R_x\left(-R_x\tilde{p}\tilde{q} \pm R_y\sqrt{(R_y^2 + R_x^2\tilde{p}^2) - \tilde{q}^2}\right)(R_y^2 + R_x^2\tilde{p}^2)^{-1}. \end{aligned} \tag{4.26}$$

Note that the right-hand side does not contain R. Since the value of R was selected arbitrarily, this is expected.

10. Problem

A circle and a parabola are given by the following equations:

$$\begin{aligned} x^2 + y^2 &= R^2, \\ y &= gx^2 + y_0. \end{aligned} \tag{4.27}$$

Then coordinates of the intersections (if any) are given by

$$\begin{aligned} y_{1,2} &= \frac{-1 \pm \sqrt{1 + 4g(gR^2 + y_0)}}{2g}, \\ y_{3,4} &= y_{1,2}, \\ x_{1,2} &= +\sqrt{R^2 - y_{1,2}^2}, \\ x_{3,4} &= -x_{1,2}, \end{aligned} \tag{4.28}$$

where subscripts $1, 2, 3, 4$ correspond to the $\pm$ signs in the right-hand side (see section A.6 for details). Solutions exist only if the expressions in the radicals are positive. Depending on the signs of the expressions in the radicals, there can be zero, two, or four roots.

Use solution (4.28) and the scaling from problem 8 above to find the coordinates of the intersections between an ellipse and a parabola.

Solution

The solution is analogous to that of problem 9. We start from equations for the ellipse and the parabola

$$\begin{aligned} \frac{\tilde{x}^2}{R_x^2} + \frac{\tilde{y}^2}{R_y^2} &= 1, \\ \tilde{y} &= \tilde{g}\tilde{x}^2 + \tilde{y}_0, \end{aligned} \tag{4.29}$$

where we use tilde notations to differentiate variables from the analogous variables (which are not marked with tildes) in the circle and parabola problem.

We need to map the ellipse and parabola problem into a scaled circle and parabola problem. To do that, we select some arbitrary radius of a circle R. Then, following the solution of problem 8, we observe that there is a scaling relationship between these two problems that is given by

$$\begin{aligned} x &= \alpha\tilde{x}, \\ y &= \beta\tilde{y}, \\ R &= \alpha R_x, \\ R &= \beta R_y. \end{aligned} \tag{4.30}$$

From $\tilde{y} = \tilde{g}\tilde{x}^2 + \tilde{y}_0$ and $y = gx^2 + y_0$ we also obtain

$$\begin{aligned} g &= \frac{\beta}{\alpha^2}\tilde{g}, \\ y_0 &= \beta\tilde{y}_0. \end{aligned} \tag{4.31}$$

The solution of the circle and parabola problem is given by equations (4.28), where the right-hand sides are functions of g, y_0, and R:

$$\begin{aligned} x_{1,2,3,4} &= f_x(g, y_0, R), \\ y_{1,2,3,4} &= f_y(g, y_0, R). \end{aligned} \tag{4.32}$$

Using the scaling relationships (4.30) and (4.31) this can be written as

$$\begin{aligned} \tilde{x}_{1,2,3,4} &= \alpha^{-1} f_x\left(\frac{\beta}{\alpha^2}\tilde{g}, \beta\tilde{y}_0, R\right), \\ \tilde{y}_{1,2,3,4} &= \alpha^{-1} f_y\left(\frac{\beta}{\alpha^2}\tilde{g}, \beta\tilde{y}_0, R\right), \end{aligned} \tag{4.33}$$

where parameters α and β are given by

$$\begin{aligned} \alpha &= \frac{R}{R_x}, \\ \beta &= \frac{R}{R_y}, \end{aligned} \tag{4.34}$$

and R is a preselected arbitrary radius of the circle. Equations (4.33) and (4.34), where functions f_x, f_y are given by the right-hand sides of equations (4.28), jointly define the solution for intersections of an ellipse and a parabola:

$$\beta\tilde{y}_{1,2} = \frac{-1 \pm \sqrt{1 + 4\tilde{g}\dfrac{\beta}{\alpha^2}\left(\tilde{g}\dfrac{\beta}{\alpha^2}R^2 + \beta\tilde{y}_0\right)}}{2\tilde{g}\dfrac{\beta}{\alpha^2}}. \tag{4.35}$$

We divide this equation by β and substitute $\alpha = R/R_x, \beta = R/R_y$ to get

$$\tilde{y}_{1,2} = \frac{-R_y^2 \pm \sqrt{R_y^4 + 4\tilde{g}R_x^2R_y^2(\tilde{g}R_x^2 + \tilde{y}_0)}}{2\tilde{g}R_x^2}. \tag{4.36}$$

For $\tilde{x}_{1,2}$ we get

$$\tilde{\alpha}\tilde{x}_{1,2} = \sqrt{R^2 - \beta^2\tilde{y}_{1,2}^2}. \tag{4.37}$$

We use $\alpha = R/R_x, \beta = R/R_y$ to get

$$\tilde{x}_{1,2} = R_x\sqrt{1 - \frac{\tilde{y}_{1,2}^2}{R_y^2}}. \tag{4.38}$$

Finally, $x_{3,4} = x_{1,2}$. Note that the solution does not contain R. Since the value of R was selected arbitrarily, this is expected.

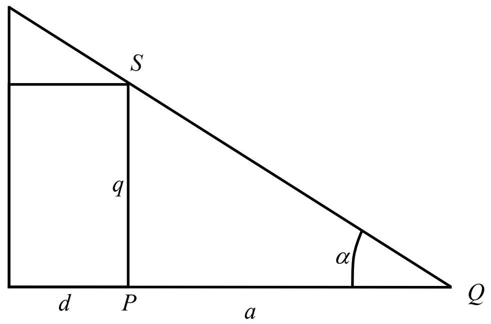

Figure 4.3
Rectangle inscribed in a right triangle: scaling

11. Problem

A rectangle is inscribed in a right triangle (see figure 4.3). One of the legs has length a, and the measure of the adjacent angle is α. The side of the rectangle that is aligned with that leg has length d.

a) Use a symmetry argument to show that for an isosceles right triangle the inscribed rectangle should have an extremum (either a maximum or a minimum) for $d = a/2$.

b) Assume that the triangle and the inscribed rectangle are scaled by the same multiplier along one leg. This scaling of an isosceles right triangle generally produces a scalene right triangle. Investigate how the scaling affects the area of the inscribed rectangle and of the triangle.

c) Using the above symmetry and scaling arguments, extend the prediction of an extremum for the rectangle's area from the case of an isosceles right triangle to any right triangle. Compare this with the results in section A.28.

Solution

a) Let's consider the area of a rectangle that is inscribed in an isosceles right triangle. Let sides of this rectangle be d and q. Because of symmetry in the triangle, if a rectangle with sides d and q is inscribed in the triangle, then a rectangle with sides q and d can also be inscribed. Therefore, the area of the rectangle as the function of one of the sides will have a symmetry with respect to a swap $d \leftrightarrow q$.

Now we consider two cases: $d_1 = q_1$ and some value $d_2 > q_2$ (see figure 4.4). Note that the first case corresponds to $d_1 = a/2$, and the second case corresponds to $d_2 > a/2, q_2 < a/2$. Suppose that the areas of these rectangles obey $A(d_1, q_1) > A(d_2, q_2)$. Then because of the symmetry we would also have $A(d_1, q_1) > A(q_2, d_2)$. Since $d_2 > d_1 > q_2$, we see that area $A(d_1, q_1)$ takes a maximum as a function of the length of one side of the rectangle. (If $A(d_1, q_1) < A(d_2, q_2)$, it would take a minimum.) Note that this argument does not specify if this is a maximum or a minimum, or that the extremum is global.

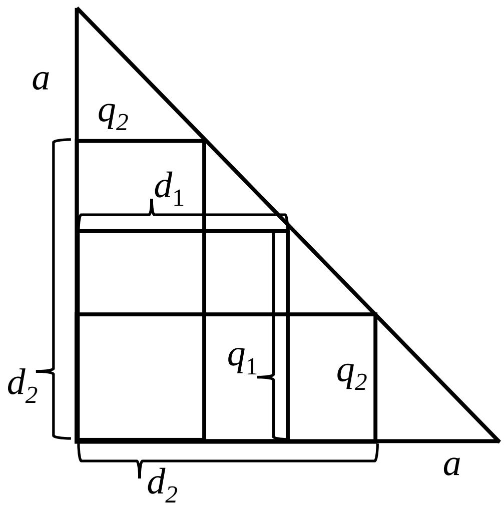

Figure 4.4
Symmetry for inscribed rectangles

b) If one leg of a right triangle is scaled by some factor μ, then the area of that triangle and any rectangle that is inscribed in it is also scaled by μ. (This is true for areas of all plane figures if we scale them with respect to one dimension.)

c) From item *b* we see that scaling preserves the relative differences in the areas of inscribed rectangles. That is, if one rectangle has a greater area than another for an isosceles triangle, the scaled versions of both rectangles will satisfy the same inequality. Therefore, if a side of a rectangle is half of the leg of the triangle and such a rectangle has a maximum area for an isosceles triangle, then it would have a maximum area for a scalene triangle. This result is consistent with one in section A.28.

12. Problem

Exercise 8 in chapter 2 deals with limiting cases for the following equation:

$$\frac{1}{x-a} - \frac{1}{x-b} = d. \tag{4.39}$$

The solution for this equation for x is given in section A.12:

$$x_{1,2} = \frac{d(a+b) \pm \sqrt{d^2(a-b)^2 + 4d(a-b)}}{2d}, \tag{4.40}$$

where subscripts 1, 2 correspond to the ± signs in the right-hand side. Use a hierarchy of scalings to show that for the limiting case $|d| \to 0$ the solution produces large values for $|x|$ unless $a \approx b$. Is this also evident from equation (4.39)?

Solution

For $|d| \to 0$ the hierarchy of scalings in equation (4.40) yields

$$\begin{aligned} x_{1,2} &= \frac{d(a+b) \pm \sqrt{d^2(a-b)^2 + 4d(a-b)}}{2d} \\ &\approx \frac{d(a+b) \pm \sqrt{4d(a-b)}}{2d} \\ &\approx \frac{\pm\sqrt{4d(a-b)}}{2d} \\ &\propto \frac{1}{\sqrt{d}}. \end{aligned} \tag{4.41}$$

Therefore $x_{1,2} \to \infty$. Note that if $a = b$, then the term $\sqrt{4d(a-b)}$ no longer dominates the term $d(a+b)$, and $x_{1,2}$ no longer goes to infinity for $d \to 0$.

Next, we consider equation (4.39) for $|d| \to 0$. The right-hand side approaches zero. If $a \neq b$, the two terms in the left-hand side cannot cancel or nearly cancel each other. Then both of them must approach zero. For finite values of a, b this requires $|x| \to \infty$.

13. Problem

Section A.4 solves the problem of finding the intersections between a circle and an ellipse (see figure 4.5). The circle and the ellipse are given by the following equations:

$$\begin{aligned} x^2 + y^2 &= R^2, \\ \frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} &= 1. \end{aligned} \tag{4.42}$$

Use scaling for this problem to flag the wrong solutions among the following options:

a) $x = \pm\sqrt{R_x^4 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $\quad y = \pm\sqrt{R_y^4 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$

b) $x = \frac{1}{4} \pm \sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $\quad y = \frac{1}{4} \pm \sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$

c) $x = \pm\sqrt{R_x^2 \frac{R_y^2 - R^2}{R_y^2 - R_x^2}}$; $\quad y = \pm\sqrt{R_y^2 \frac{R_x^2 - R^2}{R_x^2 - R_y^2}}$

d) $x = \pm\sqrt{R_x^2 \frac{R_y^2 - R^{-2}}{R_y^2 - R_x^2}}$; $\quad y = \pm\sqrt{R_y^2 \frac{R_x^2 - R^{-2}}{R_x^2 - R_y^2}}$

Solution

If all radii R_x, R_y, and R are scaled by the same amount, we expect x, y to scale by the same amount. This scaling property does not hold for options *a, b*, and *d*.

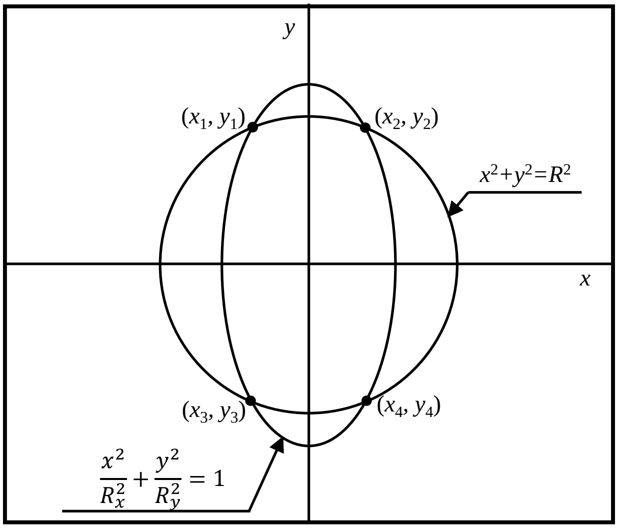

Figure 4.5
A circle and an ellipse: scaling

14. Problem

Consider a circle and a parabola that are defined by the following equations:

$$\begin{aligned} x^2 + y^2 &= R^2, \\ y &= gx^2 + y_0. \end{aligned} \tag{4.43}$$

The vertical coordinates of the intersections between these two curves are given by (see section A.6):

$$y_{1,2} = \frac{-1 \pm \sqrt{1 + 4g(gR^2 + y_0)}}{2g}. \tag{4.44}$$

Which one of the following scaling options is valid? (Check them for both the formulation and the solution of the problem.)

a) $x \leftrightarrow ax; R \leftrightarrow aR; y \leftrightarrow ay; y_0 \leftrightarrow ay_0; g \leftrightarrow a^{-1}g$

b) $x \leftrightarrow ax; R \leftrightarrow a^{-1}R; y \leftrightarrow ay; y_0 \leftrightarrow ay_0; g \leftrightarrow ag$

c) $x \leftrightarrow ax; R \leftrightarrow aR; y \leftrightarrow a^{-1}y; y_0 \leftrightarrow a^{-1}y_0; g \leftrightarrow a^{-1}g$

d) $x \leftrightarrow a^{-1}x; R \leftrightarrow aR; y \leftrightarrow ay; y_0 \leftrightarrow ay_0; g \leftrightarrow ag$

Solution

We substitute these scalings in the equations for the circle and parabola equations and for the solution:

a) For the circle:

$$(ax)^2 + (ay)^2 = (aR)^2. \tag{4.45}$$

This yields

$$x^2 + y^2 = R^2. \tag{4.46}$$

For the parabola:

$$ay = (a^{-1}g)(ax)^2 + ay_0. \tag{4.47}$$

This yields

$$y = gx^2 + y_0. \tag{4.48}$$

For the solution:

$$ay_{1,2} = \frac{-1 \pm \sqrt{1 + 4a^{-1}g(a^{-1}ga^2R^2 + ay_0)}}{2ga^{-1}}. \tag{4.49}$$

This yields

$$y_{1,2} = \frac{-1 \pm \sqrt{1 + 4g(gR^2 + y_0)}}{2g}. \tag{4.50}$$

For this option scaling holds.

b) For the circle:

$$(ax)^2 + (ay)^2 = (a^{-1}R)^2. \tag{4.51}$$

This yields

$$x^2 + y^2 = a^{-4}R^2. \tag{4.52}$$

For the parabola:

$$ay = ag(ax)^2 + ay_0. \tag{4.53}$$

This yields

$$y = a^2gx^2 + y_0. \tag{4.54}$$

For the solution:

$$ay_{1,2} = \frac{-1 \pm \sqrt{1 + 4ag(aga^{-2}R^2 + ay_0)}}{2ag}. \tag{4.55}$$

This yields

$$y_{1,2} = \frac{-1 \pm \sqrt{1 + 4ag(a^{-1}gR^2 + ay_0)}}{2a^2 g}. \tag{4.56}$$

For this option scaling fails.

c) For the circle:

$$(ax)^2 + a^{-2}y^2 = (aR)^2. \tag{4.57}$$

This yields

$$x^2 + a^{-4}y^2 = R^2. \tag{4.58}$$

For the parabola:

$$a^{-1}y = a^{-1}g(ax)^2 + a^{-1}y_0. \tag{4.59}$$

This yields

$$y = a^2 g x^2 + y_0. \tag{4.60}$$

For the solution:

$$a^{-1}y_{1,2} = \frac{-1 \pm \sqrt{1 + 4a^{-1}g(a^{-1}ga^2R^2 + a^{-1}y_0)}}{2a^{-1}g}. \tag{4.61}$$

This yields

$$y_{1,2} = a^2 \frac{-1 \pm \sqrt{1 + 4a^{-1}g(agR^2 + a^{-1}y_0)}}{2g}. \tag{4.62}$$

For this option scaling fails.

d) For the circle:

$$a^{-2}x^2 + a^2y^2 = (aR)^2. \tag{4.63}$$

This yields

$$a^{-4}x^2 + y^2 = R^2. \tag{4.64}$$

For the parabola:

$$ay = aga^{-2}x^2 + ay_0. \tag{4.65}$$

This yields

$$y = a^{-2}gx^2 + y_0. \tag{4.66}$$

For the solution:

$$ay_{1,2} = \frac{-1 \pm \sqrt{1 + 4ag(aga^2R^2 + ay_0)}}{2ag}. \tag{4.67}$$

This yields

$$y_{1,2} = a^{-2}\frac{-1 \pm \sqrt{1 + 4ag(a^3gR^2 + ay_0)}}{2g}. \tag{4.68}$$

For this option scaling fails.

15*. Problem

The solution of problem 6 in chapter 1 gives formulas for the velocity of gravity waves in deep water (in the case of $\lambda \ll h$) and in shallow water (in the case of $\lambda \gg h$):

$$\begin{aligned} V_{\text{deep}} &= \sqrt{\frac{g\lambda}{2\pi}}, \\ V_{\text{shallow}} &= \sqrt{gh}, \end{aligned} \tag{4.69}$$

where λ is the wavelength (the distance between two consecutive wave crests), h is the depth of the water basin in the area of wave propagation, and $g \approx 9.8$ m/s^2 is the gravity acceleration.

a) How does the wave velocity scale with the wavelength in the deep ocean?

b) Consider shallow water waves. What happens with the wave velocity as the wave approaches a shore, where the depth is getting progressively smaller? If the depth difference causes the rear part of the wave to travel at a speed that is different from the speed of the front of the wave, what happens to the wave crest? Have you observed this effect on the beach?

c) A tsunami is an ocean wave that may be generated by an underwater earthquake. Since the area of the ocean floor that is affected by an earthquake is large, a tsunami may have a wavelength of hundreds of kilometers. Is a tsunami described by the shallow-water equation or by the deep-water equation?

d) Explain why a tsunami travels much faster than the ocean waves that you may see at the beach. (This, of course, makes a tsunami particularly dangerous.)

Solution

a) The wave velocity scales as the square root of the wavelength.

b) The wave velocity scales as the square root of the depth. As the wave approaches a beach, the rear part of the wave is at a higher depth and therefore travels faster. As a result, the wave becomes squeezed in the direction of its propagation. This makes the wave steeper.

c) For a tsunami, $\lambda \gg h$. Therefore, tsunami is a shallow water wave (even if it may originate in a deep part of the ocean).

d) Since for a shallow water wave the speed of the wave scales as $\sqrt{h}$, and the ocean depth can be several kilometers, a tsunami travels much faster than the waves that you see on the beach.

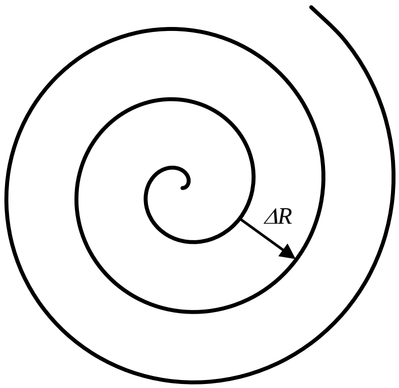

Figure 4.6
Archimedes's spiral: scaling

As the tsunami approaches the shore, it is squeezed in the direction of propagation (see item *b*) and becomes steeper. This makes it particularly dangerous.

16. Problem

The radius of Archimedes's spiral increases linearly with the turn angle (figure 4.6). The arc length of the spiral is given by the following equation:

$$L = \frac{\Delta R}{4\pi}\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right), \tag{4.70}$$

where ΔR is the distance between the adjacent loops and θ is the total turn angle.

a) What is the scaling of the arc length with respect to ΔR?

b) Identify two scaling behaviors of the arc length with respect to angle θ that correspond to the two additive terms in the right-hand side of equation (4.70).

c) Which one of these two scalings dominates the result for large values of θ?

Solution

a) The arc length scales linearly with ΔR. This is expected: if we change the scale of both coordinates, all distances and lengths should scale uniformly.

b) The two additive terms in the right-hand side are $\theta\sqrt{1+\theta^2}$ and $\ln\left(\theta + \sqrt{1+\theta^2}\right)$. For large θ the first term scales as θ^2 and the second one scales as $\ln\theta$.

For small θ the first term scales as

$$\theta\sqrt{1+\theta^2} \propto \theta. \tag{4.71}$$

The scaling of the logarithmic term for small θ is a bit more tricky to determine. One option is to use a Taylor expansion, which yields $\ln\left(\theta + \sqrt{1+\theta^2}\right) \propto \theta$. If we do not want to use a Taylor series, then we note that for small θ the quadratic term θ^2 in the square root can be neglected:

$$\ln\left(\theta + \sqrt{1+\theta^2}\right) \propto \ln(1+\theta). \tag{4.72}$$

Now we need to consider the scaling of $\ln(1+\theta)$. To do that, we consider $\ln(1+\theta)^2$. On one hand, we get

$$\begin{aligned} \ln(1+\theta)^2 &= \ln(1+2\theta+\theta^2) \\ &\propto \ln(1+2\theta). \end{aligned} \tag{4.73}$$

On the other hand, we have

$$\ln(1+\theta)^2 = 2\ln(1+\theta). \tag{4.74}$$

We see that

$$\ln(1+2\theta) \propto 2\ln(1+\theta), \tag{4.75}$$

which suggests that for small θ the expression $\ln(1+\theta)$ scales linearly with θ.

c) From the hierarchy of scalings we know that a power law always dominates a logarithm. Therefore, for large θ we have

$$L \propto \frac{\Delta R}{4\pi}\theta^2. \tag{4.76}$$

17. Problem

The number of pairs that can be selected from N objects is given by

$$M = \frac{N(N-1)}{2}. \tag{4.77}$$

a) How does M scale as a function of N for large values of N?

b) Solve equation (4.77) for N. How does N scale as a function of M for large values of M?

Solution

a) The scaling of M with respect to N for large values of N is quadratic:

$$M \propto \frac{N^2}{2}. \tag{4.78}$$

b) The solution of equation (4.77) for N is as follows. We write

$$N^2 - N - 2M = 0. \tag{4.79}$$

Then

$$N_{1,2} = \frac{1 \pm \sqrt{1 + 8M}}{2}. \tag{4.80}$$

Since N must be positive, we select the plus sign in this solution. For large M we have

$$\begin{aligned} N &\propto \frac{\sqrt{8M}}{2} \\ &= \sqrt{2M}. \end{aligned} \tag{4.81}$$

Note that scalings (4.78) and (4.81) are consistent.

18. Problem

In a room there are N people that have been randomly selected from the residents of a city to participate in a focus group. A probability p of the group having at least two people there who know each other scales approximately quadratically versus the number of people: $p(N) \propto N^2$.

a) Does this scaling work for unlimited values of N, or does it break down for large values of N?

b) Suppose that, for $N = 2$, the probability that these two people know each other is $p(2)$. Can you obtain an accurate result for probability $p(3)$ (that is, for the case of three people in the room) using the quadratic scaling for p? Why? (Hint: Use the results of problem 17.)

Solution

a) The scaling must break down for large values of N, at least because we must have $p(N) \leq 1$ for any N.

b) A group of three contains three different pairs of people. This follows from problem 17 in this chapter; also, for people A, B, and C there are pairs AB, BC, and CA. Therefore, if the probability for people in any pair to know each other is $p(2)$ and $p(2)$ is small, the probability for a group of three people to have at least two people who know each other is approximately $3p(2)$. (with some underlying assumptions, the exact formula is $3p(2) - 3p^2(2) + p^3(2) \propto 3p(2)$ for small values of $p(2)$.) This is somewhat different from the quadratic scaling estimate that is given by

$$\begin{aligned} p(3) &= \left(\frac{3}{2}\right)^2 p(2) \\ &= 2.25p(2). \end{aligned} \tag{4.82}$$

For $N \gg 1$ and $p(2) \ll 1/N^2$ the quadratic scaling is more accurate.

19. Problem

A chemical reaction between two different gases requires a collision between a molecule of one gas and a molecule of another gas. The speed of the reaction is directly proportional to (that is, scales linearly with) the number of such collisions per unit time.

a) How does the speed of the reaction scale if the density of either of the gases is scaled by a factor of a, with other parameters remaining constant?

b) Suppose we have a fixed amount of gas in a cylinder that is slowly compressed or expanded using a piston. How does the gas density scale versus its volume? (Hint: Note that gas density is measured in $\mathrm{kg/m^3}$ and the volume is measured in $\mathrm{m^3}$.)

c) How does the speed of the reaction scale if the volume of a two-gas mixture is scaled by a factor b, while keeping the total amount of gas constant? (Hint: Note a similarity with problem 18 or consider the combined effects of scaling in tasks 19a and 19b for this problem.)

Solution

a) The number of collisions is proportional to the density of each of the gases. Therefore, if the density of one of the gases is scaled by a factor of a, the speed of reaction will increase by the same factor.

b) The gas density is inversely proportional to the volume. Indeed, the product of the gas density and the volume is equal to the total mass of the gas, which remains constant.

c) As the volume of the mixture is scaled by a factor b, the density of each gas is scaled as b^{-1}. Then the speed of the chemical reaction for this mixture of gases scales as b^{-2}.

20. Problem

A bank promises to pay interest r on any deposit, which means that M_0 dollars in a savings account grow to $M_1 = (1 + r)M_0$ dollars in a year. This establishes a linear scaling for the amount M_1 at the end of the year versus amount M_0 in the beginning of the year. Use the derivation in section 4.6 as a template to determine how the account balance varies over time.

Solution

At the end of the second year the account balance will be $M_2 = (1 + r)^2 M_0$. At the end of the N–th year it will reach $M_N = (1 + r)^N M_0$. For continuous time we expect to have

$$M(t) = (1 + r)^t M_0, \tag{4.83}$$

where time t is measured in years. We rewrite this equation as

$$M(t) = e^{\rho t} M_0, \tag{4.84}$$

where $\rho = \ln(1 + r)$. The account balance scales exponentially with respect to time and linearly with respect to the initial deposit.

21. Problem

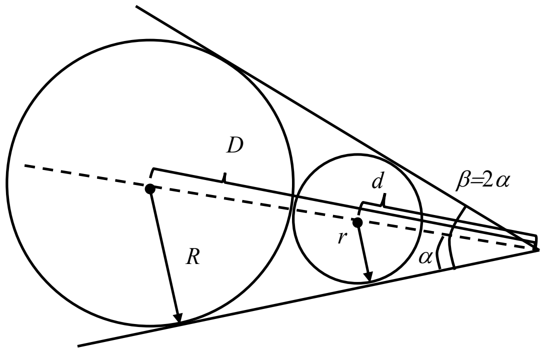

Figure 4.7
Two circles inscribed in an angle: scaling

Two circles are inscribed in an angle in such a way that they touch each other (figure 4.7). Section A.26 shows that the ratio of the radii of these circles is given by

$$\frac{R}{r} = \frac{1 + \sin\frac{\beta}{2}}{1 - \sin\frac{\beta}{2}}. \tag{4.85}$$

Equation (4.85) establishes a linear scaling for the radii of the two circles: $R \propto r$. Consider a modification of this problem where there are more than two circles inscribed in an angle. Adjacent circles touch each other. If there are N circles with radii $r_1, \ldots, r_N$, what is the ratio r_N/r_1? How does it scale with the value of N?

Solution

For three circles, the ratio of the radii of the third and the first circles is given by

$$\frac{R_3}{R_1} = \left(\frac{1 + \sin\frac{\beta}{2}}{1 - \sin\frac{\beta}{2}}\right)^2. \tag{4.86}$$

Similarly for N circles we have

$$\frac{R_N}{R_1} = \left(\frac{1 + \sin\frac{\beta}{2}}{1 - \sin\frac{\beta}{2}}\right)^{N-1}. \tag{4.87}$$

The scaling with respect to N is exponential.

22. Problem

Section A.15 solves the following equation:

$$\frac{\cos(\alpha + x)}{\cos(\alpha - x)} = \frac{p}{q}. \tag{4.88}$$

Use scaling analysis to flag the wrong solutions among the following options:

a) $x = \tan^{-1}\left(\cot\alpha \cdot \frac{q-p}{p+q}\right) + pq + n\pi.$

b) $x = \tan^{-1}\left(\cot\alpha \cdot \frac{q-p}{p+q}\right) + n\pi.$

c) $x = \tan^{-1}\left(\cot\alpha \cdot \frac{pq}{p+q}\right) + n\pi.$

Solution

Equation (4.88) remains invariant with respect to scaling $\tilde{p} = ap; \tilde{q} = aq$. Among the three options, only option *b* retains this scaling.

23. Problem

The force between two neutral atoms or molecules is commonly modeled using the *Lennard–Jones model*:

$$F(r) = \frac{24\epsilon}{\sigma}\left(2\left(\frac{\sigma}{r}\right)^{13} - \left(\frac{\sigma}{r}\right)^{7}\right), \tag{4.89}$$

where r is the distance between the atoms and σ, ϵ are positive parameters. A positive value for the force means that it is repulsive, and a negative value means it is attractive.

a) Use the hierarchy of scalings to prove that atoms attract at large distances. This is the reason that molecules in liquids and solids stay together, except at high temperatures.

b) Prove that the force becomes repulsive at small distances. This is the reason that liquids are nearly uncompressible.

Solution

a) For large distances, the term with r^{-7} dominates the term with r^{-13}. Therefore, $F(r) < 0$ and the force is attractive.

b) For small distances, the term with r^{-13} dominates the term with r^{-7}. Therefore, $F(r) > 0$ and the force is repulsive.

24. Problem

In chemistry and biology, the Q_{10} temperature coefficient tells us that a biological process or a chemical reaction runs Q_{10} times faster if the temperature is increased by 10° C.[1]

a) What type of scaling is present for the rate of a chemical reaction with respect to the temperature?

1. Flipping this definition, a biological process or a chemical reaction runs Q_{10} times *slower* if the temperature is *decreased* by 10° C. This is why food stays fresh longer if refrigerated.

b) Suppose that two chemical reactions in a human cell have temperature coefficients Q_{10} and Q'_{10}. Assume that $Q'_{10} > Q_{10}$ and that at a normal temperature these two reactions have equal rates. Which reaction will run faster than the other one if the person has a fever?

Solution

a) As the temperature is increased by a fixed amount, the speed of a process or a chemical reaction is multiplied by a fixed factor. That is a characteristic of exponential scaling.

b) If a person has a fever, speeds of both reactions increase. However, for the reaction with temperature coefficient Q'_{10} this increase will be larger, and that reaction will run faster.

25. Problem

If an object is freely falling in the air, its velocity increases until the force of gravity becomes balanced by the air drag force. This steady fall velocity is called the terminal velocity; for small objects, it is given by the following equation:[2]

$$mg = CR\eta V_t, \tag{4.90}$$

where m is the mass of the object, g is the gravity acceleration, C is a coefficient, η is the air viscosity, R is the size of the object, and V_t is its terminal velocity. Consider a steady fall scenario and investigate the scaling with respect to the size of the object:

a) If the density of the object is constant, what is the scaling of the object's mass with its size?

b) Knowing the scaling for mass m, determine the scaling for V_t versus the object's size that satisfies the equation above.

c) What does this scaling tell us for extremely small objects? Explain why specks of dust that you see in a ray of sunlight do not seem to fall down.

Solution

a) The volume of the object scales as R^3. For a constant density, the mass also scales as R^3.

b) We use $m = DR^3$, where D is a constant. Then equation (4.90) yields

$$DR^2 g = C\eta V_t, \tag{4.91}$$

which indicates that the terminal velocity scales as the square of the object's size.

c) For extremely small objects the terminal velocity is very low. That is why specks of dust that you may see in a ray of sunlight seem floating in the midair.

26. Problem

Section 4.5 noted that the volume of the space between two concentric spheres of radii R and $R + \delta R$ scales as R^2 (see figure 4.3*).

2. For a sphere, $C = 6\pi$, the formula known as Stokes's law (see section 4.2).

a) Prove this quadratic scaling from the formula for the volume of a sphere (assume that $\delta R \ll R$):

$$V = \frac{4}{3}\pi R^3. \tag{4.92}$$

b) How does the volume between the two spheres scale as a function of δR if $\delta R \ll R$?

c) How does the volume between the two spheres scale as a function of δR if $\delta R \gg R$?

d) What is the condition for δR that corresponds to the transition between these two last scalings?

Solution

a) The volume of the larger sphere is given by

$$V_l = \frac{4}{3}\pi(R + \delta R)^3. \tag{4.93}$$

The volume of the smaller sphere is given by

$$V_s = \frac{4}{3}\pi R^3. \tag{4.94}$$

Then the volume of the shell is computed as follows:

$$\begin{aligned} \delta V &= \frac{4}{3}\pi(R + \delta R)^3 - \frac{4}{3}\pi R^3 \\ &= \frac{4}{3}\pi(3R^2\delta R + 3R\delta R^2 + \delta R^3). \end{aligned} \tag{4.95}$$

For small δR, the term $3R^2\delta R$ in the parentheses is dominant. We obtain

$$\begin{aligned} \delta V &\approx \frac{4}{3}\pi 3R^2\delta R \\ &= 4\pi R^2\delta R. \end{aligned} \tag{4.96}$$

Note that this value for the shell volume can be obtained if we just multiply the area of the sphere $4\pi R^2$ by the shell thickness δR, as if it were the volume of a flat figure with the base area $4\pi R^2$ and thickness δR. This shows that for thin shells (that is, when $\delta R \ll R$) the curvature of the sphere does not affect the volume computation for the shell.

b) The volume scales linearly with respect to δR as long as $\delta R \ll R$.

c) If $\delta R \gg R$, the volume of the shell is approximately equal to the volume of the larger sphere, whose radius is approximately equal to δR. Therefore, the volume of the shell scales as δR^3.

d) The transition between these two scalings is evident from equation (4.95), where the dominant term has the lowest power of δR for small δR and the largest power of δR for large δR. The transition occurs for $\delta R \sim R$.

27. Problem

The discussion of equations (4.6*) and (4.7*) in section 4.2 explores competing scalings with respect to velocity for an object that is moving in the air. Another variable used in both equations is the size of the object. Consider a hailstone that starts forming in a cloud as a tiny speck and then grows to a larger ball. As it moves through the air, it is subject to the air drag force.

Which of the two mathematical models (given by equations (4.6*) and (4.7*)) describes the air drag force for the initial stages, and which is appropriate for the later stages of the hailstone growth? (Hint: Take into account that radius R and the cross section of the hailstone A in these two mathematical models are linked. Assume that the speed of the fall increases as the hailstone grows.)

Solution

As the hailstone falls, both its radius and velocity increase. We compare two mathematical models:

$$F_l = 6\pi R\eta V \tag{4.97}$$

and

$$F = \frac{1}{2}C_D\rho A V^2, \tag{4.98}$$

where notations are given in section 4.2. In the last equation the cross-section area of the hailstone is computed as $A = \pi R^2$. We see that F is quadratic with respect to both R and V, and F_l is linear with respect to these variables. The motion of the hailstone will be described by the larger of the forces F and F_l. Therefore, for small R and V the motion of the hailstone will be described by the formula for F_l, and for large R and V it will be described by the formula for F.

28*. Problem

Consider a plot of the function $y = 1/x$ for positive values of x.

a) Using scaling, prove that the area $S(a)$ under the curve $y = 1/x$ for $1 \le x \le a$ is the same as the area under the same curve for $c \le x \le ca$ for any positive value of c.

b) Consider the value of $S(a^2)$, which is the area under the same curve for $1 \le x \le a^2$. Split this area in two sections: area $S(a)$ for $1 \le x \le a$ and area $S'(a)$ for $a < x \le a^2$. Using the result from task 28a, express area $S(a^2)$ through $S(a)$. This establishes a scaling for function $S(a)$.

c) Which standard function exhibits the same scaling property as function $S(a)$?

Solution

a) Figure 4.8 shows two shapes. The first one is the area below the curve $1/x$ and in between $x = 1$ and $x = 2$. Its area is given by $S(2)$. The second one is also below the curve $1/x$, but in between $x = 4$ and $x = 8$. Its area corresponds to $a = 2, c = 4$.

The second shape can be obtained from the first one as a result of two operations:

i. Stretching it in the horizontal dimension by the factor c.

ii. Squeezing it in the vertical dimension by the same factor.

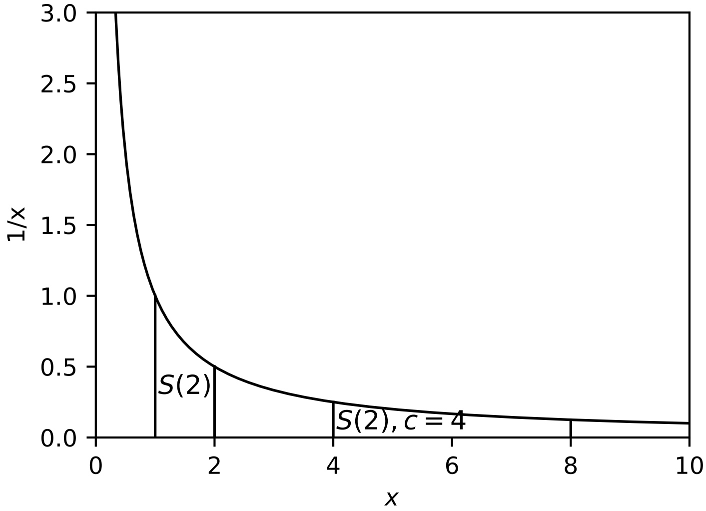

Figure 4.8
Scaling of areas under the curve $y = 1/x$.

The first operation increases the area by a factor c. The second operation decreases the area by the same factor. The net effect of these two operations leaves the area of the figure intact. Therefore, the area $S(a)$ under the curve $y = 1/x$ for $1 \leq x \leq a$ is the same as the area under the same curve for $c \leq x \leq ca$ for any positive value of c.

b) We set $c = a$ in task 28a to prove that $S(a) = S'(a)$. We also know that $S(a^2) = S(a) + S'(a)$. Therefore,

$$S(a^2) = 2S(a). \tag{4.99}$$

This scaling is the same as for the logarithm:

$$\ln a^2 = 2 \ln a, \tag{4.100}$$

which suggests that $S(a) = p \ln a$, where p is a constant multiplier. Indeed, the area under the curve $f(x) = 1/x$ between $x = 1$ and $x = a$ is given by $\ln a$.

29. Problem

As workers gain more experience, they perform their tasks faster. The first quantification of this effect is known as Wright's law. In 1936, T. P. Wright observed that as the production of airplanes at the Curtiss-Wright factory doubled, the time required to manufacture each airplane decreased by 20 percent. Express Wright's law as a scaling relationship.

Solution

We denote the airplane production as p and the time required to build each plane as T. We consider two cases: before (without tildes) and after (with tildes) the production of airplanes is doubled. We express Wright's law as follows:

$$\begin{aligned} \tilde{p} &= \alpha p, \\ \tilde{T} &= \beta T. \end{aligned} \tag{4.101}$$

From the condition of the problem we know that if $\alpha = 2$, then $\beta = 0.8$. This implies that there is a functional dependence between these two variables: $\beta = f(\alpha)$. Unfortunately, it is impossible to define function f from one data point $\alpha = 2, \beta = 0.8$. However, it is reasonable to interpret Wright's law in such a way, that it applies to any level of production. For example, consider what happens if we again double the production to get $\tilde{\tilde{p}} = \alpha\tilde{p} = \alpha^2 p$. We should get $\tilde{\tilde{T}} = \beta\beta T = \beta^2 T$. In terms of the functional dependence between β and α this means that $\beta^2 = f(\alpha^2)$. If the same is true for all values of production, we will see that $\beta^n = f(\alpha^n)$. The only function that satisfies this condition is the power law. Indeed, if $\beta = \alpha^q$, then

$$(\alpha^n)^q = (\alpha^q)^n = \beta^n. \tag{4.102}$$

Next, we must make sure that $\beta = 0.8$ for $\alpha = 2$. Condition $2^q = 0.8$ becomes an equation for q. Its solution is given by $q = \ln 0.8/\ln 2 \approx -0.322$.

30*. Problem

The depressed cubic equation is given by

$$x^3 + px + q = 0. \tag{4.103}$$

Its three solutions can be expressed through trigonometric functions (see section A.29):

$$x_k = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right), \tag{4.104}$$

where $k = 0, 1, 2$.

a) Show that equation (4.103) obeys the scaling property:

$$\begin{aligned} x' &= ax, \\ p' &= a^2 p, \\ q' &= a^3 q. \end{aligned} \tag{4.105}$$

b) Show that this scaling applies also to solution (4.104).

Solution

a) We substitute x', p', and q' in the depressed cubic equation:

$$\begin{aligned} x'^3 + p'x' + q' &= a^3x^3 + a^2pax + a^3q \\ &= a^3(x^3 + px + q) \\ &= 0. \end{aligned} \tag{4.106}$$

This confirms that the depressed cubic equation is invariant with respect to scaling (4.105).

b) To check if solution (4.104) is also invariant with respect to this scaling we consider the argument and the coefficient for the trigonometric functions separately. For the argument of $\cos^{-1}$ we get

$$\begin{aligned} \frac{3q'}{2p'}\sqrt{-\frac{3}{p'}} &= \frac{3qa^3}{2pa^2}\sqrt{-\frac{3}{pa^2}} \\ &= \frac{3q}{2p}\sqrt{-\frac{3}{p}}. \end{aligned} \tag{4.107}$$

The argument of $\cos^{-1}$ is invariant. Then the value of the cos function in equation (4.104) is also invariant with respect to scaling (4.105). For the coefficient for the cosine we get

$$2\sqrt{-\frac{p'}{3}} = 2a\sqrt{-\frac{p}{3}}. \tag{4.108}$$

This is consistent with the scaling for x_k in the left-hand side of equation (4.104). This means that the entire solution (4.104) is invariant with respect to scaling (4.105).

5 Order of Magnitude Estimates

The exercises below are different from those in other chapters. A key part of making an order of magnitude estimate is planning a simple approach that uses known or easily obtainable numbers. It would defeat the purpose of this chapter to suggest a working approach to solve each problem, or even to suggest all the data you may use. Moreover, many problems can be correctly solved multiple ways. It is up to you to chart a course for a solution and to hunt for the necessary data. As a result, some or all input data may be missing in the formulation of the exercises below.

You are allowed to speculate and to use reference books and web searches to come up with the missing data needed to solve each problem. Of course, this does not mean that you can simply Google the final result—you still have to produce an estimate from some other data.

1. Problem

A screenwriter is tasked with writing a script for a movie about a bank heist. The initial concept calls for the burglars getting away with $5 million in cash. The screenwriter wants to be sure that the burglars can physically haul that amount. Estimate how much $5 million may weigh and what the combined volume of sacks is needed to carry this cash. Make calculations for three different cases:

a) All the cash is in $100 bills.

b) All the cash is in $20 bills.

c) The cash is in roughly equal numbers of $100 and $20 bills.

Solution

We assume that all bills have the same weight and volume. A stack of 100 bills measures about 1 cm × 5 cm × 15 cm, which amounts to 75 cm^3. The density of the bills is higher than the density of water. That volume of water would weigh 75 g, and we estimate the stack of 100 bills to weigh 100 g. Then one bill has the volume of 0.75 cm^3 and weighs about 1 g. Now we consider the three scenarios. For each, we compute the number of bills.

a) The number of bills is given by $5 \times 10^6/100 = 5 \times 10^4$ bills. The weight would be 50,000 g or 50 kg. The volume would be about 4×10^4 cm^3. Since one cubic foot is about $30^3 \approx 27,000$ cm^3, the total volume of the bills would be about 1.5 cubic feet.

b) The number of bills increases by the factor of 5. Therefore, the weight and the volume of the cash increase by the same factor. The money would weigh about 250 kg and would need sacks with the total volume of about 7.5 cubic feet.

c) We consider pairs of bills, with each pair containing a \$100 and a \$20 bill. There will be $5 \times 10^6/120 \approx 4 \times 10^4$ such pairs. Each pair weighs 2 g and occupies 1.5 cm^3 of space. Then the total weight will be about 80 kg and the total volume will be about 2 cubic feet.

2. Problem

Mary has noticed that running on a treadmill is easier than running outside. She thought about this and decided that the reason for the difference is the air drag, which she has to overcome when running outside but not in the gym. To prepare for a race, Mary decided to set up the incline on the treadmill in such a way that her expended energy on the treadmill will be equal to that when running on a level trail outside. She read section 5.6 and followed the math there to compute the required incline. Assuming that Mary is of an average build and runs at the pace of 6 mph, what should the incline on the treadmill be?

Solution

We use formulas from section 5.6 for the power outlay to counter the air drag force

$$P_a = \frac{1}{2} C_D \rho A V^3. \tag{5.1}$$

and for the power outlay to counter the gravity on an incline

$$P_g \sim \frac{2\alpha}{\pi} mgV, \tag{5.2}$$

where notations are defined in section 5.6. From these equations we obtain a solution for the incline that is required to compensate for the lack of air drag on the treadmill:

$$\alpha \sim \frac{\pi}{4} \frac{C_D \rho A V^2}{mg} \tag{5.3}$$

We assume that Mary's weight is 60 kg, her height is 1.6 m, and the width of her body is 0.7 m, which yields Mary's cross-section area of about 1 m^2. The density of the air is $\rho = 1.2$ kg/m^3. Mary is running at 6 mph, which corresponds to about 3 m/s. Constant C_D is a dimensionless coefficient, and we use $C_D \approx 1$. Of course, $g = 9.8$ m/s^2. This yeilds $\alpha \approx 0.014$ radians, or 0.8 degrees.

3. Problem

Suppose that scientists have learned how to put a tag on individual molecules.[1] They use this technology to study the circulation of water in Earth's oceans and atmosphere. They tagged all

1. The statement of this problem is not as absurd as it may seem. As the air circulates in the atmosphere, some air parcels travel to high altitudes, where they are bombarded by cosmic rays. This turns nitrogen into carbon-14 $\left(^{14}C\right)$; further circulation brings this carbon back to Earth, where it enters the global nutrient cycle. The "tagged" ^{14}C carbon is used by scientists to perform radiocarbon dating of archaeological and biological specimens. Understanding the dynamics of global air circulation is important for accurate radiocarbon dating.

the molecules in a cup of water and then poured this cup into the Amazon River. Now scientists periodically scoop a cup of water at various places around the globe and measure the number of tagged molecules there. Provide a rough estimate for the number of molecules in a scooped cup that would indicate that the original sample has mixed approximately uniformly with the water in the world's oceans.

Solution

The number of molecules in a cup of water that is scooped from the ocean after the water has been mixed with the world ocean is estimated as NV_c/V_o, where V_c is the volume of the cup, V_o is the volume of the water in the ocean, and N is the number of molecules in the cup.

The volume of the water in the ocean is estimated as the surface area A of the ocean times the average depth D. To compute the surface area, we assume that the ocean covers about two thirds of the earth surface. Then the volume of the ocean is

$$V_o \sim \frac{2}{3} 4\pi R^2 D, \tag{5.4}$$

where R is the Earth radius. The number of molecules in the scooped cup is then estimated as

$$\begin{aligned} n &\sim \frac{V_c}{V_o} N \\ &\sim \frac{3V_c}{8\pi R^2 D} N \end{aligned} \tag{5.5}$$

We use $R = 6.4 \times 10^6$ m and $D \sim 5 \times 10^3$ m. A simple search shows that 1 mole of water has the volume of about 18 ml. Then one cup of water (0.2 l or $V_c \sim 2 \times 10^{-4}$ m^3) is about 10 moles, and must have about $N = 6 \times 10^{24}$ molecules. We substitute these values in equation (5.5) to get $n \sim 700$.

4. Problem

Find information on how many lightning storms one may observe per year in a temperate climate zone of the world, or use your own experience if you live in a temperate zone.

a) Based on this number, estimate how many lightning strikes occur per second globally.

b) Note that simply multiplying the number of strikes that one may observe by the total number of people on the planet is not a valid approach. Why?

Solution

a) We assume that the thunderstorm season lasts about 4 months or $n_d = 120$ days. A person in a temperate climate zone may observe one thunderstorm per week. During a season that would correspond to $n_t \sim n_d/7 = 17$ thunderstorms. During each thunderstorm, one may observe about $n_s \sim 20$ lightning strikes. Then the number of lightning strikes that a person may observe in a year is estimated as $n_s n_t$.

These lightning strikes are observed if they are within $r = 5$ km from the observer, that is, if they occur within a circle with the area of $a = \pi r^2$. They occur in the area that covers roughly a half of the Earth surface (we exclude high-latitude regions, deserts, and so on), or $A = 4\pi R^2/2$, where $R \sim 6.4 \times 10^6$ m is the Earth radius. This area can be partitioned into about A/a patches, with each patch having $n_s n_t$ lightning strikes annually. Then the total annual number of lightning strikes globally is estimated as

$$\begin{aligned} N &\sim \frac{A}{a} n_s n_t \\ &= \frac{2R^2 n_s n_t}{r^2} \\ &\approx 10^9. \end{aligned} \tag{5.6}$$

There is about 1 billion lightning strikes per year globally! Since there are 365 days in a year and each day has about 86 thousand seconds, this corresponds to about 30 lightning strikes per second on average. (Estimates in various sources put the average frequency of lightning at 30 to 100 strikes per second.)

b) Many lightning strikes are observed by multiple people (especially in highly populated areas). At the same time, lighting strikes in the middle of the ocean are hardly observed by anyone. Therefore, multiplying the number of strikes that one may observe by the total number of people on the planet is not a valid approach.

5. Problem

Every year, a number of people are injured by lightning strikes. Assume that most people are within their homes during a thunderstorm and that, if lightning strikes a house, its inhabitants are injured. Using the results from the previous problem, estimate the number of people who are injured by lightning annually.

Solution

During a thunderstorm most people seek shelter, and we assume that most of them are at home. If a lightning strikes a single-family house, we assume that people in that house will be injured. (Multifamily buildings often have lightning rods, which prevent injuries.) The probability that a lightning strike hits a single-family house is estimated as the ratio of the total area of all houses to the total area, where thunderstorms occur.

We assume that roughly a half of people around the world live in single-family houses, which is about 4 billion people. We assume that each household is on average 4 people, which means that there is about 1 billion single-family houses on this planet.

Some of these houses are large and some are small, but we roughly estimate the footprint of each house to be 100 m^2. This means that the total area of all single-family houses in the world is estimated at 10^{11} m^2.

The area of the world where thunderstorms occur is estimated as half of the Earth area: $A = 4\pi R^2/2 \approx 2.5 \times 10^{14}$ m^2. Therefore, for every lightning strike hitting the ground, the probability to strike a single family house is given by $p \sim 10^{11}/(2.5 \times 10^{14}) = 4 \times 10^{-4}$.

From problem 4 we know that there are about 1 billion lightning strikes per year globally. Only about a quarter of them are cloud-to-ground strikes (the rest are intra-cloud or cloud-to-cloud). Therefore, there are about 250 million cloud-to-ground strikes. Since the probability of hitting a house is $p \sim 4 \times 10^{-4}$, each year we have about 100,000 cases when lightning strikes a house.

If in each case four people are injured, there are about 400,000 such injuries per year. (The US National Institute of Health estimates that globally about 24,000 people are killed by lightning, and about 10 times more suffer injuries: https://www.ncbi.nlm.nih.gov/books/NBK441920/.)

6. Problem

How many jelly beans can fit in a 2 liter jar?

Solution

We estimate the net volume of each jelly bean at 0.5 cm^3. They cannot be packed very tightly, and each jelly bean takes takes somewhat larger volume in a jar, such as 0.7 cm^3. The volume of a 2-liter jar is 2×10^3 cm^2. Then it may hold about $2 \times 10^3/0.7 \sim 3,000$ jelly beans.

7. Problem

US counties draw their revenue primarily from real estate taxes. Select a particular county and estimate its budget. Assume that the annual tax rate is 1 percent of the cost of each house.

Solution

For a large and prosperous Fairfax County in Virginia, we estimate the total number of residents as 1 million. This corresponds to 3×10^5 households. A typical cost of a house is $500,000. If the tax rate is 1 percent, then each house generates $5,000 in taxes. All houses will therefore generate $\$1.5 \times 10^9$ per year in taxes. (For 2022, Fairfax County plans to collect about $3.2 billion dollars in real estate taxes.)

8. Problem

The Stefan–Boltzmann law predicts the power of total electromagnetic radiation (including light) emitted by a heated body from a unit area:

$$P = \sigma T^4, \tag{5.7}$$

where $\sigma \approx 5.67 \cdot 10^{-8}$ W/(m^2K^4). Here the power is measured in watts (W) and temperature T is in the Kelvin scale.

a) Use the size of the Sun and its surface temperature of 6,000 K to estimate the total energy emitted per second.

b) Use the scaling that was discussed in section 4.5 to determine the power per unit area away from the Sun at Earth's orbit.

c) Find reference information on the power that can be produced by solar panels on the ground per unit area. How does this number compare with the total power of sunlight per unit area?

Solution

a) From the Stefan–Boltzmann law we estimate the power of electromagnetic radiation emitted from the unit area of the Sun's surface to be

$$P \approx 5.67 \cdot 10^{-8} \times 6,000^4 = 7.3 \cdot 10^7 \text{ W/m}^2. \tag{5.8}$$

The radius of the Sun is $R = 7 \cdot 10^8$ m, and its area is $S = 4\pi R^2 = 6 \cdot 10^{18}$ m^2. Then the total energy emitted by the Sun every second is $P_t \sim 5 \cdot 10^{26}$ W.

b) The power per unit area scales as the inverse square of the distance. The Earth is located approximately at the distance of $D = 1.5 \cdot 10^{11}$ m from the Sun. Given the Sun's radius and the scaling, the power per unit area at the Earth can be computed as

$$P_E \approx P\left(\frac{R}{D}\right)^2 \approx 7.3 \cdot 10^7 \left(\frac{7 \cdot 10^8}{1.5 \cdot 10^{11}}\right)^2 \approx 1{,}600 \text{ W/m}^2. \tag{5.9}$$

c) Solar panels produce power on the order of 200 W/m^2. They are about 20% efficient, which means that the total energy absorbed is about 1,000 W/m^2. (Note that not all electromagnetic energy from the Sun is in the range of wavelengths that solar panels can convert into electricity, and that some of the radiation is absorbed before it reaches the solar panel.)

9. Problem

According to a popular legend, the inventor of the game of chess came to the local king and showed him the game. The king loved the game and asked the inventor to name a reward. The inventor said that he wanted one grain of rice for the first square on the board, two grains of rice for the second square, four grains for the third square, and so on. The king, apparently not being very good at math, quickly agreed.

a) Knowing that a chessboard has 64 squares, estimate the total weight of rice that was to be received by the inventor.

b) How does it compare with today's global annual rice production?

Solution

a) The dimensions of one grain of rice are approximately $3 \times 1.5 \times 1.5$ mm. This corresponds to the volume of about 7 mm^3, or $7 \cdot 10^{-9}$ m^3. We know that rice sinks in water; we may assume that the density of rice is double the density of water. Since 1 m^3 of water weighs 10^3 kg, the weight of one grain of rice is estimated as $7 \cdot 10^{-9} \times 2 \cdot 10^3 = 1.4 \cdot 10^{-5}$ kg.

The last square of the chess board would have 2^{63} grains of rice. All the squares will have almost the double of that or $\approx 2^{64}$ grains of rice. The total weight of this rice comes to $1.4 \cdot 10^{-5} \times 2^{64} \approx 2.6 \cdot 10^{14}$ kg or $2.6 \cdot 10^{11}$ tons.

b) The world rice production today is about $7 \cdot 10^8$ tons. The inventor of chess wanted to get the amount that exceeds the modern rice production by three orders of magnitude.

10. Problem

A spherical cap is the part of a sphere that lies above a plane that crosses this sphere (see section A.30 and figure 5.1).

a) Produce order of magnitude estimates for the volume and the surface area of the spherical cap as expressed through h and R. (Hint: For the surface area, approximate the shape of the cap as a disk with radius a. For the volume, approximate the shape of the cap as a cylinder with radius a and height h. Then use the Pythagorean theorem to solve for a through h and R.)

b) Try to improve your estimates by approximating the shape of the cap as a cone instead of a cylinder. The volume and the area of a cone (net of the area of the base) are given by $V_{\text{cone}} = \pi a^2 h/3; \quad S_{\text{cone}} = \pi a \sqrt{h^2 + a^2}$.

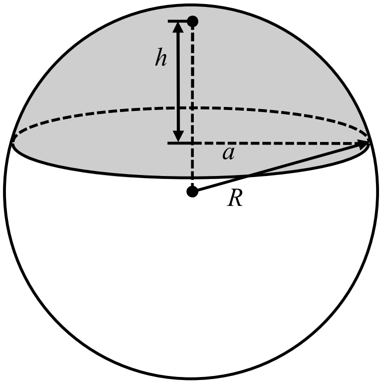

Figure 5.1
A spherical cap: order of magnitude estimates

c) Compare the two sets of estimates with the exact formulas in section A.30.

Solution

We start from considering the case $0 \leq h \leq R$. (The case of $R < h \leq 2R$ will be considered later.) We use tildes for the estimates in the cylinder model, and bars for the estimates in the cone model.

a) To express the surface area and the volume of a spherical cap through R and h, we need first to express a as a function of these two variables. From the figure we see that

$$a = \sqrt{R^2 - (R-h)^2}. \tag{5.10}$$

Then the surface area of a spherical cap as approximated by the area of a disk of radius a is given by:

$$\begin{aligned} \tilde{S}(h) &\sim \pi a^2 \\ &= \pi(R^2 - (R-h)^2) \\ &= 2\pi Rh\left(1 - \frac{h}{2R}\right). \end{aligned} \tag{5.11}$$

The volume of a cylinder is computed as the product of the area of the base and the height. We already have an estimate for the area of the base. Then an estimate for the volume of a spherical cap is

$$\tilde{V}(h) \sim 2\pi Rh^2\left(1 - \frac{h}{2R}\right). \tag{5.12}$$

b) For the surface of the cone (excluding the area of its base) we get:

$$\bar{S}(h) = \pi a \sqrt{h^2 + a^2}. \tag{5.13}$$

We use $a^2 = 2Rh - h^2$ to get

$$\bar{S}(h) \sim 2\pi Rh \sqrt{1 - \frac{h}{2R}}. \tag{5.14}$$

We get the following estimate for the volume of the cone:

$$\bar{V}(h) = \frac{\pi}{3}a^2 h = \frac{2\pi}{3}Rh^2\left(1 - \frac{h}{2R}\right). \tag{5.15}$$

Next, we proceed to the case $R < h \leq 2R$.

If $r = R$, the spherical cap occupies a half of the sphere, and if $h = 2R$, the spherical cap coincides with the entire sphere. Therefore, as h varies from R to $2R$, the surface area and the volume of the spherical cap double. This gives us a way to produce estimates for the surface area and the volume of the spherical cap for $R < h \leq 2R$ using a linear model:

$$\begin{aligned}
\tilde{S}(h) &\sim \tilde{S}(R) + (h - R)\tilde{S}(R),\\
\tilde{V}(h) &\sim \tilde{V}(R) + (h - R)\tilde{V}(R),\\
\bar{S}(h) &\sim \bar{S}(R) + (h - R)\bar{S}(R),\\
\bar{V}(h) &\sim \bar{V}(R) + (h - R)\bar{V}(R).
\end{aligned} \tag{5.16}$$

c) Performance of these rough estimates is compared to the exact formulas in figures 5.2 and 5.3. We see that both the flat disk and the cone model underestimate the surface area of the spherical cap, but the cone model is doing a better job. This is expected from the shapes of the disk and the cone as compared to the shape of the spherical cap.

The cylinder model overestimates, and the cone model underestimates the volume. Again, this performance is expected from the shapes we used for producing the estimates.

11. Problem

In the seventeenth century, Peter Minuit orchestrated the notorious purchase of Manhattan from a native American tribe.

a) Estimate the total value of real estate in Manhattan in today's prices.

b) Using this value, estimate the total current cost of land on the island.

c) Assume that the purchase occurred in 1626 and the land was purchased for the equivalent of \$1,000 in today's money. Estimate the rate of return for the original purchase price to grow to the value of the Manhattan land today. This assumes that a value grows exponentially in time as $I = I_0 e^{rt}$, where I_0 is the initial amount and r is the rate of return.

d) How would the estimate for the rate of return change if we assume that Peter Minuit paid \$24 for the purchase?

e) How does it compare with the 7–8 percent growth that many people expect to see in their retirement accounts?

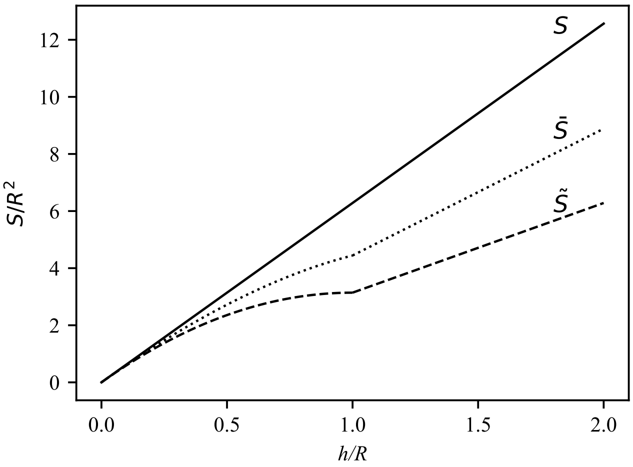

Figure 5.2
Estimates for the surface area of a spherical cap

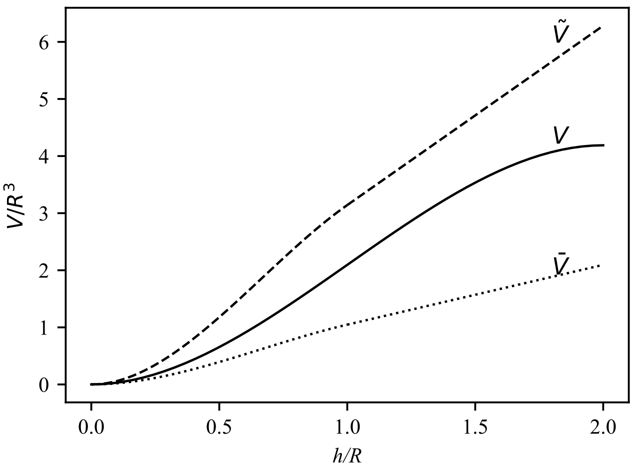

Figure 5.3
Estimates for the volume of a spherical cap

Solution

a) Let's assume that the price of residential real estate in Manhattan is about \$1,000 per square foot (this corresponds to a \$1 million price tag for a 1,000 square foot apartment). There are about 1.6 million people living in Manhattan, which is about half a million households. If each household lives in a 1,000 square foot apartment, the total price of residential real estate is about $\$5 \cdot 10^{11}$ or about half a trillion dollars. To this value we should add the price of commercial real estate. The total value is estimated at $\$10^{12}$.

b) Let's assume that the land has the value that is a half of the total value of real estate, or $\$5 \cdot 10^{11}$.

c) For the annual rate of return r from 1626 to 2022 we have:

$$10^3 \cdot e^{r(2022-1626)} = 5 \cdot 10^{11}. \tag{5.17}$$

This yields

$$r = \frac{\ln(5 \cdot 10^8)}{396} \approx 0.05. \tag{5.18}$$

The estimated rate of return is 5 percent.

d) Using the same model, we get $r \approx 6$ percent for the purchase price of \$24.

e) Many people expect to beat the 5-6 percent rates of return in their investment accounts. Note that if one could sustain the 7-8 percent growth in an investment account for hundreds of years, that could produce a gargantuan value! H. G. Wells based on this premise the plot of his science fiction novel "The Sleeper Awakes".

12. Problem

A torus is a donut-shaped body (see figure 5.4). Use an order of magnitude estimate to flag the incorrect formulas for the volume of a torus below. (Hint: Estimate the volume of a torus as the volume of a cylinder you would get if you cut the torus at one place and unrolled it to a cylinder.)

a) $V = 2Rr^2$

b) $V = 2\pi^2 Rr^2$

c) $V = 12\pi^2 Rr^2$

Solution

The volume of a cylinder with the radius of the base r and the height h is given by $\pi r^2 h$. If we cut and unroll a torus we would get a cylinder with the height of $2\pi R$. Therefore, the estimate for the volume of the torus is $V = 2\pi^2 Rr^2$. This value is given by option b in the condition of the problem. Formulas a and c differ from this value by an order of magnitude each and therefore are likely to be wrong. (In fact, they are.)

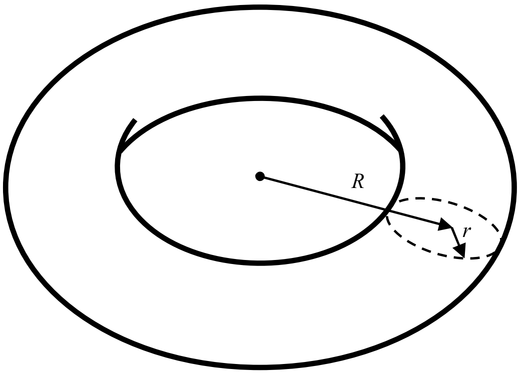

Figure 5.4
A torus: order of magnitude estimates

13. Problem

A powerful explosion creates a supersonic[2] spherical shock wave. As the wave propagates outward, it slows down, and at some point its speed falls below the speed of sound. Use equation (4.38*) in section 4.8 to estimate the maximum radius of the supersonic shock wave from an explosion caused by 10 kg of TNT that releases the energy $E \approx 4 \cdot 10^7$ kg · m^2 · s^{-2}.

Solution

We use the following equation for the speed of the shock wave:

$$V = C\sqrt{\frac{E}{\rho R^3}}, \tag{5.19}$$

where E is the energy released by the explosion, ρ is the density of the air, R is the radius of the shock wave, and C is a dimensionless constant. We solve this equation for R to get

$$R = \sqrt[3]{\frac{C^2 E}{\rho V^2}}. \tag{5.20}$$

We use $\rho = 1.2$ kg/m^3, $E \approx 4 \cdot 10^7$ kg · m^2 · s^{-2}, and $C \sim 1$. As the radius of the shock wave increases, the wave slows down. The maximum radius of the supersonic shock wave is achieved

2. A shock wave is supersonic if it travels at a speed that exceeds the speed of sound.

when the speed of the shock wave is equal to the speed of sound in the air, $V \approx 340$ m/s. This yields $R \sim 6.6$ m.

14. Problem

The velocity of gravity waves in deep water (in the case of $\lambda \ll h$) and in shallow water (in the case of $\lambda \gg h$) is given by (see problem 15 in chapter 4):

$$\begin{aligned} V_{\text{deep}} &= \sqrt{\frac{g\lambda}{2\pi}}, \\ V_{\text{shallow}} &= \sqrt{gh}, \end{aligned} \tag{5.21}$$

where λ is the wavelength (the distance between two consecutive wave crests), h is the depth of the water basin, and $g \approx 9.8$ m/s^2 is the gravity acceleration. Tsunamis are large gravity waves in the ocean that are often caused by underwater earthquakes. A tsunami wavelength can be on the order of the size of the region on the ocean floor affected by the earthquake, or about 10^2 km.

An engineer is tasked to come up with a preliminary design of a tsunami warning system. The key factor in the design is the travel time of tsunamis across the ocean.

a) Which of the equations in system (5.21) should be used to estimate the speed of a tsunami?

b) How much in advance can a satellite warning system notify the people who live on the Pacific Rim for a tsunami that originates in the middle of the ocean?

Solution

a) For this problem we have $\lambda \gg h$, which means that tsunami is a shallow-water wave (even though it may originate in a deep part of the ocean).

b) For $g = 9.8$ m/s^2 and $h = 5 \cdot 10^3$ m we get $V \sim 200$ m/s, which is on the order of the speed of a jet plane!

For the propagation distance of 5,000 km (that is, $5 \cdot 10^6$ m) the travel time is $\sim 5 \cdot 10^6/200 = 2.5 \cdot 10^4$ seconds or about 7 hours.

15. Problem

Zoo employees Jim and Jack are preparing for the arrival of their first adult African bush elephant to the zoo. They have no experience caring for elephants and are frantically trying to figure out what and how they should feed him. Jim has a lot of experience in caring for rabbits and knows an excellent supplier of rabbit food. After some brainstorming, Jim and Jack decide that the elephant might enjoy rabbit food but would obviously need a lot of it. They apply Klieber's law (see equation (4.2*) in section 4.1) to estimate how much food an elephant needs. What estimate may they come up with?

Solution

The weight f of the food is directly proportional to the energy yield from that food. To sustain an animal, the daily amount of food must be proportional to the metabolic rate B of that animal

$$B = cf, \tag{5.22}$$

where c is a coefficient. Generally, values of c would be different for different types of food. However, if the elephant is kept on the rabbit's diet, the proportionality coefficients in equation (5.22) for the elephant and the rabbit have the same value. Therefore, we can apply Klieber's law to the weight of the food rather than to the animal's metabolic rate. That way, we get an estimate for the amount of food that is needed to feed an elephant:

$$f_e = f_r \left(\frac{M_e}{M_r} \right)^{\frac{3}{4}}, \tag{5.23}$$

where M is the weight of an animal, f is the weight of the food for that animal, and subscripts e, r refer to the elephant and rabbit.

A 3 kg rabbit may eat 50 g of pellets and 3 cubic liters of hay per day. The density of hay is about 10 pounds per cubic foot, which corresponds to about 0.17 kg per cubic liter. Three cubic liters of hay would weigh about 0.5 kg. Along with the pellets, the total weight of a daily ration for a rabbit is about 0.55 kg. An elephant weighs about 5 tons. We substitute these values to get $f_e \sim 140$ kg. (According to National Geographic, an adult elephant may consume up to 300 pounds of food in a day, which corresponds to about 135 kg.)

16. Problem

The radius r of Archimedes's spiral increases linearly with the angle θ of the turn (figure 5.5). Mathematically, this is expressed in polar coordinates as

$$r = a\theta. \tag{5.24}$$

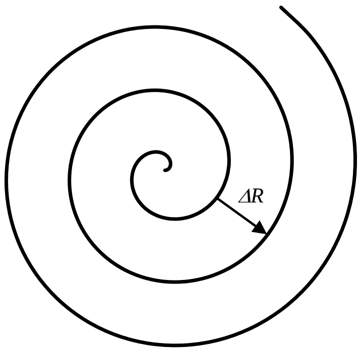

Figure 5.5
Archimedes's spiral: order of magnitude estimates

Emma found a formula on the web for the arc length of the spiral:

$$L = \frac{a}{2}\left(\theta\sqrt{1+\theta^2} + \ln\left(\theta + \sqrt{1+\theta^2}\right)\right). \tag{5.25}$$

She knows that the spiral is fully defined by two parameters: the total turn angle (dimensionless) and the distance between the adjacent loops (measured in meters). By applying dimensional analysis to equation (5.25), Emma concluded that a may be denoting the distance between the adjacent loops. To check this conjecture, she decided to compare equation (5.25) to a rough estimate for the arc length. Reproduce Emma's estimate for the arc length of the Archimedes's spiral and analyze the result:

a) Derive a rough estimate for the arc length of the spiral as expressed through the total turn angle θ and the distance between adjacent loops ΔR. (Hint: First estimate the number of loops, and then use rule 4 in section 5.2 to estimate the length of each loop.)

b) Compare this estimate with equation (5.25), assuming that parameter a in that equation denotes the distance between adjacent loops: $a = \Delta R$. Is your rough estimate consistent with that formula?

c) Does your rough estimate include a logarithmic term that is a part of equation (5.25)? If not, is it important?

d) Refer to a more complete formulation in section 6.6. What is the correct relationship between a and ΔR?

Solution

a) If the total turn angle is θ, then the spiral will have $N \sim \theta/(2\pi)$ loops. The average radius of a loop will roughly be determined by the loop in the middle of the count. If the separation between the loops is ΔR, then loop number $N/2$ will have the radius of

$$\begin{aligned}\bar{r} &\sim \Delta R \frac{N}{2} \\ &\sim \frac{\Delta R \theta}{4\pi}.\end{aligned} \tag{5.26}$$

The total length of that loop is given by $2\pi\bar{R}$. The total length of N loops is then estimated as

$$\begin{aligned}L &\sim 2\pi\bar{r}N \\ &\sim 2\pi\frac{\Delta R\theta}{4\pi}\frac{\theta}{2\pi} \\ &= \frac{\Delta R\theta^2}{4\pi}.\end{aligned} \tag{5.27}$$

b) If we assume that $a = \Delta R$, estimate (5.27) differs from formula (5.25) in several ways: it lacks a logarithmic term, the coefficient is $1/4\pi$ instead of $1/2$, and the estimate contains θ^2 instead of $\theta\sqrt{1+\theta^2}$.

c) The rough estimate should have a better accuracy for a large number of turns N, when the effect of individual turns on the total is relatively smaller. Large N corresponds to a large θ, when the power law term in equation (5.25) dominates the logarithmic term.

d) We already determined that the logarithmic term is small for large θ. In addition, we have $\theta^2 \approx \theta\sqrt{1+\theta^2}$. Then the estimate (5.27) substantially differs from formula (5.25) in one way only: the coefficient is off by the factor 2π (that is, assuming $a = \Delta R$). This means that the estimate and formula (5.25) have different orders of magnitude, which makes either formula (5.25) or the assumption $a = \Delta R$ suspect. Indeed, consider the increase in r as θ is incremented by one complete revolution. If θ is increased by 2π, we see from equation (5.24) that r increased by $2\pi a$. Therefore, $\Delta R = 2\pi a$ and the assumption $r = a$ is incorrect. This argument reconciles the rough estimate and the exact formula.

17. Problem

Estimate the length of the curve defined by $y = \sin x$ for $0 \leq x \leq 2\pi$. Compare your result with $L \approx 7.6404$.

Solution

We split the curve $y = \sin x$ for $0 \leq x \leq 2\pi$ into four equal segments, as shown in figure 5.6. In each segment, x varies by $\pi/2$ and y varies by 1 (either going from 0 to ± 1 or from ± 1 to 0). We approximate the curve in each segment as a straight line connecting the start and the end points. The length of such a line is computed as the length of the hypotenuse of a right triangle that has legs $\pi/2$ (for x) and 1 (for y). Then the total length of all four straight line segments is given by

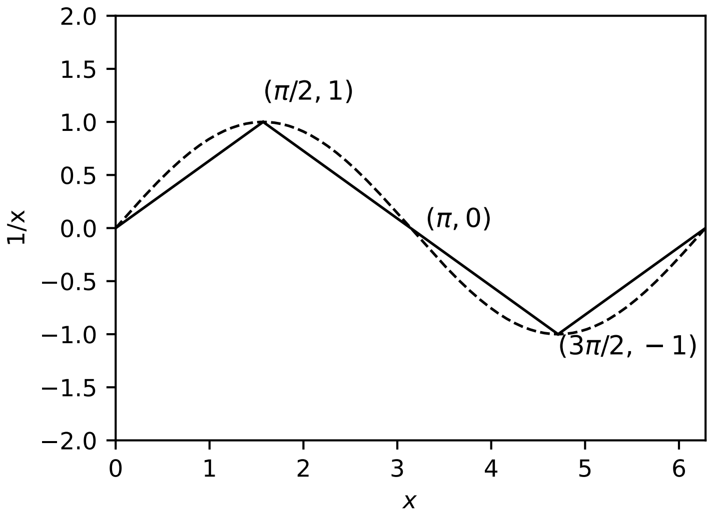

Figure 5.6
The length of the sine curve

$$\tilde{L} \sim 4\sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \approx 7.4. \quad (5.28)$$

The agreement with $L \approx 7.6404$ is excellent.

18*. Problem

Consider the curve that is a plot of function $y = bx^2$.

a) Estimate the length of this curve for $0 \leq x \leq c$.

b) Compare your estimate with the exact result:

$$L = \frac{c}{2}\sqrt{4b^2c^2 + 1} + \frac{1}{4b}\ln\left(2bc + \sqrt{4b^2c^2 + 1}\right). \quad (5.29)$$

c) Does your rough estimate include a logarithmic term? If not, is it important?

d) Plot your estimate and the exact formula for $b = 2$ as a function of c.

Solution

a) The curve connects points $x = 0, y = 0$ and $x = c, y = bc^2$. For a rough estimate, we compute the length of a straight line that connects these two points:

$$\begin{aligned}\bar{L} &= \sqrt{c^2 + b^2c^4} \\ &= c\sqrt{1 + b^2c^2}.\end{aligned} \quad (5.30)$$

For comparison with the exact formula we rewrite this equation as

$$L = \frac{c}{2}\sqrt{4b^2c^2 + 4}. \quad (5.31)$$

b) The rough estimate differs from the exact formula in two ways: the exact formula has a logarithmic term and contains a 1 instead of a 4 under the square root.

c) A parabola curves up more for small x, and for larger x the curvature is smaller. We might expect our approximation of the parabola as a straight line to be more accurate for large c, when the contribution from large x dominates the result. This would be consistent with a power law dominating a logarithmic function for large values of the argument.

d) Figure 5.7 shows plots of L (dashed line) and $\bar{L}$ (solid line). Since we approximated a curve by a straight line, $\bar{L}$ should underestimate the exact value of the length of the curve. We do see that $\bar{L} < L$, but only by a very small amount.

Even though we argued that the rough estimate should be more accurate for large values of c, the plots show a good agreement for all values of c. There are two sources of error in our estimate: the value given by (5.31) overestimates the fist term in the exact formula (5.29) but does not account for the logarithmic term there. Amazingly, for small c these two sources of error almost exactly compensate each other, producing a very accurate result. This makes our estimate quite accurate for all values of c.

19. Problem

Equation (2.94*) in section 2.12 estimates the maximum possible radius of coverage by a low-orbit satellite that transmits a signal to customers on Earth's surface. Assume you are planning a satellite constellation that must cover the entire globe and that these satellites will fly at an altitude of 400 km. How many satellites will you need?

Solution

Equation (2.94*) in section 2.12 computes the length of the arc from the center of the satellite beam to its edge on the Earth surface:

$$L = \sqrt{2RH}, \tag{5.32}$$

where R is the Earth radius and H is the orbit altitude. The beam spot forms a spherical cap, for whose area we have an exact formula in section A.30. However, for low orbit satellites the beam spot is small compared to the Earth radius and can be approximated as flat. Then the area of one spot beam is $A \approx \pi L^2 = 2\pi RH$. The total number of beams to cover the Earth surface is

$$\begin{aligned} N &\sim \frac{4\pi R^2}{2\pi RH} \\ &= \frac{2R}{H}. \end{aligned} \tag{5.33}$$

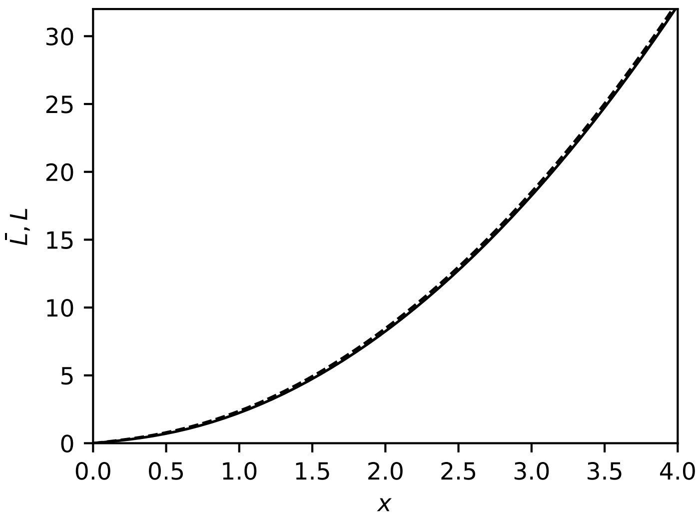

Figure 5.7
Length of the parabola: a rough estimate and the exact value

For R =6,400 km and H = 400 km we get N = 32. (The geometry of satellite orbits makes it difficult for a constellation to have the same coverage at all latitudes. Some beam spots will overlap, while other may have gaps. In addition, round beam spots are not well suited for tiling the Earth surface tightly. To work around these constraints, one may have to increase the required number of satellites.)

20. Problem

The Greenland ice sheet covers about $1.7 \cdot 10^6$ km^2 and is 3 km thick at the thickest point. Give a rough estimate for the rise of the sea level if this ice sheet completely melts as a result of climate change.

Solution

The total volume of ice in the ice sheet is estimated as its area times the average thickness. We estimate the average thickness to be a half of the maximum thickness value. Then the total volume of ice is

$$V = \frac{1}{2} A_G H_G, \tag{5.34}$$

where $A_G = 1.7 \cdot 10^6$ km^2 is the area of Greenland and $H_G = 3$ km is the largest thickness of ice there. If the ice melts, the volume of the oceans will be increased by approximately the same amount. The increase in the ocean level H_o is linked with the area of the ocean A_o and the volume of water added V as:

$$V = A_o H_o. \tag{5.35}$$

Then

$$H_o \sim \frac{A_G}{2A_o} H_G. \tag{5.36}$$

For the area of the ocean, we use two thirds of the area of the Earth, that is,

$$A_o = \frac{2}{3} 4\pi R^2, \tag{5.37}$$

where R is the Earth radius. We get $A_o \sim 3.4 \cdot 10^8$ km^2. Then

$$H_o \sim \frac{1.7 \cdot 10^6}{2 \cdot 3.4 \cdot 10^8} 3 \text{ km} = 7.5 \text{ m}. \tag{5.38}$$

21. Problem

Estimate $n! = 1 \times 2 \times \cdots \times n$ as a function of n. Check the quality of this estimate for $n = 5$ and $n = 10$. (Hint: Since $n!$ is a product of n factors, use the nth power of the average value for these factors.) Compare the functional form of your estimate to the Stirling formula, which approximates the factorial as

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \tag{5.39}$$

Note that a rough estimate may miss the Stirling formula value or the exact value by more than one order of magnitude for a large n but still captures the rough behavior of $n!$. The reason for the crudeness of this estimate is the extremely steep dependence of $n!$ on n.

Solution

For a rough estimate, we use the product of n factors, where each factor is the average of the values $1, 2, \ldots, n$. Then

$$\tilde{n}! \sim \left(\frac{n}{2}\right)^n. \tag{5.40}$$

This is different from the Stirling formula in two ways: the estimate lacks a multiplier $\sqrt{2\pi n}$ and has 2 in place of e in the denominator. For small n the effect of these two differences partially compensates each other, but for large n the wrong denominator produces a large difference between $n!$ and $\tilde{n}!$. Table 5.1 lists values for $n = 5$ and $n = 10$. The rough estimate yields a good approximation for $n = 5$, but is off by a factor of ~ 3 for $n = 10$. The Stirling formula provides a much better approximation. In fact, its relative accuracy increases with n.

22. Problem

To put a satellite into orbit, we need to design a booster. The Tsiolkovsky rocket equation (see section 4.2) links the velocity of the exhaust V_e, the total velocity gained by the rocket V, the total starting mass of the rocket including propellant m_0, and the final mass of the rocket when the propellant has been used m_f:

$$V = V_e \ln \frac{m_0}{m_f}. \tag{5.41}$$

The velocity of the exhaust leaving the rocket depends on the type of fuel used and on engineering constraints; for this problem, use $V_e = 2{,}000$ m/s. With these data in hand, start planning a satellite mission:

a) The radius of a low-Earth orbit is only slightly larger than Earth's radius. Use the results from problem 4 in chapter 1 to estimate the satellite velocity on the orbit.

b) Use the Tsiolkovsky equation (5.41) to estimate the total mass of a rocket that is needed to put a 10-ton satellite on a low-Earth orbit.

c) Compare your estimates with masses of real launch vehicles that have been used for launching satellites on low-Earth orbits, such as *Zenit-2* or *Falcon 9 v1.0*.

Solution

Table 5.1
A rough estimate, the Stirling formula and the exact value of a factorial

n	$n!$	Estimate	Stirling formula value
5	120	97.7	117.95
10	3,628,800	9,765,625	$3.599 \cdot 10^6$

a) The satellite velocity on a circular orbit is given by

$$V = \sqrt{Rg}, \tag{5.42}$$

where R is the orbit radius and g is the gravity acceleration at the orbit altitude. For a low-Earth orbit we can use the radius of the Earth for R and the gravity acceleration at the sea level for g. Then $V \sim 8 \cdot 10^3$ m/s.

b) We solve equation (5.41) for m_0 and substitute there V from equation (5.42):

$$m_0 \sim m_f e^{\sqrt{Rg}/V_e}. \tag{5.43}$$

For a satellite with a mass of $m_f = 10^4$ kg, $R = 6.4 \cdot 10^6$ m, $g = 9.8$ m/s^2, and $V_e = 2 \cdot 10^3$ m/s we obtain $m_0 \sim 5.5 \cdot 10^5$ kg, or 550 tons.

c) This has the same order of magnitude as launch vehicles *Zenit-2* (vehicle weight 445 tons, maximum payload weight 14 tons) or *Falcon 9 v1.0* (vehicle weight 333 tons, maximum payload weight 9 tons).

23. Problem

In statistics and probability theory, the so-called logistic distribution is given by

$$\tilde{P}(x) = \frac{c}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}, \tag{5.44}$$

where constant c is chosen so that the total area under the curve defined by equation (5.44) is equal to 1. Give an estimate for c.

Solution

Function $\tilde{P}(x)$ has a maximum at $x = 0$ and approaches zero for $x \to \pm\infty$. The area under the curve can be approximated as the product of the width and height of the main hump.

At the maximum, $\tilde{P}(0) = c/4$. To compute the width of the hump, we consider the value of y for which $\tilde{P}(y)$ is at the half of the maximum height:

$$\frac{c}{\left(e^{\frac{y}{2}} + e^{-\frac{y}{2}}\right)^2} = \frac{c}{8}. \tag{5.45}$$

Then

$$e^{\frac{y}{2}} + e^{-\frac{y}{2}} = \sqrt{8}. \tag{5.46}$$

We multiply this equation by $e^{y/2}$ and denote $z = e^{y/2}$ to get

$$z^2 - \sqrt{8}z + 1 = 0. \tag{5.47}$$

This is a quadratic equation for z. We look for a positive solution for it, which is $z = \sqrt{2} + 1$. Then

$$e^{y/2} = \sqrt{2} + 1, \tag{5.48}$$

which yields

$$y = 2\ln(\sqrt{2} + 1). \tag{5.49}$$

The main hump decreases to a half of its height at $x = \pm y$. Therefore, the total width of the main hump is estimated as $2y = 4\ln(\sqrt{2} + 1)$. Then the requirement for the area under the curve to be equal to 1 produces the following equation for c:

$$\frac{c}{4} 4\ln(\sqrt{2} + 1) = 1, \tag{5.50}$$

which yields

$$c = \frac{1}{\ln(\sqrt{2} + 1)} \approx 1.13. \tag{5.51}$$

(For reference, the exact value is $c = 1$.)

24. Problem

If an object is freely falling in air, its velocity increases until the force of gravity becomes balanced by the air drag force. This steady fall velocity is called the terminal velocity (see also problem 25 in chapter 4). Depending on the values of the parameters, the terminal velocity for a sphere may be given by one of the following equations:

$$\begin{aligned} mg &= \frac{1}{2} C_D \rho A V_t^2, \\ mg &= 6\pi R \eta V_t, \end{aligned} \tag{5.52}$$

where g is the gravity acceleration, m is the mass of the object, V_t is the terminal velocity with respect to the air, R is the radius, A is the cross-section area, C_D is the drag coefficient ($C_D \approx 0.47$ for a sphere), and η is called the dynamic viscosity, a property of the medium. For air, assume $\eta = 1.8 \cdot 10^{-5}\text{kg/(m} \cdot \text{s)}$. Consider a hailstone that grows slowly in a cloud. At any time, the velocity of its fall through the air is approximately equal to the terminal velocity. As the size and the speed of the hailstone increase, its fall transitions between the two regimes for the air drag. Estimate the size of the hailstone at which this transition occurs.

Solution

The transition between the two regimes occurs when both equations in (5.52) produce the same value for the air drag:

$$\frac{1}{2} C_D \rho A V_t^2 = 6\pi R \eta V_t. \tag{5.53}$$

We substitute the cross-section area of the hailstone $A = \pi R^2$ and simplify the result to get

$$\frac{C_D}{2} \rho R V_t = 6\eta. \tag{5.54}$$

This equation has two unknowns, R and V_t. Luckily, we can formulate a second equation for these unknowns, because at any moment the air drag force is approximately equal to the weight of the hailstone. For that second equation we can use either of equations (5.52), where we note that the mass of the hailstone is a function of its volume and the ice density; in turn, the volume is a function of the radius:

$$m = \frac{4}{3}\pi R^3 \rho_i, \tag{5.55}$$

where ρ_i is the density of ice. Then from equation (5.55) and the second equation in (5.52) we get:

$$\frac{4}{3}\pi R^3 \rho_i g = 6\pi R \eta V_t. \tag{5.56}$$

From this we get

$$V_t = \frac{2\rho_i g}{9\eta} R^2. \tag{5.57}$$

We substitute this into equation (5.54) and solve for R to get

$$R = 3\sqrt[3]{\frac{2\eta^2}{C_D \rho \rho_i g}}. \tag{5.58}$$

We use $\rho = 1.2$ kg/m^3, $\rho_i = 9 \cdot 10^2$ kg/m^3, and $g = 9.8$ m/s^2. Values of C_D and η are given in the formulation of the problem. This yields $R \sim 1.5 \cdot 10^{-4}$ m $= 0.15$ mm.

25*. Problem

Linear regression is an algorithm that is commonly used to interpret numerical data (see section A.33). Assume that we have N pairs of measurements x_i and y_i for variables x and y. The underlying linear relationship between these two variables is corrupted by measurement noise R:

$$y = ax + b + R. \tag{5.59}$$

The linear regression algorithm estimates the best fit for model parameters a and b from available data:

$$\begin{aligned} a &= \frac{N\sum_{i=1}^N x_i y_i - \sum_{i=1}^N x_i \cdot \sum_{i=1}^N y_i}{N\sum_{i=1}^N x_i^2 - \left(\sum_{i=1}^N x_i\right)^2}, \\ b &= \frac{\sum_{i=1}^N y_i \cdot \sum_{i=1}^N x_i^2 - \sum_{i=1}^N x_i \cdot \sum_{i=1}^N x_i y_i}{N\sum_{i=1}^N x_i^2 - \left(\sum_{i=1}^N x_i\right)^2}, \end{aligned} \tag{5.60}$$

where x_i, y_i are the available data for variables x and y.

For this problem, assume that the data points form two approximately equal clusters. The first cluster is centered at $(x_1 = 1, y_1 = 2)$, and the second cluster is centered at $(x_2 = 6, y_2 = 4)$.

In each cluster, points are located within a distance less than 2 units from the cluster center. Compute an estimate for parameters a and b of a linear fit to this data set. (Hint: Use rule 7 in section 5.2 to estimate various sums in equations (5.60).)

Solution

A simple way to obtain an estimate for a and b is just to assume that there is a big fat data point at the location of each cluster. Then linear regression should produce a line that goes through these two points. The equation for a line going through two points is as follows (we write it in two equivalent ways for the reasons that will be apparent below):

$$\begin{aligned} y &= \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) + y_1 \\ &= \frac{y_1 - y_2}{x_1 - x_2}(x - x_2) + y_2. \end{aligned} \tag{5.61}$$

This should be the same as $y = ax + b$. Therefore, we should have

$$\begin{aligned} a &\sim \frac{y_1 - y_2}{x_1 - x_2}, \\ b &\sim -\frac{x_1(y_1 - y_2)}{x_1 - x_2} + y_1 \\ &= -\frac{x_2(y_1 - y_2)}{x_1 - x_2} + y_2. \end{aligned} \tag{5.62}$$

Using the locations of clusters given in the problem, we get

$$\begin{aligned} a &\sim 0.4, \\ b &\sim 1.6. \end{aligned} \tag{5.63}$$

In addition to this simple argument, we can get the same result directly from equations (5.60). These equations contain multiple sums, which we consider separately. To estimate a sum in equations (5.60), we split it into two separate sums, one dealing with the points from the first cluster, and the other with the points from the second cluster. Then for all points in a cluster we replace all coordinates x_i and y_i with the coordinates of the cluster center. This will produce the following estimates:

$$\begin{aligned}\sum_{i=1}^{N} x_i y_i &\sim \frac{N}{2} x_1 y_1 + \frac{N}{2} x_2 y_2,\\ \sum_{i=1}^{N} x_i &\sim \frac{N}{2} x_1 + \frac{N}{2} x_2,\\ \sum_{i=1}^{N} y_i &\sim \frac{N}{2} y_1 + \frac{N}{2} y_2,\\ \sum_{i=1}^{N} x_i^2 &\sim \frac{N}{2} x_1^2 + \frac{N}{2} x_2^2.\end{aligned} \tag{5.64}$$

We substitute these estimates into equations (5.60). To make equations less cumbersome, we will do it separately for the numerators and denominators in equations (5.60). We start from the denominator, which is common to these equations:

$$\begin{aligned}N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2 &\sim N\frac{N}{2}(x_1^2 + x_2^2) - \frac{N^2}{4}(x_1 + x_2)^2\\ &= \frac{N^2}{4}(2x_1^2 + 2x_2^2 - x_1^2 - x_2^2 - 2x_1x_2)\\ &= \frac{N^2}{4}(x_1 - x_2)^2.\end{aligned} \tag{5.65}$$

The numerator of the expression for a is computed as follows:

$$\begin{aligned}N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i &= \frac{N^2}{2}(x_1y_1 + x_2y_2) - \frac{N^2}{4}(x_1 + x_2)(y_1 + y_2)\\ &= \frac{N^2}{4}(2x_1y_1 + 2x_2y_2 - x_1y_1 - x_1y_2 - x_2y_1 - x_2y_2)\\ &= \frac{N^2}{4}(x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1)\\ &= \frac{N^2}{4}(x_1 - x_2)(y_1 - y_2).\end{aligned} \tag{5.66}$$

The value of a is computed as a ratio of the right-hand sides of equations (5.66) and (5.65):

$$\begin{aligned}a &= \frac{\frac{N^2}{4}(x_1 - x_2)(y_1 - y_2)}{\frac{N^2}{4}(x_1 - x_2)^2}\\ &= \frac{y_1 - y_2}{x_1 - x_2},\end{aligned} \tag{5.67}$$

which matches our estimate for a in equations (5.62).

The numerator of the expression for b is computed as follows:

$$\begin{aligned}\sum_{i=1}^{N} y_i \cdot \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i &= \frac{N^2}{4}((y_1 + y_2)(x_1^2 + x_2^2) - (x_1 + x_2)(x_1 y_1 + x_2 y_2)) \\ &= \frac{N^2}{4}(y_1 x_1^2 + y_1 x_2^2 + y_2 x_1^2 + y_2 x_2^2 - y_1 x_1^2 - y_2 x_1 x_2 - y_1 x_1 x_2 - y_2 x_2^2) \\ &= \frac{N^2}{4}(x_1 - x_2)(y_2 x_1 - y_1 x_2).\end{aligned} \tag{5.68}$$

Then the value of b is computed as a ratio of the right-hand sides of equations (5.68) and (5.65):

$$\begin{aligned} b &= \frac{\frac{N^2}{4}(x_1 - x_2)(y_2 x_1 - y_1 x_2)}{\frac{N^2}{4}(x_1 - x_2)^2} \\ &= \frac{y_2 x_1 - y_1 x_2}{x_1 - x_2}.\end{aligned} \tag{5.69}$$

Let's see if this value matches the expressions given by equations (5.62). Note that equation (5.69) is symmetric with respect to swapping subscripts 1 and 2, but neither of the two equivalent expressions for b in equations (5.62) has this symmetry. To compare equation (5.69) with the values given by equations (5.62), we should first impose symmetry on the latter. We compute the mean of the two expressions for b in equations (5.62) to get

$$\begin{aligned} b &= -\frac{1}{2}\frac{x_1(y_1 - y_2)}{x_1 - x_2} + \frac{1}{2}y_1 - \frac{1}{2}\frac{x_2(y_1 - y_2)}{x_1 - x_2} + \frac{1}{2}y_2 \\ &= \frac{1}{2} \cdot \frac{-x_1(y_1 - y_2) + y_1(x_1 - x_2) - x_2(y_1 - y_2) + y_2(x_1 - x_2)}{x_1 - x_2} \\ &= \frac{1}{2} \cdot \frac{2x_1 y_2 - 2x_2 y_1}{x_1 - x_2} \\ &= \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}.\end{aligned} \tag{5.70}$$

This result does match expression (5.69).

6 Successive Approximations

1. Problem

Section A.31 of this book presents a formula for mortgage payments. Given the initial loan amount D, the interest rate r, and the duration of the loan T, the annual payment rate on the mortgage is

$$p = \frac{rDe^{rT}}{e^{rT} - 1}. \tag{6.1}$$

Suppose that a customer has selected a house to buy and that she has a limited and known budget for paying the mortgage. What is the maximum interest rate that will keep the payments under budget?

The mortgage payment formula does not yield a closed-form solution for the interest rate r. Solve it for r using the MSA. Rewrite equation (6.1) as follows:

$$r = \frac{p}{D}(1 - e^{-rT}). \tag{6.2}$$

Assume that e^{-rT} serves as a small parameter here: any inaccuracy in the interest rate would be dampened by the exponent, yielding a more accurate value for r in the next approximation. Use the initial approximation $r_0 = 0.04$/year, the term of the mortgage $T = 30$ years, mortgage amount $D = \$230{,}000$, and a mortgage budget of \$14,000/year to estimate the maximum acceptable interest rate.

Solution

We successively substitute approximations for r in the right-hand side of equation (6.2). Results are presented in table 6.1. The maximum mortgage rate that would keep the payments under budget is $r \approx 0.04514$.

2. Problem

For each of the following equations, substitute numerical results for the zero-order approximation and for the highest computed order from the corresponding table of results. Draw conclusions on the accuracy of the MSA in each case.

Table 6.1
Approximations for the mortgage rate

Approximation order	r
0	0.04
1	0.04253600449230074
2	0.04387908559615115
3	0.04455006489136136
4	0.04487528210471318
5	0.04503057181508544
6	0.04510418918258567
7	0.04513896892699059

a) For equation (6.30*), substitute values from table 6.3*.

b) For equation (6.52*), substitute values from table 6.4*.

c) For equation (6.58*), substitute values from table 6.5*.

d) For equation (6.67*), substitute values from table 6.6*.

Solution

a) Equation (6.30*) computes the total turn angle of a spiral. The difference between the right and the left-hand sides of equation (6.30*) for the zero order approximation is ≈ -0.247. For the seventh-order approximation it is $\approx -10^{-15}$.

b) Equation (6.52*) models the satellite coverage for oblate Earth. The difference between the right and the left-hand sides of equation (6.52*) for the zero order approximation is ≈ -0.00234. For the third-order approximation it is $-4.3 \cdot 10^{-7}$.

c) Equation (6.58*) solves for the intersections between a circle and a parabola for small values of the linear term. We substitute x_+ values from the table. The difference between the right and the left-hand sides of equation (6.58*) for the zero order approximation is ≈ 0.032. For the fourth-order approximation it is $1.5 \cdot 10^{-10}$.

d) Equation (6.67*) solves for the intersections between a circle and a parabola for small values of the quadratic term. We substitute x_+ values from the table. The difference between the right and the left-hand sides of equation (6.67*) for the zero order approximation is ≈ 0.025. For the fourth-order approximation it is $9.17 \cdot 10^{-10}$.

In all cases the accuracy is excellent.

3. Problem

Consider the following cubic equation:

$$(x-a)(x-b)(x-c) = d. \tag{6.3}$$

a) What are the criteria for a, b, c, d for the method to be valid?

b) For $a = 1, b = 2, c = 3$, and $d = 0.01$, estimate all three roots using the MSA. Produce both analytical formulas and numerical values for the first-order approximation.

c) Check the numerical results by substituting them in the original equation (6.3).

Solution

a) We rewrite equation (6.3) as follows:

$$x_a = \frac{d}{(x-b)(x-c)} + a. \tag{6.4}$$

This produces MSA approximations for one of the roots. For the two other roots the MSA scheme is analogous. In all cases, we want the ratio in the right-hand side to be small. This requires values of parameters a, b, and c to be sufficiently different.

b) Analytical formulas for the roots are as follows:

$$\begin{aligned} x_a &= \frac{d}{(a-b)(a-c)} + a, \\ x_b &= \frac{d}{(b-a)(b-c)} + b, \\ x_c &= \frac{d}{(c-a)(c-b)} + c. \end{aligned} \tag{6.5}$$

For the first-order approximation we get $x_1 = 1.005, x_2 = 1.99, x_3 = 3.005$.

c) Substitution of these values back into equation (6.3) produces the following differences for the left and right-hand sides:

i) For x_a, we get $(x_a - a)(x_a - b)(x_a - c) - d = -7.49 \cdot 10^{-5}$.

ii) For x_b, we get $(x_b - a)(x_b - b)(x_b - c) - d = -10^{-6}$.

iii) For x_c, we get $(x_c - a)(x_c - b)(x_c - c) - d = 7.51 \cdot 10^{-5}$.

(Note that values $-7.48 \cdot 10^{-5}$ and $7.51 \cdot 10^{-5}$ for x_a and x_c are nearly symmetric. The reason for this is that values of x_a and x_c are nearly symmetric with respect to x_b.)

4. Problem

The equation

$$\sin x = a \tag{6.6}$$

has one solution on the interval $0 \le x \le \pi/2$ for $0 \le a \le 1$:

$$x = \sin^{-1} a. \tag{6.7}$$

a) Use the MSA to find approximate formulas for the solutions of a modified equation:

$$\sin x = a + bx, \tag{6.8}$$

where b is small and $0 \leq x \leq \pi/2$. Limit your analysis to the first order.

b) From the formula for the first-order approximation, find a condition for a and b when the solution exists. (Since this condition is found from an approximate solution, it is also approximate.)

c) Plot both sides of equation (6.8) for $a = 0.8, b = 0.1$. Explain the condition on a and b that you found in task 4b.

Solution

a) We apply the inverse sine to get an MSA iterative scheme:

$$x_{n+1} = \sin^{-1}(a + bx_n). \tag{6.9}$$

In the zero-order approximation, $x_0 = \sin^{-1} a$. Then

$$x_1 = \sin^{-1}(a + b\sin^{-1} a). \tag{6.10}$$

b) This solution exists if the argument of $\sin^{-1}$ does not exceed 1. This yields

$$a + b\sin^{-1} a \leq 1. \tag{6.11}$$

c) The plots are shown in figure 6.1. For small b the straight line is approximately horizontal. The solution exists if the straight line crosses the plot of the sine function below the maximum, which the sine function has at $x = \pi/2$. This means that

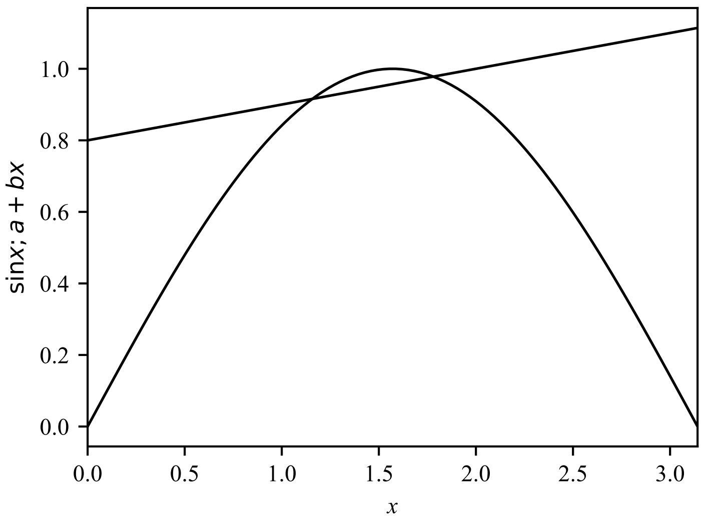

Figure 6.1
Plots of $\sin x$ and $a + bx$

$$a + b\frac{\pi}{2} \le 1. \tag{6.12}$$

Consider now the maximum value of a for which a solution exists. Since b is small, we must have $a \approx 1$ in this case. Then $\sin^{-1} a \approx \sin^{-1} 1 = \pi/2$, and in inequality (6.11) we can replace $\sin^{-1} a$ with $\pi/2$ to get inequality (6.12).

5. Problem

A radar measures a range (distance) to the object it is tracking. Assume there are two radars that detect a sea vessel at ranges R_1 and R_2. Respective coordinates of the radars are $x_1 = 0; y_1 = 0$ and $x_2 = D; y_2 = 0$ (see figure 6.2). Section A.25 computes the coordinates of a sea vessel that is detected by the radars:

$$\begin{aligned} x &= \frac{D^2 + R_1^2 - R_2^2}{2D}, \\ y &= \pm\sqrt{R_1^2 - x^2}. \end{aligned} \tag{6.13}$$

Though never mentioned explicitly, the solution in section A.25 assumes that Earth is flat. This is an approximation. Using dimensional analysis, specify a relevant small parameter or parameters for this approximation to be valid.

Solution

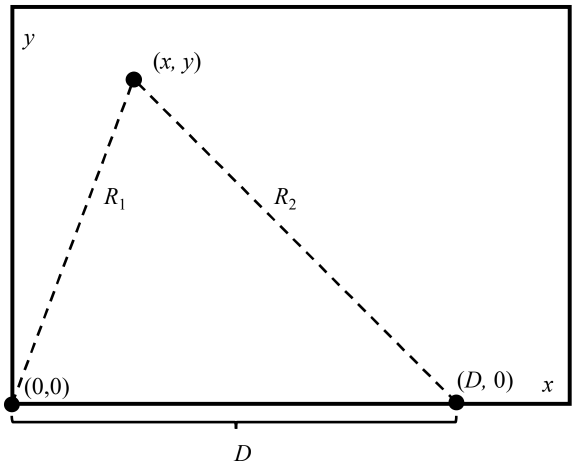

Figure 6.2
Approximations for detecting a vessel by two radars

For this solution to be valid, variables D, R_1, and R_2 must be small compared to the Earth radius.

6. Problem

You are presented with the following equation for x:

$$e^{-x} = \tan x. \tag{6.14}$$

Assume that $x \gg 1$.

a) Plot both sides of this equation. How many solutions does it have?

b) Using a zero-order approximation $e^{-x} \approx 0$, isolate three different solutions $x = n\pi$, where $n = 1, 2$, and 5, and compute first-order approximations for these three solutions. How do they differ from the zero-order approximations? Is there a pattern? If yes, why?

Solution

a) Plots are shown in figure 6.3. This equation has infinitely many solutions.

b) Since the exponent e^{-x} is small for $x \gg 1$, we can use condition $\tan x_0 = 0$ for a zero-order approximation. This yields $x_0 = \pi n$, where $n = 1, 2, 3, \ldots$. The first-order approximation will then be given by

$$x_1 = \pi n + \tan^{-1} e^{-\pi n}. \tag{6.15}$$

The values for zero and first-order approximations are presented in table 6.2.

Table 6.2
Solutions of $e^{-x} = \tan x$

n	Zero order	First order
1	3.141592653589793	3.184779702114575
2	6.283185307179586	6.285052747740495
5	15.707963267948966	15.707963418650692

We see that the difference between the zero and the first-order approximation becomes smaller for larger values of n. The reason for this is that for large n the exponent e^{-x} is very small, and the zero-order approximation is already very close to the exact solution. The first-order approximation modifies it only slightly.

7*. Problem

In this problem, you are again dealing with equation

$$e^{-x} = \tan x. \tag{6.16}$$

Now we assume that $x < 0$ and $|x| \gg 1$.

a) Plot both sides of this equation. How many solutions does it have?

b) Select one solution and find its approximate value using the MSA. (Hint: You need to transform the original equation so that you can again use the exponent in lieu of a small parameter.)

Solution

a) Plots are shown in figure 6.4. There are infinitely many solutions.

b) We see from the plot that values of the exponent are large (in contrast to exercise 6). This means that we cannot use e^{-x} as a small parameter. However, we can rewrite equation (6.16) as

$$e^x = \cot x. \tag{6.17}$$

Now for $x < 0$ and $|x| \gg 1$ we have $e^x \ll 1$, which can be used as a small parameter. Then the zero-order approximation is

$$x_0 = \cot^{-1} 0 + \pi n = \frac{\pi}{2} + \pi n. \tag{6.18}$$

Successive approximations are obtained using

$$x_{m+1} = \cot^{-1} e^{x_m} + \pi n \tag{6.19}$$

We use $\cot^{-1} x = \pi/2 - \tan^{-1} x$ to get:

$$x_{m+1} = \frac{\pi}{2} + \pi n - \tan^{-1} e^{x_m}, \tag{6.20}$$

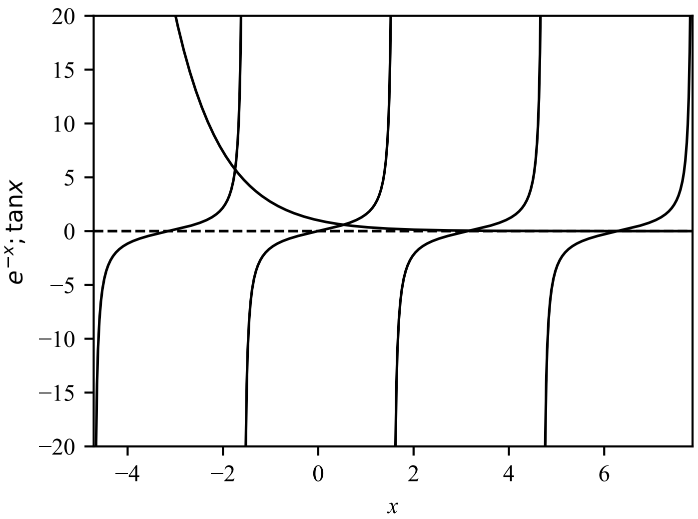

Figure 6.3
Plots of e^{-x} and $\tan x$ for $x > 0$

where we have $n < 0$. For example, if we select $n = -3$ we get

$$\begin{aligned} x_0 &= -7.853981633974483, \\ x_1 &= -7.854369837158909, \\ x_2 &= -7.854369686486459. \end{aligned} \tag{6.21}$$

This solution quickly converges.

8. Problem

Problem 23 in chapter 4 introduced the Lennard–Jones model for atom interaction. It describes the force between two neutral atoms:

$$F(r) = \frac{24\epsilon}{\sigma}\left(2\left(\frac{\sigma}{r}\right)^{13} - \left(\frac{\sigma}{r}\right)^{7}\right), \tag{6.22}$$

where r is the distance between the atoms and σ, ϵ are positive parameters. A positive value for the force means that it is repulsive and a negative value means it is attractive.

At large distances the magnitude of both terms becomes small, which means that the attraction force is weak. In that case, an atom on the surface of a liquid can break away from the surface because it is no longer constrained by the attraction force from other atoms. This process is known as evaporation.

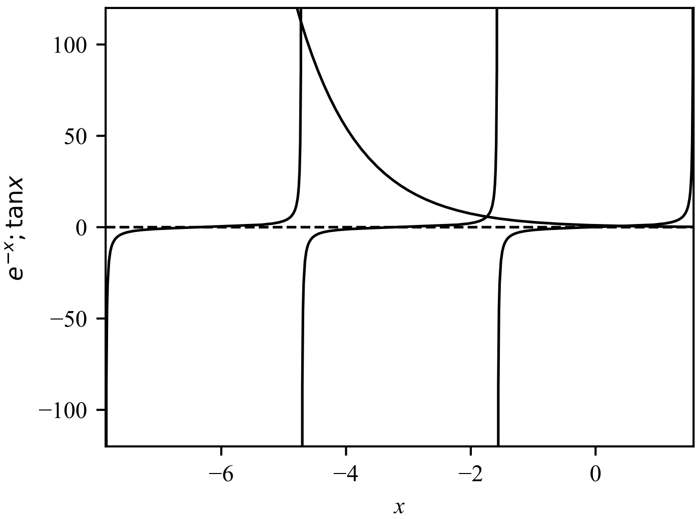

Figure 6.4
Plots of e^{-x} and $\tan x$ for $x < 0$

a) Observe that both terms in the Lennard–Jones model exhibit power law scaling. Which term has a smaller magnitude for large values of r?

b) Assume that an atom has a high chance of leaving the surface of a liquid if $F(r) \geq -0.24\epsilon/\sigma$. Leverage the small value of one of the terms to find distances r at which this condition is true.

Solution

a) The higher-order term will be small for large r:

$$2\left(\frac{\sigma}{r}\right)^{13} \ll \left(\frac{\sigma}{r}\right)^{7}. \tag{6.23}$$

b) Condition $F(r) \geq -0.24\epsilon/\sigma$ leads to

$$-2\left(\frac{\sigma}{r}\right)^{13} + \left(\frac{\sigma}{r}\right)^{7} \leq d, \tag{6.24}$$

where $d = 0.01$. We seek the smallest r that satisfies this inequality, which corresponds to the equal sign there. We denote $\rho = \sigma/r$. If the higher-order term is small, we can use it as a small parameter. Then the iterative scheme is obtained by solving for ρ in the lower-order term:

$$\rho_{n+1} = \sqrt[7]{d + 2\rho_n^{13}}. \tag{6.25}$$

For the zero-order approximation we select $\rho_0 = \sqrt[7]{d} \approx 0.517947$. Then $\rho_1 \approx 0.520758$. This yields

$$r = \frac{\sigma}{\rho} \approx 1.920\sigma. \tag{6.26}$$

Atoms will have a high chance leaving the surface of the liquid if r is equal or exceeds this value.

9. Problem

Among other requirements, a driverless car must quickly respond to emergency vehicles' sirens. In a driving range test, a siren is transmitted starting exactly at time $t_s = 0$ from a point with coordinates x_s, y_s. A test vehicle is moving in a circular loop. Its coordinates x_v, y_v are given by

$$\begin{aligned} x_v(t) &= R\cos\frac{Vt}{R}, \\ y_v(t) &= R\sin\frac{Vt}{R}, \end{aligned} \tag{6.27}$$

where R is the radius of the circular loop and V is the vehicle speed. To measure the latency of the vehicle's response, we need to compute the time T of the signal's arrival at the vehicle location. This time is delayed from $t_s = 0$ by the time needed for the sound to travel from the signal source to the vehicle. If the speed of sound is V_s, the signal travel time is given by

$$T = \frac{1}{V_s}\sqrt{(x_s - x_v(T))^2 + (y_s - y_v(T))^2}. \tag{6.28}$$

Substitution of $x_v(T), y_v(T)$ yields an equation for the time of the arrival of the signal:

$$T = \frac{1}{V_s}\sqrt{\left(x_s - R\cos\frac{VT}{R}\right)^2 + \left(y_s - R\sin\frac{VT}{R}\right)^2}. \tag{6.29}$$

This is a transcendental equation for T that cannot be solved by conventional methods. With this mathematical model in hand, do the following (note that cars do not drive close to supersonic speeds, and therefore $V \ll V_s$):

a) Using the MSA, produce formulas for the zero-order (which corresponds to $V_s \to \infty$), first-order, and second-order approximations for T.

b) Compute numerical values for the first- and second-order approximations using $R = 100$ m, $V_s = 340$ m/s, $V = 20$ m/s, and for the following two cases:

- **i.** $x_s = 300$ m; $y_s = 100$ m
- **ii.** $x_s = 100$ m; $y_s = 300$ m

c) What is the difference between the first- and second-order approximations for each location of the signal source? For one location, the first- and second-order approximations are closer than for the other location. Why?

Solution

a) In the limit $V_s \to \infty$ equation (6.29) produces $T_0 = 0$ for the zero-order approximation. The next two approximations are

$$T_1 = \frac{1}{V_s}\sqrt{(x_s - R)^2 + y_s^2},$$
$$T_2 = \frac{1}{V_s}\sqrt{\left(x_s - R\cos\frac{VT_1}{R}\right)^2 + \left(y_s - R\sin\frac{VT_1}{R}\right)^2}, \tag{6.30}$$

b) The first two approximations for the two scenarios are as follows:

i. $T_1 = 0.657667$ s, $T_2 = 0.643676$ s.

ii. $T_1 = 0.882353$ s, $T_2 = 0.830731$ s.

c) The difference between the first and the second-order approximations is $T_1 - T_2 \approx 0.014$ s for case *i*, and is ≈ 0.052 s for case *ii*, which points to a slower convergence for case *ii*. The reason for this is that in that second case the car moves roughly in the direction to the signal source, and any error in the position of the car will strongly affect the time of signal arrival. In case *i* the car moves in the direction, which is nearly perpendicular to the direction to the signal source, and an error in the car position has a smaller effect.

10. Problem

This problem shows that a wrong application of the MSA may produce a diverging solution.

In chapter 7 we consider rare random events. In particular, we deal with the following formula for the probability of rare random events (see equation (7.11*) in section 7.6):

$$\tilde{P}_n(x) \approx \frac{\sigma}{(x-\mu)\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{6.31}$$

If the events in question are rare, the probability of their occurrence is small: $\tilde{P}_n(x) \ll 1$. For brevity, we denote $y = (x-\mu)/\sigma$ to get

$$\tilde{P}_n(y) \approx \frac{1}{y\sqrt{2\pi}} e^{-\frac{y^2}{2}}. \tag{6.32}$$

For many applications, it is important to solve the last equation for y. Assume probability $\tilde{P}_n = 10^{-7}$, which corresponds to truly rare events.

a) Rewrite equation (6.32) as

$$y = \frac{1}{\tilde{P}_n\sqrt{2\pi}} e^{-\frac{y^2}{2}}. \tag{6.33}$$

Compute three MSA iterations using $y_0 = 5$.

b) Solving for y in the exponent of equation (6.32) produces

$$y = \sqrt{-2\left(\ln\left(\sqrt{2\pi}\tilde{P}_n\right) + \ln y\right)}. \tag{6.34}$$

Compute three MSA iterations using $y_0 = 5$.

c) Which approach converges, and which does not?

Solution

a) For approach a we have

$$\begin{aligned} y_0 &= 5, \\ y_1 &= 14.867195, \\ y_2 &= 4.0188076 \cdot 10^{-42}, \\ y_3 &= 3989422.8. \end{aligned} \tag{6.35}$$

b) For approach b we have

$$\begin{aligned} y_0 &= 5, \\ y_1 &= 5.2133903, \\ y_2 &= 5.2053678, \\ y_3 &= 5.2056636. \end{aligned} \tag{6.36}$$

c) We see that approach b converges and approach a does not.

11. Problem

Solve the following equation for t, assuming that ω is a large parameter (rather than small):

$$\omega(t - t_0) = A \sin(ft). \tag{6.37}$$

Obtain analytical formulas for the zero- and first-order approximations.

Solution

To have a small parameter, we rewrite equation (6.37) as

$$t = t_0 + \frac{A}{\omega} \sin(ft). \tag{6.38}$$

The zero-order approximation $t = t_0$ corresponds to $\omega \to \infty$. For the first-order approximation we have

$$t_1 = t_0 + \frac{A}{\omega} \sin(ft_0). \tag{6.39}$$

12. Problem

The equation for x below is a combination of the equations for the sum of trigonometric functions (see section A.13) and for the product of two linear expressions (see section A.7):

$$f \sin x + g \cos x = c(x - a)(x - b). \tag{6.40}$$

Find analytical expressions for the zero- and first-order approximations for x for two cases:

a) Parameter c is small.

b) Parameter c is large.

c) If you roughly follow the MSA implementation in section 6.4 for the case of large values of c, the solution will break down if $a \approx b$. Provide a solution that is valid even if $a \approx b$.

Solution

a) We use the derivation in section A.13 to solve for x in the left-hand side of equation (6.40):

$$x = -\tan^{-1}\frac{g}{f} + (-1)^n \sin^{-1}\left(\frac{c(x - a)(x - b)}{\sqrt{f^2 + g^2}}\right) + \pi n. \tag{6.41}$$

In the zero-order approximation for small values of c we have:

$$x_0 = -\tan^{-1}\frac{g}{f} + \pi n. \tag{6.42}$$

The first-order approximation is given by

$$x_1 = -\tan^{-1}\frac{g}{f} + (-1)^n \sin^{-1}\left(\frac{c(x_0 - a)(x_0 - b)}{\sqrt{f^2 + g^2}}\right) + \pi n. \tag{6.43}$$

b) If c is large we can use $1/c$ as a small parameter. We rewrite equation (6.40) as

$$(x-a)(x-b) = \frac{1}{c}(f\sin x + g\cos x). \tag{6.44}$$

One way to apply the MSA here is to follow the derivation in section 6.4. We consider a solution that is close to a and construct successive approximations:

$$x_{n+1} = a + \frac{f\sin x_n + g\cos x_n}{c(x_n-b)}. \tag{6.45}$$

There is also a solution that is closed to b; it is obtained similarly.

c) Another way to solve for x is to view equation (6.44) as a quadratic equation. We expand the parentheses in the left-hand side and group the terms to get:

$$x^2 - x(a+b) + ab - \frac{1}{c}(f\sin x + g\cos x) = 0. \tag{6.46}$$

Two solutions of this equation are given by

$$x^{(1,2)} = \frac{(a+b) \pm \sqrt{(a-b)^2 + \frac{4}{c}(f\sin x + g\cos x)}}{2}. \tag{6.47}$$

This produces the following MSA scheme:

$$x_{n+1}^{(1,2)} = \frac{(a+b) \pm \sqrt{(a-b)^2 + \frac{4}{c}(f\sin x_n^{(1,2)} + g\cos x_n^{(1,2)})}}{2}. \tag{6.48}$$

13. Problem

You are given the following equation:

$$\sin e^{-x} = \sin x. \tag{6.49}$$

a) How many solutions does it have?

b) Select a solution with $x \gg 1$ and compute the zero- and first-order approximations for x.

Solution

a) Equation (6.49) is equivalent to

$$e^{-x} = (-1)^n x + \pi n, \tag{6.50}$$

where n is an integer. If we plot $y = e^{-x}$ and the family of lines $y = (-1)^n x + \pi n$ for different values of n, we'll see that the exponent and the straight lines intersect at infinite number of points. Therefore, this equation has infinitely many solutions.

b) For $x \gg 1$ the exponent is small, and we must have $(-1)^n x \approx -\pi n$. We select $x = 4\pi$ as the zero-order approximation (this corresponds to $n = -4$). Then the first order approximaion is

$$x_1 = e^{-x_0} + 4\pi. \tag{6.51}$$

The numerical values are: $x_0 = 12.566370614359172, x_1 = 12.566374101701529$.

14. Problem

The Lotka–Volterra equations describe the dynamics for two species, a prey (for example, rabbits) and a predator (for example, foxes). These equations have a stationary point, when the numbers of the prey animals and of the predators are constant, that is, do not vary over time. It is achieved when the decrease in the number of each species (for example, because of restricted food supply or being eaten by predators) is perfectly balanced with births. The stationary equations are

$$\begin{aligned} x(\alpha - \beta y) &= 0, \\ -y(\gamma - \delta x) &= 0, \end{aligned} \tag{6.52}$$

where x is the number of prey, y is the number of predators, and α, β, γ, and δ are constants. In this model, the equilibrium is given by the solution of the above equations:[1]

$$\begin{aligned} y &= \frac{\alpha}{\beta}, \\ x &= \frac{\gamma}{\delta}. \end{aligned} \tag{6.53}$$

A researcher has come up with a more accurate version of the Lotka–Volterra equations, which modifies the stationary conditions as follows:

$$\begin{aligned} x(\alpha - \beta y) + \epsilon x^2 y^2 &= 0, \\ -y(\gamma - \delta x) + \epsilon x^2 y^2 &= 0. \end{aligned} \tag{6.54}$$

For these modified stationary equations, find solutions for x and y, assuming that ϵ is small:

a) Use solution (6.53) as the zero-order approximation in the first equation of (6.54) to get the first-order approximation for y. Then use the zero-order approximation for x and the first-order approximation for y in the second equation of (6.54) to get the first-order approximation for x.

b) Use the zero-order approximation for x and y in both equations (6.54) simultaneously to get the first-order approximation for both variables.

c) Use $\alpha = 1, \beta = 2, \gamma = 3, \delta = 4$, and $\epsilon = 10^{-2}$ to compute two different versions of first-order approximations, as explained in items 14a and 14b above.

1. The second stationary point is given by $x = 0; y = 0$. Zero numbers for both species at some time moment would obviously remain zero in the future, also making this solution stationary.

d) Substitute the numerical values in equations (6.54) and determine which method yields better accuracy for the first-order approximation.

Solution

a) For the nontrivial solution we get from the first equation in (6.54):

$$y = \frac{\alpha}{\beta} + \frac{\epsilon}{\beta} x y^2. \tag{6.55}$$

We substitute the zero-order approximation (6.53) in the right-hand side to get the first-order approximation:

$$y_1 = \frac{\alpha}{\beta} + \epsilon \frac{\gamma}{\delta} \frac{\alpha^2}{\beta^3}. \tag{6.56}$$

From the second equation (6.54) we get:

$$x = \frac{\gamma}{\delta} - \frac{\epsilon}{\delta} y x^2. \tag{6.57}$$

In the right-hand side we substitute the zero-order approximation for x and the first-order approximation for y to get the first-order approximation for x:

$$x_1 = \frac{\gamma}{\delta} - \epsilon \frac{\gamma^2}{\delta^3} \left(\frac{\alpha}{\beta} + \epsilon \frac{\gamma}{\delta} \frac{\alpha^2}{\beta^3} \right). \tag{6.58}$$

b) We again use equations (6.55) and (6.57), but this time we substitute the zero-order approximations given by equations (6.53) into their right-hand sides to get

$$\begin{aligned} \tilde{y}_1 &= \frac{\alpha}{\beta} + \epsilon \frac{\gamma}{\delta} \frac{\alpha^2}{\beta^3}, \\ \tilde{x}_1 &= \frac{\gamma}{\delta} - \epsilon \frac{\gamma^2}{\delta^3} \frac{\alpha}{\beta}. \end{aligned} \tag{6.59}$$

c) Numerical values for x_1, y_1 for both methods are listed in table 6.3.

d) Values of y_1 and $\tilde{y}_1$ are identical: $y_1 = \tilde{y}_1 = 0.5009375$. This is not surprising, because both are obtained using the zero-order approximation for x and y. Values of x_1 and $\tilde{x}_1$ differ slightly, because the former was obtained using y_1 and the latter used $\tilde{y_0}$. Indeed, if we look at equations (6.56), (6.58), and (6.59) we will see that $y_1 = \tilde{y}_1$, but the expression for x_1 differs from that for $\tilde{x}_1$ by a term $\sim \epsilon^2$. Since y_1 is obtained using y_0 and x_1, it is essentially a mix of a first- and second-order approximations. Generally, mixing approximations of different orders does not necessarily improve the accuracy of the solution. We substitute values of the two sets of first order approximations back in equations (6.54) and introduce new notations for the error from applying the first-order approximations:

$$\begin{aligned} d_y &= x_1(\alpha - \beta y_1) + \epsilon x_1^2 y_1^2, \\ d_x &= -y_1(\gamma - \delta x_1) + \epsilon x_1^2 y_1^2. \end{aligned} \tag{6.60}$$

Values of errors d_x, d_y are also listed in table 6.3 for both methods of computing the first-order approximation.

Table 6.3
Results for solutions of the Lotka–Volterra equations

Variable	First method	Second method (tilded variables)
x_1	0.74929556	0.749296875
y_1	0.5009375	0.5009375
d_y	$3.94888 \cdot 10^{-6}$	$3.9513656 \cdot 10^{-6}$
d_x	$-2.6503329 \cdot 10^{-6}$	$-3.7125163 \cdot 10^{-9}$

We see that even though y_1 is obtained using a better estimate x_1 (as compared to $\tilde{y}_1$ that uses x_0), it does not produce lower errors when substituted back in modified Lotka-Volterra equations (6.54).

15. Problem

In section 2.4 we investigated limiting cases for the quadratic equation

$$ax^2 + bx + c = 0. \tag{6.61}$$

We have determined that for $c \to 0$ one of the roots approaches zero. Present this equation in the form:

$$x = -\frac{c}{ax + b}. \tag{6.62}$$

a) Use $a = 1, b = 2$, and $c = 10^{-2}$ and compute numerical values for MSA approximations of the orders 1 through 5. Use $x_0 = 0$ as the zero-order approximation.

b) Compute the exact value of x using the quadratic formula.

c) Compute the error in each of the approximations and plot it versus the approximation order number using the logarithmic scale for the vertical axis.

d) How does this error scale with the order number?

e) Section 6.5 presented a different way to solve the quadratic equation using the MSA. Use equations (6.27*) to compute the numerical values for the first three approximations, and compare the results with those you obtained from applying MSA iterations to equation (6.62).

Solution

a) Values for x_n are given in table 6.4.

b) The quadratic formula yields $x_e = -0.005012562893380035$ for this root.

c) A plot of the error $|x_n - x_e|$ versus the approximation order is shown in figure 6.5 using circles.

d) The error scales exponentially with the order of approximation.

Table 6.4
Approximations for a root of a quadratic equation (small c)

Approximation order n	x_n
0	0
1	-0.005
2	-0.005012531328320802
3	-0.005012562814070352
4	-0.005012562893180773
5	-0.005012562893379545

e) Section 6.5 uses the following scheme to approximate the same root:

$$x_{n+1} = -\frac{c + ax_n^2}{b}. \tag{6.63}$$

Errors from this method of computing successive approximations are shown in the same figure 6.5 using stars. In both cases, the error decreases exponentially as a function of n, but the convergence for equation (6.62) is a bit faster.

16. Problem

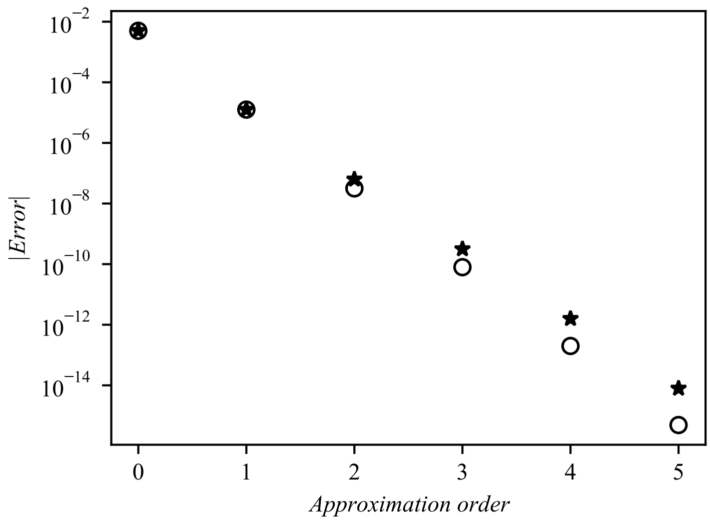

Figure 6.5
Errors for MSA iterations for equation (6.62) (circles) and for the method presented in section 6.5 (stars)

We again solve the quadratic equation:

$$ax^2 + bx + c = 0. \tag{6.64}$$

Rewrite this quadratic equation as

$$x = -\frac{ax^2 + c}{b}. \tag{6.65}$$

a) Use the quadratic formula to obtain both exact solutions for x when $c = 2, b = 1$, and $a = 10^{-3}$. Note that this case is different from the one we have investigated in section 6.5 and in exercise 15: we now assume a to be small instead of c.

b) Use $x_0 = 0$ and compute the first three approximations. Do they approach the exact solution?

c) Set $x_0 = -10^3$, which happens to be close to the other exact solution. Compute the first- and second-order approximations. Why do they not approach the exact solution, even though x^2 in the right-hand side is multiplied by a small parameter?

Solution

a) The exact (to within computation accuracy) solutions of this quadratic equation are $x^{(1)} = -2.004016080450699$ and $x^{(2)} = -997.9959839195493$.

b) Values for $x_n^{(1)}$ are given in table 6.5.

Table 6.5
Approximations for a root of a quadratic equation (small a)

Approximation order n	$x_n^{(1)}$
0	0
1	-2.0
2	-2.004
3	-2.004016016

We see that the solution converges to the exact value.

c) Values for the second root are given in table 6.6.

This solution does not converge to the exact value $x^{(2)} = -997.9959839195493$. With every next iteration, the gap between the estimate and the exact value widens. The reason for this is that the value of x is large enough to cancel the benefit of having a small multiplier a in the expression ax^2 in equation (6.65).

Table 6.6
Approximations for the second root of a quadratic equation (small a)

Approximation order n	$x_n^{(2)}$
0	-1000.0
1	-1002.0
2	-1006.004

17. Problem

In section 6.1 we observed that the MSA works for the Achilles and the tortoise problem only if $\left|\frac{V_T}{V_A}\right| < 1$. Reconcile this inequality with the criterion for MSA convergence that we formulated in section 6.3.

Solution

Section 6.1 shows that successive approximations for this problem are given by truncating the sum in the right-hand side of the following series:

$$T = \frac{D_0}{V_A}\left(1 + \frac{V_T}{V_A} + \left(\frac{V_T}{V_A}\right)^2 + \left(\frac{V_T}{V_A}\right)^3 + \ldots\right). \tag{6.66}$$

The recursive scheme of getting T_{n+1} from T_n is then given by

$$T_{n+1} = \frac{D_0}{V_A} + T_n \frac{V_T}{V_A}. \tag{6.67}$$

This defines T_{n+1} as a function of T_n. This function is linear, and its slope is V_T/V_A. According to the convergence criterion in 6.3, such an MSA scheme converges if the slope is less than 1, that is, $V_T/V_A < 1$.