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## Preface

When teaching a graduate topology course, it's tempting to rush through the point-set topology, or even skip it altogether, and do more algebraic topology, which is more fun to teach and more relevant to today's students. Many point-set topology ideas are already familiar to students from real analysis or undergraduate point-set topology courses and may seem safe to skip. Also, point-set ideas that might be unfamiliar but important in other subjects, say the Zariski topology in algebraic geometry or the  $p$ -adic topology in number theory, can be picked up later when they are encountered in context.

An alternative to rushing through point-set topology is to cover it from a more modern, categorical point of view. We think this alternative is better for several reasons. Since many students are familiar with point-set ideas already, they are in a good position to learn something new about these ideas, like the universal properties characterizing them. Plus, using categorical methods to handle point-set topology, whose name even suggests an old-fashioned way of thinking of spaces, demonstrates the power and versatility of the methods. The category of topological spaces is poorly behaved in some respects, but this provides opportunities to draw meaningful contrasts between topology and other subjects and to give good reasons why some kinds of spaces (like compactly generated weakly Hausdorff spaces) enjoy particular prevalence. Finally, there is the practicality that point-set topology is on the syllabus for our first-year topology courses and PhD exams. Teaching the material in a way that both deepens understanding and prepares a solid foundation for future work in modern mathematics is an excellent alternative.

This text contains material curated from many resources to present elementary topology from a categorical perspective. In particular, we cover some of the same topics as Ronnie Brown (2006), although our outlook is, from the outset, more categorical. The result is intentionally less comprehensive but more widely useful. We assume that students know linear algebra well and have had at least enough abstract algebra to understand how to form the quotient of a group by a normal subgroup. Students should also have some basic knowledge about how to work with sets and their elements, even as they endeavor to work with arrows instead. Students encountering diagrams and arrows for the first time may

want to spend a little extra time reading the preliminaries where the objects (sets) are presumably familiar but the perspective may be new.

Covering spaces, homology, and cohomology are not in this book, but students will be ready to learn more algebraic topology after reading through our text. The omitted topics are likely included in whichever algebraic topology book is used afterward, including Massey (1991), Rotman (1998), May (1999), Hatcher (2002), and tom Dieck (2008), for which the reader will be well prepared. When we teach the first semester topology course in our PhD program, we usually cover the classification of compact surfaces. While this classification theorem is not in the text, an instructor may wish to cover it in their course, and it is hard to beat Conway's ZIP proof or the proof in Massey (1991).

With detailed descriptions of topological constructions emphasizing universal properties; filter-based treatment of convergence; thorough discussions of limits, colimits, and adjunctions; and an early emphasis on homotopy, this book guides the student of topology through the important transition from an undergraduate with a solid background in analysis or point-set topology to a graduate student preparing to work on problems in contemporary mathematics.