Mage Merlin's Unsolved Mathematical Mysteries

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contents

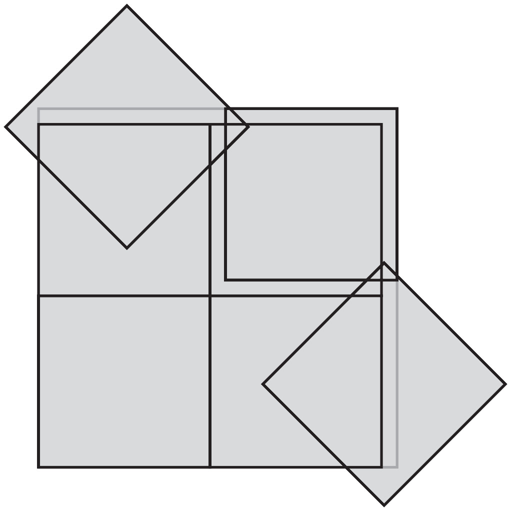
I was summoned to Camelot in the middle of the night.

The full moon shone brightly upon the majestic castle. Arriving in the Great Hall, the resting place of the mighty sword Excalibur, I noticed that the resplendent stained glass window, a perfect 201 × 201 square, lay shattered on the floor.

There were six 100 × 100 square tiles nearby, each inlaid with an ancient design. King Arthur asked if I could cover the window opening using these tiles, protecting Excalibur from the dangers outside. I could arrange and overlap the six sacred tiles, but certainly not break them.

All night I toyed with the tiles, positioning and repositioning them in various configurations, but even with my powers of magic and logic, I could not succeed.

Four 1 × 1 squares fit together to exactly cover a 2 × 2 window. If this square window is slightly larger, the four squares will not be enough to cover it. In 2000, Trevor Green showed how to use seven tiles to cover a slightly larger square.



Green's covering

In 2009, Janusz Januszewski proved that five squares are not enough to cover the square window, no matter how they are arranged. As Merlin suggests, the question of whether six squares is enough remains unsolved. The six 100×100 tiles have a total area of 60,000, significantly more than the area of the 201×201 window, which is 40,401. However, it seems impossible to position the first few tiles so that they do not leave long narrow gaps.

Although Merlin's problem is quite specific, it is one of a number of similar questions: What is the minimum number of 1×1 squares that are needed to cover an $n \times n$ square? How many squares are needed to cover rectangles or triangles of various sizes? Given a fixed number of tiles, what is the largest square that they can cover? In 2005, Alexander Soifer conjectured the following:

UNSOLVED: The largest square that can be covered with $n^2 + 1$ squares of size 1×1 will be of size $n \times n$.

Definitive answers to these questions are few and far between. Computers have had some success finding efficient covers, but even in those cases, it is not known whether the solutions are optimal.

I was summoned to Camelot, where I received joyous news.

King Arthur and Queen Guinevere were now parents to a second daughter, whom they had named Vivian. Unlike Princess Anna, their firstborn, Princess Vivian's hair was wild and windblown, a sure indication of her fiery temperament. Remarkably, Vivian was born on the same day as Anna, but exactly two years later.

To celebrate this marvelous blessing, the king and queen decided to enact a new tradition in Camelot: every year on their shared birthday, each daughter would light a red candle if turning a prime age and a white candle otherwise. On the years when both daughters lit a red candle, a grand banquet would be held throughout Camelot.

Guinevere noticed that these feasts would happen less and less frequently as her daughters got older. She wondered whether the celebrations would eventually stop if this custom were to continue forever.

I played with this puzzle for years, trying to find patterns among primes, but even with my powers of magic and logic, I was never able to answer her.

One of the earliest findings on primes comes from the great Greek geometer Euclid. Around 300 BCE, Euclid compiled much of what was known about mathematics at the time into a book called *The Elements*. In it, he provided the first known proof that there are infinitely many prime numbers. Euclid argued that if there were only finitely many primes, then it would be possible to multiply all the primes together and then add one. He was able to show that the resulting number would be neither prime nor not prime. Since that is impossible, Euclid concluded that the set of primes could not be finite.

The story of the princesses' birthdays involves pairs of primes that differ by two. These are called *twin primes*. They occur frequently among small numbers: 3–5, 5–7, 11–13, 17–19, but less frequently among larger numbers. Guinevere's question is a natural follow-up to Euclid's result: since there are infinitely many primes, are there infinitely many twin primes? The question was formally posed by Alphonse de Polignac in 1846 as a conjecture:

UNSOLVED: There are infinitely many twin primes.

For a long time, little progress was made toward proving or disproving the conjecture. But in 2013, a previously unheralded mathematician named Yitang Zhang surprised the mathematical world with a major breakthrough. Zhang established the first finite bound on gaps between primes, proving that there are infinitely many prime pairs of the form $(p, p + N)$, where N is some number less than 70 million.

Following a flurry of activity, the upper bound N was significantly narrowed to be less than 246, yet it remains to be seen whether it reaches 2, thereby solving the twin prime conjecture.