Exercises for “Macroeconomic Analysis”

Exercises for "Macroeconomic Analysis"

Dirk Niepelt with Lorenz Driussi, Charles Lardy, and Fabienne Schneider

The MIT Press
Cambridge, Massachusetts
London, England

Contents

1 Microeconomic Foundations

1.1 Microeconomics

[1] Characterize the competitive equilibrium in an economy with:

i. one household, one good, preferences $u(x)$, and endowment $\bar{x}$;
ii. one household, two goods, preferences $u(x, y)$, and endowments $\bar{x}, \bar{y}$;
iii. one household, two goods, one firm, preferences $u(x, y)$, endowments $\bar{x}, \bar{y}$, and production function $y = f(x)$ where f is strictly increasing and concave and satisfies Inada conditions.

[2] Who chooses the market clearing prices in a competitive equilibrium?
[3] How helpful is the Pareto criterion to guide political decisions?

1.2 Primitives

[4] (Discounting) Suppose that $T \to \infty$, the period utility function u is bounded, and $\beta \in [0, 1)$. Show that the discounted utility, $\sum_{t=0}^{\infty} \beta^t u(c_t)$, is finite for all consumption paths $\{c_t\}_{t=0}^{\infty}$.
[5] Show that $\lim_{\sigma \to 1} \frac{c^{1-\sigma}-1}{1-\sigma} = \ln(c)$.
[6] Show that the intertemporal elasticity of substitution of a CIES utility function equals σ^{-1}.
[7] Compute the slope of an indifference curve when $\sigma \to 0$.
[8] (Homotheticity) Show that $U(c) = g(h(c))$ is homothetic if g is strictly increasing and h is homogeneous of degree $n \in \mathbb{N}$; c denotes the consumption vector $(\ldots, c_t, \ldots, c_s, \ldots)$.
[9] (Euler's theorem) Let $n \in \mathbb{N}$ and consider a function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ taking arguments $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $z \in \mathbb{R}^n$. Suppose that F is homogenous of degree $m \in \mathbb{N}$ in x and y, i.e., for any $\lambda \geq 0$ and $z \in \mathbb{R}^n$,

$$F(\lambda x, \lambda y, z) = \lambda^m F(x, y, z). \tag{1.1}$$

Suppose moreover that F is differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Show that the following condition holds:

$$mF(x, y, z) = x\frac{\partial F}{\partial x}(x, y, z) + y\frac{\partial F}{\partial y}(x, y, z).$$

Moreover, show that the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are homogenous of degree $m - 1$ in x and y.

[10] Show that with CRTS and competitive factor markets, output of a neoclassical firm coincides with total factor payments.

[11] How large are a firm's profits when its production function exhibits CRTS and factor markets are competitive?

[12] Show that with CRTS, output per worker as well as marginal products depend on the capital-labor ratio, $k \equiv K/L$, not on K and L separately.

[13] Show that the CES production function,

$$f(K, L) = \left(\alpha K^{1-\frac{1}{\theta}} + (1-\alpha)L^{1-\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

has a constant elasticity of substitution, θ.

[14] Show that with competitive factor markets, the Cobb-Douglas production function implies constant factor shares.

2 Consumption and Saving

2.1 Consumption Smoothing

[15] (Sufficient Condition) Consider a household that maximizes $u(c_0) + \beta u(c_1)$ subject to the intertemporal budget constraint where u is strictly concave and increasing. Show that the consumption plan that solves the Euler equation and the intertemporal budget constraint is the globally optimal consumption plan.

[16] What happens to the consumption smoothing motive if the felicity function is linear rather than strictly concave?

[17] Characterize the wealth, income, and substitution effects on c_0 in the two-period model with CIES preferences. What happens if the felicity function is logarithmic?

[18] Suppose that $\beta R_t = 1$ for all t. Solve for the optimal consumption path. Do you need to make functional form assumptions about u? What if $\beta R_t \neq 1$?

[19] (Lagrangian) Would the household's program be well defined if the terminal condition (stipulating a lower bound on a_{T+1}) were dropped?

[20] (Lagrangian) Interpret the first-order conditions

$$\beta^t u'(c_t) = \lambda_t, \ t = 0, \ldots, T,$$
$$\lambda_t = \lambda_{t+1} R_{t+1}, \ t = 0, \ldots, T-1.$$

[21] (Dynamic Programming) Assume that $u(c) = \ln(c)$ and $w_t = 0$ for all t. Derive $V_{T-1}(a_{T-1})$ and the policy function for a_{T-1}.

[22] Consider the household's savings decision at date t. Let $\alpha_t^\star = a_{t+1}^\star/(a_t R_t + w_t)$ denote the fraction of the "cash at hand" at date t $(a_t R_t + w_t)$ that the household carries into the next period. Intuition would suggest that the optimality condition characterizing the consumption-savings tradeoff is given by

$$u'(c_t) = \beta R_{t+1} \left\{ (1 - \alpha_{t+1}^\star) u'(c_{t+1}) + \alpha_{t+1}^\star \beta R_{t+2} u'(c_{t+2}) \right\}$$

since only a share $1 - \alpha_{t+1}^\star$ of the return on saving at date t is consumed at date $t+1$. However, as we know, the optimality condition reads $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. Do the two conditions contradict each other? Explain.

[23] (Natural Bounds on a_{t+1}) Assuming that consumption cannot be negative, show that at all dates the household's assets satisfy

$$-\sum_{s=t+1}^{T} \frac{q_s}{q_t} w_s \leq a_{t+1} \leq \frac{a_0 R_0}{q_t} + \sum_{s=0}^{t} \frac{q_s}{q_t} w_s .$$

Interpret these bounds.

[24] Show that a change in an exogenous variable θ affects optimal consumption as follows:

$$\frac{1}{c_t} \frac{\partial c_t}{\partial \theta} = -\frac{1}{\sigma(c_t)} \left(\frac{1}{q_t} \frac{\partial q_t}{\partial \theta} + \frac{1}{\lambda} \frac{\partial \lambda}{\partial \theta} \right),$$

where $\sigma(c) = -\frac{u''(c)c}{u'(c)}$ is the inverse of the intertemporal elasticity of substitution and λ denotes the Lagrange multiplier of the intertemporal budget constraint.

[25] (Infinite Horizon, No-Ponzi-Game Condition) Consider an infinitely lived household with assets $a_0 < 0$ at date $t = 0$. The household compares different options of servicing the debt, as specified below. For each option, compute the present value (discounting at the gross interest rate R) of the debt service and check whether the household satisfies the condition $\lim_{T\to\infty} a_{T+1} \geq 0$ and/or the no-Ponzi-game condition $\lim_{T\to\infty} q_T a_{T+1} \geq 0$:

i. At date 0, fully repay all debt, that is pay $-Ra_0$.
ii. At each date t, pay $-xa_t$ where $0 < x < R$. Distinguish between the case $x \leq R - 1$ and the case $x > R - 1$.
iii. At each date t, pay nothing (i.e., pay $-xa_t$ with $x = 0$).

[26] Consider an infinitely lived household with CIES preferences. Solve for c_0. Characterize the effect of a change in the interest rate at date $T > 0$ on c_0.

[27] (Infinite Horizon Dynamic Programming) Assume that $u(c) = \ln(c)$, $w_t = 0$, and $R_t = R$ at all dates; the planning horizon is infinite. The Bellman equation thus reads

$$V(a) = \max_{a'} \ln(aR - a') + \beta V(a') .$$

An educated guess for the functional form of V is $V(a) = E + F \ln(a) + G \ln(R)$, where E, F, and G are real constants. Apply the method of undetermined coefficients to determine the value of these constants (and thereby verify the guess).

[28] Write computer code to solve the household's infinite horizon dynamic programming problem for arbitrary preferences.

[29] (Uniqueness of Optimal Consumption Plan) Suppose that the felicity function is strictly concave. Show that the optimal consumption plan is unique.

[30] (Interior Optimal Consumption Plan) Show that any nontrivial optimal consumption plan is interior (that is, $c_t > 0$ at all dates) if the utility function is concave and differentiable (with finite derivative) for all strictly positive consumption levels and the Inada condition is satisfied, $\lim_{c\to 0} u'(c) = +\infty$.

2.2 Extensions

[31] (Borrowing Constraint) Consider a three-period setting and assume that $\beta = R_1 = R_2 = 1, u'(c) > 0, u''(c) < 0$. The household may not borrow. Conjecture the optimal consumption path if $a_0 = a_3 = 0$ and

i. $w_0 = w_1 = w_2$;
ii. $w_0 > w_1 = w_2$;
iii. $w_0 < w_1 = w_2$;
iv. $w_0 = w_1 < w_2$.

Verify your conjecture.

[32] (Nongeometric Discounting and Time Consistency) Consider a three-period setting with logarithmic utility and constant interest rates, $R = 1$. At date $t = 0$, the household maximizes $u(c_0)+\beta(u(c_1)+\delta u(c_2))$ with $0 < \beta \leq \delta$. At date $t = 1$, the household maximizes $u(c_1)+\beta u(c_2)$. Derive the ex-ante optimal consumption plan (c_0, c_1, c_2) at date $t = 0$. Derive the optimal consumption plan at date $t = 1$; show that the latter differs from (c_1, c_2) if $\beta < \delta$. Frame the household's program at date $t = 0$ as a game between the "date-$t = 0$ self" and the "date-$t = 1$ self." Derive equilibrium consumption.

[33] Consider the consumption index

$$c_t(d_t, e_t) = \left(\delta^{\frac{1}{\theta}} d_t^{1-\frac{1}{\theta}} + \varepsilon^{\frac{1}{\theta}} e_t^{1-\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \ \delta + \varepsilon = 1, \ \theta > 0, \ \theta \neq 1.$$

Show that the elasticity of substitution between d_t and e_t equals θ.

[34] Consider the consumption index

$$c_t(d_t, e_t) = \left(\delta^{\frac{1}{\theta}} d_t^{1-\frac{1}{\theta}} + \varepsilon^{\frac{1}{\theta}} e_t^{1-\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \ \delta + \varepsilon = 1, \ \theta > 0, \ \theta \neq 1.$$

Let p_t denote the price of e_t in terms of d_t. Find the optimal choice of d_t and e_t subject to the budget constraint, $z_t = d_t + p_t e_t$. Show that this choice satisfies

$$d_t = \delta p_t^{\theta} \frac{z_t}{\delta p_t^{\theta} + \varepsilon p_t}, \ \ e_t = \varepsilon \frac{z_t}{\delta p_t^{\theta} + \varepsilon p_t}, \ \ c_t = p_t^{\frac{\theta}{1-\theta}} \left(\delta p_t^{\theta} + \varepsilon p_t\right)^{\frac{1}{\theta-1}} z_t.$$

Explain the optimal choice of $\frac{d_t}{e_t}$.

[35] (Price Index) Consider the price index

$$\mathcal{P}_t = \left(\delta + \varepsilon p_t^{1-\theta}\right)^{\frac{1}{1-\theta}}, \ \theta > 0, \ \theta \neq 1.$$

Analyze the behavior of the index for $p_t \to \infty$ depending on whether θ is larger or smaller than unity. Explain.

3 Dynamic Competitive Equilibrium

3.1 Representative Agent and Capital Accumulation

[36] Show that with CRTS, the representative firm's first-order conditions map factor prices into the capital-labor ratio in the firm sector.
[37] The resource constraint reads

$$k_{t+1} = k_t(1-\delta) + f(k_t, 1) - c_t.$$

Rewrite the constraint to establish the following: (i) Gross investment plus consumption equals output; (ii) savings equals net investment.
[38] Suppose that the Inada conditions are satisfied. Show that the Robinson Crusoe problem implies the equilibrium conditions in the decentralized economy.
[39] Consider the Robinson Crusoe economy and assume that preferences are logarithmic; the production function is Cobb-Douglas, $f(k,1) = k^\alpha$; and $\delta = 1$. Show that the associated dynamic programming problem can be written as

$$V(k_\circ) = \max_{k_+ \in [0, k_\circ^\alpha]} \ln(k_\circ^\alpha - k_+) + \beta V(k_+).$$

Show that the policy function g, where $k_+ = g(k_\circ)$, satisfies the following functional equation:

$$\frac{1}{k_\circ^\alpha - g(k_\circ)} = \beta \frac{\alpha g(k_\circ)^{\alpha-1}}{g(k_\circ)^\alpha - g(g(k_\circ))}.$$

Conjecture a solution of the form $g(k_\circ) = \varphi k_\circ^\alpha$. Show that $\varphi = \alpha\beta$.
[40] (Negishi) Consider an infinite-horizon endowment economy populated by n households indexed by i. Households maximize lifetime utility from consumption subject to intertemporal budget constraints. At date t, household i receives endowment w_{it} and consumes c_{it}. At date $t = 0$, households can trade claims on date-s consumption at price q_s.

i. Show that in competitive equilibrium the following first-order condition holds for all i and t:

$$\beta^t u'(c_{it}) = q_t \theta_i,$$

where θ_i denotes the Lagrange multiplier on agent i's intertemporal budget constraint.

ii. Set up the social planner's problem of maximizing the weighted sum of utilities (for positive welfare weights $\{\alpha_i\}_{i=1,\dots,n}$) subject to the resource constraints. Show that the following first-order condition holds for all i and t:

$$\alpha_i \beta^t u'(\hat{c}_{it}) = \lambda_t,$$

where λ_t denotes the Lagrange multiplier on the resource constraint and $\hat{c}_{it}$ denotes consumption in the social planner allocation.

iii. Show that the competitive equilibrium allocation and the social planner allocation coincide if and only if $\alpha_i q_t \theta_i = \lambda_t$.

iv. Assume that the previous condition holds and normalize (without loss of generality) the weights to $\sum_{i=1}^n \alpha_i = 1$. Prove the following relations:

$$\lambda_t = q_t \left(\sum_{j=1}^n \theta_j^{-1} \right)^{-1}, \quad \alpha_i = \frac{\theta_i^{-1}}{\sum_{j=1}^n \theta_j^{-1}}.$$

v. Show that for any competitive equilibrium allocation there exist positive weights $\{\alpha_i\}_{i=1,\dots,n}$ with $\sum_{i=1}^n \alpha_i = 1$ such that the social planner allocation coincides with the competitive equilibrium allocation.

[41] Why does steady-state consumption in the decentralized equilibrium fall short of consumption at the golden-rule capital stock—although the equilibrium allocation is Pareto efficient?

[42] Suppose that at date $t = 0$ the capital stock is at the steady state value associated with the production function f. Describe and explain the equilibrium paths of consumption and capital when (i) productivity permanently increases at date $t = 0$ (f changes to γf with $\gamma > 1$); (ii) productivity permanently increases at date $t = T$ where $T > 0$ and this is anticipated; (iii) productivity increases at date $t = T$ and reverts back to the original value at date $t = s$ where $s > T > 0$ and this is anticipated.

[43] (Saddle Path) Recall the linearized system dynamics around the steady state:

$$\begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = M \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix},$$

where

$$M \equiv \begin{bmatrix} 1 - \frac{c}{\sigma}\beta f_{KK} & f_{KK}\frac{k}{\sigma} \\ -\frac{c}{k} & \beta^{-1} \end{bmatrix}.$$

Let $\psi \equiv 1 + \beta^{-1} - c\frac{\beta}{\sigma} f_{KK}(k, 1)$ and $\chi \equiv -1 + \beta^{-1} + c\frac{\beta}{\sigma} f_{KK}(k, 1)$. Show that the eigenvalues of M are given by ρ_1 and ρ_2, where $0 < \rho_1 < 1 < \rho_2$, $\rho_1\rho_2 = 1/\beta$, and $\rho_1 = \frac{\psi}{2} - \sqrt{\frac{\psi^2}{4} - \beta^{-1}}$. Solve the linearized dynamic system for $\hat{c}_t$ and $\hat{k}_t$. Compute the slope of the saddle path. Compute the slope when $\sigma \to \infty$.

[44] Why do the equilibrium conditions in the decentralized economy give rise to an unstable eigenvalue (in addition to the stable one) in the linearized dynamic system?

[45] (Population Growth) Suppose that the number of household members grows at gross rate ν per period. Derive the Euler equation.

3.2 Overlapping Generations and Capital Accumulation

[46] Show that with CIES preferences and CRTS technology the equilibrium law of motion for capital is given by

$$k_{t+1} = \frac{f_L(k_t, 1)}{1 + \beta^{-\frac{1}{\sigma}} (1 + f_K(k_{t+1}, 1) - \delta)^{1-\frac{1}{\sigma}}}.$$

With logarithmic preferences this simplifies to

$$k_{t+1} = \frac{\beta}{1+\beta} f_L(k_t, 1).$$

[47] Show that with logarithmic preferences and a Cobb-Douglas production function, there exists a unique strictly positive steady-state capital stock, which is globally stable.

[48] (Continuum of Steady States; Hiraguchi (2012)) Consider the baseline model with logarithmic preferences. The production function satisfies $f(K, L) \equiv g(K/L)L$ for some function g. Show that any steady state $k > 0$ must be a solution of the differential equation

$$k = \frac{\beta}{1+\beta} \left(g(k) - kg'(k)\right). \tag{3.1}$$

Define the function g^* as

$$g^*(k) = \begin{cases} 0 & \text{if } k = 0 \\ A\left(2k - k \ln k\right) & \text{if } k \in (0, 1] \\ A\left(2 + \ln k\right) & \text{if } k > 1 \end{cases},$$

where $A = (1+\beta)/\beta$. Observe that g^* is positive, twice continuously differentiable, strictly increasing, concave, and satisfies $g^*(0) = 0$, $\lim_{k \downarrow 0} g^{*\prime}(k) = +\infty$; the production function $f^*(K, L) \equiv g^*(K/L)L$ satisfies the Inada conditions.

Verify that g^* satisfies the differential equation (3.1) on $(0, 1)$. Show that $k_t = k_0$ for all t if $k_0 \in [0, 1]$; and $\lim_{t\to\infty} k_t = 1$ for all $k_0 > 1$. What do you conclude about the nature of the steady state(s)?

[49] (Aggregate Saving and Capital) Suppose that young households save a constant fraction, ϕ, of their labor income. Define savings of young and old households. Derive aggregate saving and capital and their growth rates if cohorts grow at gross rate ν_t and wages at rate γ_t?

[50] (Capital Share and Inefficiency) Consider the case with logarithmic preferences and Cobb-Douglas technology. Let $\beta = 0.98^{25}$ and $\delta = 1 - (1 - 0.05)^{25}$ (these values are meant to be "realistic" for a 25-year period). Suppose that the capital share is either high ($\alpha = 0.3$) or low ($\alpha = 0.2$). In which of the two scenarios is the steady-state capital stock lower? In which is it inefficiently high?

[51] Can the steady-state capital stock in the representative agent model exceed the golden rule capital stock? Argue (i) based on the phase diagram and (ii) based on the household's intertemporal budget constraint. Contrast this with the situation in the overlapping generations model.

[52] Suppose that the world ends at date $t = T$. At this date, the young only live for one period and total resources available for consumption (or investment) consist of the wages of the young and the gross return on savings of the old. What would happen to consumption of young households at date $t = T$ when a social security system were put in place? Would this lead to a Pareto improvement? What does this imply about the possibility of dynamic inefficiency in a finite-horizon economy?

[53] Consider an infinite-horizon overlapping generations model without population growth. Capital does not contribute to production, i.e., $f(K_t, L_t) = L_t$ or $f(k_t, 1) = 1$, but may be stored from one period to the next subject to a depreciation rate δ. Show that the steady state of this economy is inefficient. Suppose that in each period, young households transfer Δ to old households. When does a marginal increase in Δ lead to a Pareto improvement? Is this condition satisfied?

[54] (Population Growth) Derive the resource constraint and the condition for dynamic inefficiency in the model with population growth.

4 Risk

4.1 Consumption, Saving, and Insurance

[55] Prove the law of iterated expectations, $\mathbb{E}[\mathbb{E}[x|y,z]|z] = \mathbb{E}[x|z]$.

[56] Analyze the incomplete-markets savings problem in an economy with three periods and two states of nature at dates $t = 1$ and $t = 2$. Both the return on saving and the wage may be history-contingent. What are the choice variables in this problem? Set up the household's program using the intertemporal budget constraints and form a Lagrangian. Derive the three Euler equations.

[57] Consider an infinite-horizon savings problem with incomplete markets. The return on savings is constant; the wage is random and identically and independently distributed over time. What are the state variables? Write down the Bellman equation and derive the Euler equation.

[58] (Precautionary Saving, Two Periods) Consider a two-period economy with incomplete markets where households face wage risk in the second period. The household maximizes

$$\max_{a_1} u(w_0 - a_1) + \beta \mathbb{E}_0 \left[u(a_1 R + w_1(\epsilon^1)) \right].$$

Let a_1^* denote the optimal savings choice. Let $\bar{a}_1$ denote the optimal savings choice in a modified problem where the second-period wage is deterministic and equal to $\bar{w}_1 \equiv \mathbb{E}_0 \left[w_1(\epsilon^1) \right]$. Show that $\bar{a}_1 < a_1^*$ if u' is (decreasing and) strictly convex.

[59] (Precautionary Saving, CARA Preferences; Caballero (1990)) Consider an infinite-horizon savings problem with risky wages. The dynamic budget constraint at date t reads (dropping histories for legibility) $a_{t+1} = a_t R + w_t - c_t$. The household's felicity function exhibits constant absolute risk aversion, $u(c) = -\theta^{-1} \exp(-\theta\, c)$, $\theta > 0$, and $\beta R = 1$. The wage follows an invertible ARMA process with MA representation

$$w_t = \psi_0 \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-2} + \ldots,$$

where the innovations are i.i.d. We conjecture that the optimal consumption choice satisfies $c_{t+1} = c_t + \gamma + \phi \epsilon_{t+1}$. Verify the conjecture and derive $\gamma > 0$ and $\phi > 0$. [Hint: Disregard the non-negativity constraint on consumption. Show that the Euler equation implies a restriction between γ and ϕ. Derive the intertemporal budget constraint, take expectations, and use the conjecture to derive an expression for c_t. Then use the intertemporal budget constraint to derive ϕ and, from the Euler equation, γ.] Argue that optimal consumption equals permanent income minus a correction term that reflects precautionary savings in the face of wage risk.

[60] (Precautionary Saving, CARA preferences) Consider an infinite-horizon savings problem with risky wages. The dynamic budget constraint at date t reads (dropping histories for legibility) $a_{t+1} = (a_t - c_t)R + w_t$. Wages are i.i.d. and the wage in a period is realized only after the household has made its consumption choice. The Bellman equation reads

$$V(a) = \max_c u(c) + \beta\mathbb{E}\left[V\left(R(a-c)+\tilde{w}\right)\right],$$

where $\tilde{w}$ denotes the random wage. The household's felicity function exhibits constant absolute risk aversion, $u(c) = -\theta^{-1}\exp(-\theta\, c),\ \theta > 0$.

Prove that there exists a constant A such that $V(a) = Au(\rho a)$ solves the Bellman equation, where $\rho \equiv (R-1)/R$. [Hint: Substitute the guess in the Bellman equation and use the change of variable $z = \rho a - c$.] Show that given a, optimal consumption is given by

$$c = \frac{R-1}{R}a - \frac{\ln(\beta R)}{\theta(R-1)} + \frac{u^{-1}\left(\mathbb{E}\left[u\left(\rho\tilde{w}\right)\right]\right)}{R-1}.$$

Show that assets follow a random walk with a drift. Suppose that $\beta R \geq 1$. What is the expected rate of growth of assets? What if $\beta R < 1$?

[61] (Certainty Equivalence) Consider the three-period model with quadratic utility and $\beta R = 1$. Establish the Hall (1978) random walk result, $\mathbb{E}_0[c_1(\epsilon^1) - c_0] = 0$. What is the sign and magnitude of $c_1(\epsilon^1) - c_0$? Is the sign of $c_1(\epsilon^1) - c_0$ the same as the sign of $w_1(\epsilon^1) - \mathbb{E}_0[w_1(\epsilon^1)]$?

[62] (Quadratic Utility, Permanent Income) Consider an infinitely-lived household that chooses $\{c_t(\epsilon^t), a_{t+1}(\epsilon^t)\}_{t\geq 0}$ to maximize

$$\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t u(c_t(\epsilon^t))$$

subject to the dynamic budget constraint, $a_{t+1}(\epsilon^t) = a_t(\epsilon^{t-1})R + w_t(\epsilon^t) - c_t(\epsilon^t)$. (From now on, we suppress histories.) The wage follows an AR(1) process, $w_t = \theta w_{t-1} + (1-\theta)\bar{w} + \nu_t$, where $\theta \in [-1, 1]$ and the innovation ν_t is i.i.d. with mean zero. Suppose that $\beta R = 1$ and that utility is quadratic, $u(c) = -(c-\bar{c})^2$ for some sufficiently large $\bar{c} > 0$. Note that the utility function is increasing (for $c \leq \bar{c}$) and strictly concave. Show that the household's consumption follows a martingale, $\mathbb{E}_t[c_{t+1}] = c_t$. Show that consumption can be expressed as

$$c_t = \frac{R-1}{R}a_t R + \bar{w} + \frac{R-1}{R-\theta}(w_t - \bar{w}).$$

[63] (Complete vs. Incomplete Markets) Consider the model with two periods, two states, and a general return structure (see subsection 4.1.2.1 in the textbook). The Euler equations read

$$\begin{aligned} u'(c_0) &= \beta(R_1^1(h)\eta(h)u'(c_1(h)) + R_1^1(l)\eta(l)u'(c_1(l))), \\ u'(c_0) &= \beta(R_1^2(h)\eta(h)u'(c_1(h)) + R_1^2(l)\eta(l)u'(c_1(l))). \end{aligned}$$

Under what assumptions about the returns $(R_1^1(h), R_1^1(l), R_1^2(h), R_1^2(l))$ are markets incomplete and the first asset dominates the second in return? Under what assumptions are markets complete and neither asset dominates the other?

[64] Consider a two-period model with two states of nature in the second period. There are two assets, with return vectors $[1\ r]$ and $[s\ 1]$. (If $r = s = 0$, then the assets are Arrow securities.) Are markets complete if $r = s = 1$ or if $r = 1/s$?

4.2 Risk Sharing

[65] (Planner Problem) Consider an infinite-horizon endowment economy populated by n households indexed by i. Household i has felicity function u_i and receives a risky endowment $w_{it}(\epsilon^t)$ at date t. The planner attaches weight $\alpha_i > 0$ to i's utility and chooses consumption sequences $\{\{c_{it}(\epsilon^t)\}_i\}_{t\geq 0}$ to maximize

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \sum_{i=1}^{n} \alpha_i u_i(c_{it}(\epsilon^t)) \right]$$

subject to the resource constraint

$$\sum_{i=1}^{n} c_{it}(\epsilon^t) = \sum_{i=1}^{n} w_{it}(\epsilon^t).$$

at any date t and history ϵ^t. Show that the optimal allocation is a function of the aggregate endowment and the Pareto weights only (it does not depend on the distribution of endowments in a specific date and history).

4.3 Uninsurable Labor Income Risk and Capital Accumulation

[66] (Markov Chain) A household faces wage risk and saves in an asset with risk-free return. The wage may take two values, $\bar{w}$ or $\underline{w}$; the transition matrix for the wage is given by

$$P = \begin{bmatrix} \alpha & (1-\alpha) \\ \beta & (1-\beta) \end{bmatrix},$$

where α (β) denotes the probability that next period's wage equals $\bar{w}$ given that the current wage equals $\bar{w}$ ($\underline{w}$); $\alpha, \beta \in (0, 1)$. (This implies, for example, that $\mathbb{E}[w_{t+1}|w_t = \bar{w}] = \alpha\bar{w} + (1-\alpha)\underline{w}$.) The household may choose between carrying assets $\bar{a}$ or $\underline{a}$ into the next period. Suppose that the optimal savings behavior amounts to choosing $\bar{a}$ when the wage equals $\bar{w}$, and $\underline{a}$ when the wage equals $\underline{w}$. What is the dimension of the state space? Derive the equilibrium transition matrix for the state. Derive the invariant distribution of the state. What is the probability that state j is realized two periods after state i?

[67] Write computer code to determine the equilibrium in the model with uninsurable labor income risk and capital accumulation.

5 Asset Returns and Asset Prices

5.1 Euler Equation

[68] Consider the equilibrium in an economy with two periods and risk in the second period. Markets are complete, the number of states is finite, and all states occur with strictly positive probability. Show that the stochastic discount factors of any two households l and n coincide in each history, $m^l(\epsilon^1) = m^n(\epsilon^1)$.

5.2 Excess Returns

[69] Let X be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a non increasing function such that $\mathbb{E}[g(X)]$ exists. Show that $\mathbb{C}\text{ov}\,[X, g(X)] \leq 0$.

[70] Consider an environment with two assets: asset f is risk-free while asset r is risky with a strictly positive excess return. The household is risk averse and its income from sources other than assets is risk-free. Show that the household chooses a strictly positive exposure to the risky asset.

[71] (Portfolio Choice with Borrowing Constraints) Consider the effect of a borrowing constraint on the household's portfolio choice in a two-period setting. The household's program reads

$$\begin{aligned} \max \quad & u(c_0) + \beta\mathbb{E}\left[u(c_1(\epsilon^1))\right] \\ \text{s.t.} \quad & c_0 = w_0 - \sum_i a^i,\ c_1(\epsilon^1) = w_1(\epsilon^1) + \sum_i a_1^i R_1^i(\epsilon^1)\ \ \forall \epsilon^1, \\ & a_1^i \geq \bar{a}^i\ \ \forall i. \end{aligned}$$

i. Derive the Euler equation for asset i. Explain the effect of a binding borrowing constraint.

ii. Show that the asset allocation is distorted by binding borrowing constraints. [Hint: Combine two Euler equations.]

iii. What happens if the borrowing constraint applies to total assets, i.e., $\sum_i a_1^i \geq \bar{a}$?

iv. Rewrite the Euler equation by expressing returns in terms of asset prices and dividends. Interpret the condition.

[72] (CAPM) A household can invest in multiple risky assets and a safe asset; in equilibrium, the latter is in zero net supply. The household values the mean return on its portfolio (positively) and the return variance (negatively). Derive the first-order condition for the portfolio shares invested in the risky assets. Assume further that the household is representative such that in equilibrium, the household holds the market portfolio. Derive the CAPM.

[73] (Wealth Portfolio) Consider the dynamic budget constraint of an investor with no sources of income other than the n assets in the household's portfolio which are indexed by i. Suppressing histories for simplicity, the constraint at date t reads $\sum_{i=1}^{n} a_{t+1}^{i} = \sum_{i=1}^{n} a_t^i R_t^i - c_t$. Define total wealth at date t by $a_t \equiv \sum_{i=1}^{n} a_t^i$. Show that the dynamic budget constraint can be written as $a_{t+1} = a_t R_t^w - c_t$. How do you interpret R_t^w?

[74] (Factor Model) Suppose that consumption $c_t(\epsilon^t)$ depends on n stochastic "factors," denoted $z_t^1(\epsilon^t), \ldots, z_t^n(\epsilon^t)$. That is, in equilibrium, $c_t(\epsilon^t) = C\left(z_t^1(\epsilon^t), \ldots, z_t^n(\epsilon^t)\right)$ for some smooth function C. The factors are stationary; the steady-state values of consumption and the factors are $c, z^1, \ldots, z^n$, respectively.

i. Let $\hat{c}_{t+1} \equiv c_{t+1}/c - 1$ and $\hat{z}_{t+1}^i \equiv z_{t+1}^i/z^i - 1$ denote relative deviations from steady state (for simplicity, we suppress histories). Show that up to a first-order approximation, $\hat{c}_{t+1}$ is a linear combination of $\hat{z}_{t+1}^i, i = 1, \ldots, n$, where the weight for factor i, $\eta^i \equiv (z^i/c)(\partial C(z^1, \ldots, z^n)/\partial z^i)$, is the elasticity of consumption with respect to factor i.

ii. Show that up to a first-order approximation the stochastic discount factor equals

$$m_{t+1} = \beta\left(1 - \sigma \sum_{i=1}^{n} \eta^i (\hat{z}_{t+1}^i - \hat{z}_t^i)\right),$$

where $\sigma \equiv -u''(c)c/u'(c)$.

iii. Show that up to a first-order approximation the rate of return, R_{t+1}, satisfies

$$\mathbb{E}_t[R_{t+1}] = \mu_t\left(1 + \beta\sigma \sum_{i=1}^{n} \eta^i \mathbb{C}\mathrm{ov}_t\left[\hat{z}_{t+1}^i, R_{t+1}\right]\right).$$

Give an interpretation of μ_t.

iv. Show that up to a first-order approximation the excess return satisfies

$$\mathbb{E}_t[R_{t+1}] - R_{t+1}^f = \mu_t\beta\sigma \sum_{i=1}^{n} \eta^i \mathbb{C}\mathrm{ov}_t\left[\hat{z}_{t+1}^i, R_{t+1}\right].$$

5.3 Asset Prices

[75] Show that under risk neutrality, the equilibrium asset price adjusted for dividends and time discounting follows a first-order expectational difference equation. Specify the general form of solutions for this equation. What happens if we exclude bubbles? What happens if, in addition, dividend growth is deterministic and constant, $d_{t+1} = d_t\gamma$ (we suppress histories for simplicity and assume that $\gamma < \beta^{-1}$)?

[76] Consider an environment with two periods and a finite number of states in the second period, $s = 1, 2, \ldots, S$. The market's stochastic discount factor in history s is $m_1(s)$. Derive the equilibrium price of an Arrow security for state $\tilde{s}$; of a risk-free bond; of a stock that pays a contingent return equal to s; and of a call option on that stock with strike price K.

[77] Consider an environment with two periods and two, equally likely states of nature in the second period. There are two assets. Asset 1 pays $(1, 1)$ across the two states and its price equals 1. Asset 2 pays $(2, 0)$ and its price equals 1 as well. Derive the market's stochastic discount factor and characterize equilibrium consumption.

5.4 Term Structure of Interest Rates

[78] Suppose that equilibrium consumption of the representative consumer is i.i.d. Compute the yield to maturity of a k-period zero-coupon risk-free bond that pays off unity. Characterize the slope of the term structure.

[79] (Stochastic Volatility; adapted from Ljungqvist and Sargent (2018, p. 619)) Consider an economy with a representative consumer and an exogenous consumption process. The consumer values the consumption stream $\{C_t\}_{t\geq 0}$ according to

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right],$$

where $u(C) = \frac{C^{1-\gamma}-1}{1-\gamma}$ and where we suppress histories. The logarithm of consumption, $c_t \equiv \ln(C_t)$, follows the stochastic process

$$c_{t+1} = \mu + c_t + \sigma_t\eta_{t+1},$$

where c_0 is an initial condition, $\mu > 0$, $\eta_{t+1} \sim \mathcal{N}(0, 1)$ is i.i.d., and the volatility measure σ_t is the date-t realization of a Markov process. At each date t, σ_t takes one of n possible values, $\left\{\sigma^1, \ldots, \sigma^n\right\}$; the one-period transition probability from σ^i to σ^j is π_{ij}.

i. Define the consumer's stochastic discount factor, m_{t+1}, and express it as a function of σ_t and η_{t+1}.
ii. Solve for $\mathbb{E}_t[m_{t+1}]$. Provide intuition for the result.
iii. Solve for $\mathbb{E}_t[m_{t+1}m_{t+2}]$ conditional on $\sigma_t = \sigma^i$. How does the volatility of σ_{t+1} shape the term structure of interest rates?

5.5 Equilibrium Asset Prices in an Endowment Economy

[80] Derive all competitive equilibrium conditions in the Lucas (1978) tree model. Derive the equilibrium price of a tree when utility is logarithmic.

[81] Consider the previous model and assume that the representative household has logarithmic preferences. Dividends follow a Markov process with two states, d^h (high) or d^l (low), and symmetric transition probabilities of one half. Compute the price of a tree and the risk-free one-period interest rate in each state, p^h, p^l, R^{fh}, and R^{fl}, respectively. Determine the excess return on trees.

[82] (Prescott and Mehra (1980)) Consider the Lucas (1978) tree model with CIES preferences. Dividends grow randomly at gross rate $\gamma \in \{\gamma^1, \gamma^2, \ldots, \gamma^I\}$. The growth rate follows a Markov process; π_{ij} is the probability of growth rate γ^j at date t conditional on growth rate γ^i at date $t-1$. The state in this economy is the contemporaneous dividend and growth rate, (d, γ).

i. Show that the risk-free one-period interest rate in state (d, γ^i) is given by

$$R^{fi} = \left(\beta \sum_{j=1}^{I} \pi_{ij} \cdot (\gamma^j)^{-\sigma}\right)^{-1}.$$

How do β, σ, and the growth rates affect R^{fi}?

ii. Determine the price of a tree, $p(d, \gamma^i)$. [Hint: Use the fact that $p(d, \gamma^i)$ is homogeneous of degree one in d such that we can write $p(d, \gamma^i) = d\tilde{p}(\gamma^i)$ where $\tilde{p}(\gamma^i)$ denotes the price-dividend ratio.]

iii. Suppose that dividend growth is i.i.d., that is, the transition probabilities π_{ij} do not depend on i. Show that the price-dividend ratio and the risk-free interest rate are constant. Show that the excess return on a tree is strictly positive and equals

$$-\frac{\mathbb{C}\text{ov}[\gamma^{-\sigma}, \gamma]}{\beta \mathbb{E}[\gamma^{1-\sigma}]\mathbb{E}[\gamma^{-\sigma}]}.$$

Approximate γ around $\mathbb{E}[\gamma]$ to the first order and express the excess return as

$$\frac{\sigma}{\beta} \frac{\mathbb{V}\text{ar}[\gamma]}{\mathbb{E}[\gamma]^2} \mathbb{E}[\gamma]^{\sigma}.$$

6 Labor Supply, Growth, and Business Cycles

6.1 Goods versus Leisure Consumption

[83] Consider a static model with $u(c, x) = \ln(c) + v(x)$. How does x respond to a wage change when the wage is the only source of income? Explain your result with reference to income and substitution effects.

[84] Consider a static model with labor supply choice at the extensive margin. Household preferences are given by $u(c, x) = \ln(c) - v(\ell)$ where $v(0) = 0$ and ℓ denotes labor supply of a worker. Compute the extensive-margin labor supply (Frisch) elasticity. Compute the elasticity of aggregate labor supply (the share of household members working) with respect to the wage when labor income is the only source of income.

[85] Consider a model with two periods. The felicity function is of the form

$$u(c, x) = \xi \ln(c) + \frac{x^{1-\varphi} - 1}{1 - \varphi},\ \xi > 0, \varphi > 0, \varphi \neq 1.$$

Letting $w \equiv w_1/w_0$, combine the optimality conditions to find

$$\xi x_0^{\varphi}(1 + \beta) + x_0 \left(1 + \frac{\left(\frac{\beta R_1}{w}\right)^{\frac{1}{\varphi}} w}{R_1}\right) = 1 + \frac{w}{R_1}.$$

What happens when $\varphi \to 1$?

[86] Consider the model of wage inequality and risk sharing analyzed in subsection 6.1.3 in the textbook.

i. Set up the planner's problem for date $t = 1$ and derive the optimality conditions. Show that consumption is equalized within a group whereas labor supply varies across individuals.
ii. Derive the expressions for equilibrium $\ln(c_{gi1})$ and $\ln(\ell_{gi1})$.
iii. Compute the covariance between $\ln(c_{gi1})$ and $\ln(w_{gi1})$, and between $\ln(c_{gi1})$ and $\ln(\ell_{gi1})$.

6.2 Growth

[87] (Balanced Growth Path Restrictions on Growth Rates) Show that

$$k_T(\nu\gamma_k - 1 + \delta) = y_T(\gamma_y/\gamma_k)^{t-T} - c_T(\gamma_c/\gamma_k)^{t-T}$$

and $y_T, k_T, c_T > 0$ implies $\gamma_y = \gamma_k = \gamma_c$.

[88] Consider a Cobb-Douglas production function with labor augmenting ("Harrod-neutral") technical progress. Show that this production function can be reformulated to feature capital augmenting ("Solow-neutral") technical progress or technical progress that augments both production factors ("Hicks-neutral").

[89] (Balanced Growth Path Restrictions on Preferences) Show that existence of a balanced growth path equilibrium imposes restrictions on preferences, namely

$$u(c, x) = \begin{cases} \frac{c^{1-\sigma}v(x)}{1-\sigma}, & \sigma > 0,\ \sigma \neq 1 \\ \ln(c) + v(x), & \sigma = 1 \end{cases} .$$

[90] (Euler Equation with Detrended Variables) Let a bar denote detrended (by population and technological growth) variables. Show that with CIES preferences, the first-order condition

$$\nu\gamma u_c(\bar{c}_t, x_t) = \beta^\star(1 - \delta + f_K(\bar{k}_{t+1}, 1 - x_{t+1}))u_c(\bar{c}_{t+1}, x_{t+1})$$

is equivalent to the standard Euler equation,

$$u_c(c_t, x_t) = \beta(1 - \delta + f_K(\bar{k}_{t+1}, 1 - x_{t+1}))u_c(c_{t+1}, x_{t+1}) = \beta R_{t+1} u_c(c_{t+1}, x_{t+1}).$$

[91] (Solow (1956) Model) Show that with logarithmic preferences, full depreciation after one period, Cobb-Douglas technology, and no labor-leisure choice, the neoclassical growth model reduces to the Solow (1956) model with a constant consumption-output ratio.

[92] Consider the balanced growth path in an overlapping generations economy. The cohort size grows at gross rate ν and productivity grows at gross rate γ per period. Show that the gross interest rate satisfies

$$R = \beta^{-1}\psi^{\sigma},$$

where ψ denotes old- relative to young-age consumption of a cohort. Compare this to the interest rate in an economy with infinitely lived homogenous households.

[93] (Growth Rate in *Ak* Model) Show that in the equilibrium of the *Ak* model, capital and consumption grow at the same constant rate.

[94] (Two Sector Model) Consider the model with two sectors, one producing consumption goods with a Cobb-Douglas technology and the other producing capital with an *Ak* technology. Interpret the equilibrium condition

$$u'(c_t)\alpha(k_t^c)^{\alpha-1} = u'(c_{t+1})\alpha(k_{t+1}^c)^{\alpha-1}\beta(1 - \delta + A).$$

[95] (Two Sector Model, Relative Price Dynamics) Consider the model with two sectors, one producing consumption goods with a Cobb-Douglas technology and the other producing capital with an Ak technology. Derive the equilibrium conditions with CIES preferences when the economy grows along a balanced growth path. State the conditions for positive growth and bounded utility along a balanced growth path. Show that the price of consumption relative to investment goods increases.
[96] (Externalities, Equilibrium Allocation) Consider the endogenous growth model with externalities. Show that the programs of price taking representative households and firms yield the same equilibrium conditions as the program

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & k_{t+1} = k_t(1-\delta) + A_t k_t^{\alpha} - c_t, \;\; k_0 \text{ given}, \;\; k_{t+1} \geq 0. \end{aligned}$$

[97] (Externalities, Social Planner Allocation) Consider the endogenous growth model with externalities. Characterize the social planner allocation.

6.3 Business Cycles

[98] Consider the real business cycle model.

i. Write down the planner's program using sequence notation and derive the planner's optimality conditions.
ii. Write down the household's program in the decentralized equilibrium using sequence notation and derive all equilibrium conditions.
iii. Show that the conditions in i. and ii. coincide.

[99] Consider the real business cycle model and let $f(K, L) = K^{\alpha} L^{1-\alpha}$, $\delta = 1$, and $u(c, x) = \ln(c) + h(x)$. Guess and verify that in equilibrium, consumption and next period's capital stock are proportional to output and leisure is constant.
[100] Write computer code to solve for the equilibrium in the real business cycle model for arbitrary preferences and technology.
[101] (Sunspot Model) Consider the model with sunspot-driven business cycles.

i. Derive (for CIES preferences) the balanced-growth-path equilibrium conditions given in the text,

$$\begin{aligned} \nu\mu\bar{k}_{t+1} &= \bar{k}_t(1-\delta) + \mu^{-t} f(\bar{k}_t\mu^t, \ell_t\gamma^t)^{1+\chi} - \bar{c}_t, \\ u_x(\bar{c}_t, x_t)\mu^t &= f_L(\bar{k}_t\mu^t, \ell_t\gamma^t)\gamma^t f(\bar{k}_t\mu^t, \ell_t\gamma^t)^{\chi} u_c(\bar{c}_t, x_t), \\ \nu\mu u_c(\bar{c}_t, x_t) &= \beta^{\star}\mathbb{E}_t[\{1-\delta + f_K(\bar{k}_{t+1}\mu^{t+1}, \ell_{t+1}\gamma^{t+1}) f(\bar{k}_{t+1}\mu^{t+1}, \ell_{t+1}\gamma^{t+1})^{\chi}\} u_c(\bar{c}_{t+1}, x_{t+1})]. \end{aligned}$$

ii. Show that with a Cobb-Douglas production function and logarithmic utility of consumption a balanced growth path exists. Derive the growth rate given in the text,

$$\mu = \gamma^{\frac{(1-\alpha)(1+\chi)}{1-\alpha(1+\chi)}}.$$

iii. Repeat i. and ii. under the assumption that the externality derives from the aggregate factor inputs rather than from per-capita inputs. That is, assume that $A_t = f(k_t\nu^t, \ell_t\gamma^t\nu^t)^\chi$ rather than $A_t = f(k_t, \ell_t\gamma^t)^\chi$.

7 The Open Economy

7.1 Current Account and Net Foreign Assets

[102] Consider an endowment economy and suppose that output grows at the gross rate γ. What trade balance as a share of output is needed to stabilize the net-foreign-assets-to-output ratio? By how much must the steady-state trade balance as a share of output increase if the country's foreign debt equals 50 percent of output, output grows by 2 percent per period, and the net interest rate rises from 5 to 10 percent?
[103] Consider an open economy with production. Show that $ca_t = nfa_{t+1} - nfa_t$.
[104] Consider a production economy and assume that the international gross interest rate is constant, $R_t \equiv R$. Define the representative agent's permanent wage at date t, $\tilde{w}_t$, as the value that satisfies the condition $\tilde{w}_t \sum_{s=0}^{\infty} R^{-s} = \sum_{s=0}^{\infty} R^{-s} w_{t+s}$, where w_t denotes the wage at date t. Show that, absent profits,

$$k_t R + \tilde{w}_t \sum_{s=0}^{\infty} R^{-s} = \sum_{s=0}^{\infty} R^{-s} \left(f(k_{t+s}, 1) - k_{t+s+1} + k_{t+s}(1-\delta) \right).$$

[105] Consider a small open economy without production; the representative household has CIES preferences. The initial level of net foreign assets (before interest payments) equals nfa_0.

i. Suppose that the wage, w, and the international gross interest rate, R, are constant over time with $\beta R = 1$. Show that equilibrium net foreign assets and consumption are constant as well, at level nfa_0 and $c \equiv (R-1)nfa_0 + w$, respectively.
ii. Suppose that the household consumes c at date $t = 0$ and carries assets $nfa_1 = nfa_0$ into the subsequent period. At date $t = 1$, the household learns that the interest rate at date $t = 1$ equals R_1 rather than R and the wage w_1 rather than w; in all future periods the interest rate equals R and the wage w. What is the difference between consumption at date $t = 1$, $\bar{c}$, and c?

[106] Consider a small open, infinite horizon, endowment economy. The representative household has CIES preferences and the international interest rate is constant. Characterize equilibrium consumption when $\beta R \neq 1$. How does the current account and net foreign assets evolve?

7.2 Real Exchange Rate

[107] Derive the household optimality condition $u'(c_t)/\mathcal{P}_t = \beta R_{t+1} u'(c_{t+1})/\mathcal{P}_{t+1}$.

[108] Derive the conditions $\hat{A}_t^T = \hat{w}_t \sigma_t^T$ and $\hat{p}_t + \hat{A}_t^N = \hat{w}_t \sigma_t^N$ given in the text. Deduce the relation

$$\hat{p}_t = \hat{A}_t^T \frac{\sigma_t^N}{\sigma_t^T} - \hat{A}_t^N.$$

[109] Derive the conditions $\hat{p}_t = \frac{dr_t}{p_t f^N}(k_t^N - k_t^T) = \hat{r}_t(\sigma_t^T - \sigma_t^N)/\sigma_t^T$ given in the text.

7.3 Gains From Trade

[110] Suppose that capital market frictions prevent a small open economy from accumulating assets such that $a_t = a_{t+1}$. Can the country still exploit gains from trade? Suppose next that capital market frictions prevent a small open economy from swapping physical capital against financial assets. Can the country still exploit gains from trade?

7.4 International Risk Sharing

[111] Show that with complete financial markets,

$$\beta^s \frac{u'(c_{t+s}(\epsilon^{t+s}))/\mathcal{P}_{t+s}(\epsilon^{t+s})}{u'(c_t(\epsilon^t))/\mathcal{P}_t(\epsilon^t)}$$

is equalized internationally. Why is

$$\beta^s \frac{u'(c_{t+s}(\epsilon^{t+s}))}{u'(c_t(\epsilon^t))}$$

generally *not* equalized internationally?

8 Frictions

8.1 Capital Adjustment Frictions

[112] Consider the equilibrium conditions in the model with convex capital adjustment costs and $\delta = 0$. Show that absent adjustment costs, $q_t = 1$ and the usual first-order condition characterizing a firm's demand for capital holds.

[113] Consider the model with convex capital adjustment costs and $\delta = 0$. Assume that the adjustment cost function takes the form $\Xi(I, K) = \xi I^2/(2K)$ and that labor supply is fixed and equals L. Construct the phase diagram in (K, q) space.

[114] Consider the model with convex capital adjustment costs. Why is a larger stock of installed capital associated with a faster rise in its shadow price?

[115] Consider the model with convex capital adjustment costs. At date $t = 0$, a firm with steady-state capital stock installed learns about a temporary interest rate increase between dates $t = 0$ and $t = T$. Using the phase diagram, analyze the effect on the firm's capital stock and q.

[116] Carefully explain figure 8.2.

[117] Consider a variant of the two-period model with irreversible investment. We now allow for investment to be liquidated after a period, generating revenue $L \in [0, RI]$. Let $\rho^l = 0$; this implies that it is optimal to liquidate first-period investment in the second period if the low state is realized at date $t = 1$. Characterize the optimal investment policy as a function of L.

8.2 Labor Market Frictions

[118] (Duration of Job Search) Suppose that conditional on having started job search at date $t = 0$ and not having found a job until the beginning of date $t = s$, $s \geq 0$, the probability of finding a job at date $t = s$ equals α. What is the expected duration until finding a job?

[119] Consider the model with matching frictions in the labor market. Derive the equilibrium wage,

$$w_t = \gamma c_t \left(1 - \frac{1-s}{\theta_t \eta(\theta_t)}\right) + \phi \left\{ f_y(A_t, k_t, y_t(1-\nu_t)) \left(1 + \frac{1-s}{\eta(\theta_t)}\right) - \gamma c_t \left(1 - \frac{1-s}{\theta_t \eta(\theta_t)}\right) \right\}.$$

[120] Consider the model with matching frictions in the labor market. Describe the opportunity cost of a marginal hire for the firm and for the household as they appear in the condition for the equilibrium wage.

[121] Consider the model with matching frictions in the labor market. Show that the allocation in the decentralized equilibrium and the social planner allocation coincide if the Hosios condition is satisfied.

[122] Consider the model with matching frictions in the labor market. Suppose that the matching function has the Cobb-Douglas form, $g(z_t, v_t) = z_t^{\varphi} v_t^{1-\varphi}$. Derive the elasticity of the matching function as well as the elasticities of the matching probabilities for job seekers and vacancies. Explain your findings and discuss the implications for the bargaining weight, ϕ, under the Hosios condition.

[123] Consider the model with matching frictions in the labor market, without capital. Solve for the steady state or describe how to solve for it.

[124] Consider the model with matching frictions in the labor market, without capital and leisure. Prove that the Beveridge curve is decreasing and convex in (v, z) space.

[125] (McCall (1970) Search Model) Consider an unemployed worker who searches for a job. As long as the worker is unemployed she receives one job offer in each period and chooses between two options: Either she accepts the job and keeps it with the fixed wage forever; or she rejects, receives an unemployment benefit, b, and receives a new job offer with a new wage in the subsequent period. The offered wage, w, is independent over time and drawn from a distribution H that satisfies $\text{prob}(w \leq x) = H(x)$, $H(0) = 0$, and $H(\bar{w}) = 1$. The worker maximizes $\sum_{t=0}^{\infty} \beta^t c_t$ where $c_t = b$ when unemployed and $c_t = w$ when employed at wage w. Characterize the minimum wage offer (the reservation wage) that induces the unemployed worker to accept a job offer. How does b affect this reservation wage?

8.3 Financial Frictions

[126] Consider the costly state verification model. The mechanism design problem is given by

$$\begin{aligned}
\max_{\rho^h,\rho^l,\psi^h,\psi^l,\theta^h,\theta^l} \quad & \sum_{s=h,l} \eta^s \left(\theta^s \psi^s + (1-\theta^s)\rho^s\right) \\
\text{s.t.} \quad & \sum_{s=h,l} \eta^s \left(R^s - \theta^s(\psi^s + \gamma) - (1-\theta^s)\rho^s\right) \geq R(z - nw), \\
& \rho^h, \rho^l, \psi^h, \psi^l \geq 0, \\
& \theta^h \psi^h + (1-\theta^h)\rho^h \geq \theta^l 0 + (1-\theta^l)\left(\rho^l + R^h - R^l\right), \\
& 0 \leq \theta^h, \theta^l \leq 1.
\end{aligned}$$

Show that this problem is equivalent to the program

$$\begin{aligned}\min_{\rho^h,\theta^l} \quad & \gamma\eta^l\theta^l \\ \text{s.t.} \quad & \eta^h(R^h-\rho^h)+\eta^l(R^l-\theta^l\gamma)=R(z-nw), \\ & \rho^h \geq (1-\theta^l)(R^h-R^l), \\ & 0 \leq \theta^l \leq 1.\end{aligned}$$

[127] Consider the costly state verification model. Assume that the project return in the low state falls short of the required return on the loan, $R^l < R(z-nw)$. Show that there exists no feasible mechanism with $\theta^l = 0$.

[128] Consider the costly state verification model. Assume that the project return in the low state falls short of the required return on the loan, $R^l < R(z-nw)$. Show that in the optimal mechanism both the participation constraint and the incentive constraint are satisfied with equality.

[129] Consider the costly state verification model. Plot the participation constraint and the incentive constraint (with equality) in (θ^l, ρ^h) space. Analyze the effect of an increase in R^h, starting from $R^h = R^l$.

[130] Consider the model with collateral and asset prices. Derive the effect of a change in the asset price on net worth.

[131] Consider the model with collateral and asset prices. What is the gross growth rate of the investor's net worth if the investor borrows the maximum amount? How does this growth rate compare to R?

[132] Consider the steady state in the general equilibrium model with collateral and asset prices and suppose that the investor is collateral constrained. Show that the investor chooses maximal leverage when $e > y(\delta^{-1}-1)$ (using the fact that $\delta R < 1$).

[133] Consider the general equilibrium model with collateral and asset prices. Derive equations (8.19)–(8.21) in the textbook.

[134] Consider the model with financial constraints and pecuniary externalities. Suppose that the financial constraints do not bind in equilibrium and markets are complete. Use the solution of the social planner problem to show that the pecuniary externalities have no first-order welfare consequences in this case.

[135] Consider a two-period model with a continuum of mass one of homogeneous investors who maximize expected second-period consumption. Investor i has an endowment w, may borrow or save at the gross interest rate R, and may invest in capital, k^i. The return on capital can be high, ρ^h, or low, ρ^l. The expected return is $\bar{\rho}$ with $\bar{\rho} > R > \rho^l$. The investor faces the borrowing constraint $k^i\rho^l \geq \phi \cdot (\text{maturing debt}) = \phi \cdot (\max[0, k^i - w]R)$. The investor's problem reads

$$\max_{k^i} k^i\bar{\rho} + (w-k^i)R \ \text{ s.t. } \ k^i\rho^l \geq \phi \max[0, k^i - w]R.$$

The aggregate capital stock, k, increases ϕ, for instance because lenders fear a fire sale in the low return state: $\phi = \varphi(k)$ with $\varphi'(k) > 0, \varphi''(k) > 0$. Investors take k as given.

i. Derive the first-order condition of the representative investor. Characterize the equilibrium allocation.
ii. Derive the first-order condition of a social planner that acts on behalf of investors (and faces the same borrowing constraint). Characterize the social planner allocation.
iii. Discuss the pecuniary externality and compare the two allocations.
iv. Suppose that there is a third asset which pays the safe gross return z, $\bar{\rho} > z > R$. Characterize the equilibrium allocation and the social planner allocation.

[136] A continuum of homogeneous investors of mass one maximizes $f(x^i, y^i)$ subject to the constraint that $g(x, y)$ be larger than $h(x^i, y^i)$. Here, i indexes an individual investor and function f incorporates the budget constraint. In equilibrium $x = x^i$ and $y = y^i$. Characterize the equilibrium allocation and the allocation chosen by a social planner.

9 Money

9.1 Unit of Account

[137] (Fisher Equation) Consider a real bond with risk-free gross real rate of return, $R^f_{t+1}(\epsilon^t)$, and a nominal bond with risk-free gross nominal rate of return, $I^f_{t+1}(\epsilon^t)$. Derive the Fisher equation. What happens when there is no inflation risk? What happens when inflation positively covaries with consumption?

[138] (Term Structure of Nominal Interest Rates) Denote by $q^{fs}_t(\epsilon^t)$ and $Q^{fs}_t(\epsilon^t)$, respectively, the prices at date t, history ϵ^t of a risk-free real and nominal bond; the former (latter) is quoted in units of consumption (money) and delivers one unit of consumption (money) at date $t + s$.

i. Show that the prices satisfy

$$Q^{fs}_t(\epsilon^t) = q^{fs}_t(\epsilon^t)\mathbb{E}_t\left[\left(\frac{P_{t+s}(\epsilon^{t+s})}{P_t(\epsilon^t)}\right)^{-1}\right] + \beta^s\mathbb{C}\text{ov}_t\left[\frac{u'(c_{t+s}(\epsilon^{t+s}))}{u'(c_t(\epsilon^t))}, \left(\frac{P_{t+s}(\epsilon^{t+s})}{P_t(\epsilon^t)}\right)^{-1}\right].$$

ii. Suppose that there is no inflation risk and the price level grows at the constant rate π. Show that the relative spread, $(I^{fs}_{t+1} - R^{fs}_{t+1})/R^{fs}_{t+1}$, equals $(1 + \pi)^s - 1$.

iii. Show that ex post, the gross real rate of return on the nominal bond is higher than on the real bond if

$$\left(\frac{P_{t+s}(\epsilon^{t+s})}{P_t(\epsilon^t)}\right)^{-1} - \mathbb{E}_t\left[\left(\frac{P_{t+s}(\epsilon^{t+s})}{P_t(\epsilon^t)}\right)^{-1}\right] > \left(q^{fs}_t(\epsilon^t)\right)^{-1}\mathbb{C}\text{ov}_t\left[m_{t+1}\ldots m_{t+s}, \left(\frac{P_{t+s}(\epsilon^{t+s})}{P_t(\epsilon^t)}\right)^{-1}\right].$$

Explain the factors that can generate an ex-post excess return.

[139] (Factor Model) Recall the factor model from the exercise in section 5.2. As shown there the stochastic discount factor can be approximated as

$$m_{t+1}(\epsilon^{t+1}) = \beta\left(1 - \sigma\sum_{i=1}^{n}\eta^i(\hat{z}^i_{t+1}(\epsilon^{t+1}) - \hat{z}^i_t(\epsilon^t))\right).$$

Suppose that there exists a factor $z^\star$ (e.g., commodity prices) which follows an autoregressive process and drives inflation. Specifically, assume that $\Pi^{-1}_{t+1} = 1 - \pi - \gamma\hat{z}^\star_{t+1}$ where

$z^\star_{t+1} = \rho^\star z^\star_t + (1-\rho^\star)z^\star + \xi^\star_{t+1}$; π and γ are positive constants and $\hat{z}^\star_{t+1} \equiv z^\star_{t+1}/z^\star - 1$. Show that

$$Q^{f1}_t(\epsilon^t) = q^{f1}_t(\epsilon^t)\left(1 - \pi - \gamma\rho^\star \hat{z}^\star_t\right) + \beta\sigma\gamma \sum_{i=1}^{n} \eta^i \mathbb{C}\text{ov}_t\left[\hat{z}^i_{t+1}, \hat{z}^\star_{t+1}\right].$$

Interpret.

[140] (Monetary Model of the Exchange Rate (Frenkel, 1976; Mussa, 1976)) Assume that the demand for real balances satisfies

$$\ln\left(\frac{M_{t+1}(\epsilon^t)}{P_t(\epsilon^t)}\right) = \phi\ln(c_t(\epsilon^t)) - \gamma i_{t+1}(\epsilon^t),$$

where $\phi, \gamma > 0$.

i. Positing purchasing power parity and letting $\psi_t(\epsilon^t) \equiv \ln(M_{t+1}(\epsilon^t)) - \ln(P^\star_t(\epsilon^t)) - \phi\ln(c_t(\epsilon^t)) + \gamma i^\star_{t+1}(\epsilon^t)$ represent "fundamentals," show that

$$\psi_t(\epsilon^t) = \ln(E_t(\epsilon^t)) - \gamma(i_{t+1}(\epsilon^t) - i^\star_{t+1}(\epsilon^t)).$$

ii. Log-linearize the uncovered interest parity condition and use it to derive the following representation of the exchange rate (assuming stationarity):

$$\begin{aligned}\ln(E_t(\epsilon^t)) &\approx \frac{1}{1+\gamma}\psi_t(\epsilon^t) + \frac{\gamma}{1+\gamma}\mathbb{E}_t[\ln(E_{t+1}(\epsilon^{t+1}))] \\ &= \frac{1}{1+\gamma}\sum_{j=0}^{\infty}\left(\frac{\gamma}{1+\gamma}\right)^j \mathbb{E}_t[\psi_{t+j}(\epsilon^{t+j})].\end{aligned}$$

Interpret this result.

9.2 Store of Value

[141] Consider the model with borrowing constrained, infinitely lived households. Suppose that markets are complete. Derive c^e and c^o. Show that the equilibrium allocation is Pareto optimal.

[142] Consider the model with borrowing constrained, infinitely lived households. Suppose that households may not borrow and that a bubble is traded. Derive c^e_t in even and odd periods.

[143] Consider the model with borrowing constrained, infinitely lived households. Suppose that households may not borrow and that a bubble is traded. What is wealth of a household in equilibrium?

9.3 Medium of Exchange

[144] Consider the model with money in the utility function and abstract for simplicity from risk. Based on the first-order condition

$$\frac{u_z(c_t, z_{t+1})}{u_c(c_t, z_{t+1})} = \frac{i_{t+1}}{I_{t+1}},$$

derive the elasticity of money demand with respect to consumption. Show that this elasticity is positive when u is additively separable and strictly concave.

[145] Suppose that purchasing the quantity of consumption goods c requires “shopping time” $\ell(c, z)$ where z denotes real balances and $\ell_c > 0$, $\ell_z \leq 0$. Suppose in addition that the household values consumption and time not spent shopping, $x = 1 - \ell$; the utility function $u(c, x)$ is strictly increasing and concave in both arguments. When are consumption and real balances complements?

[146] Consider the model with money in the utility function.

i. Derive the asset pricing condition

$$\frac{u_c(c_t(\epsilon^t), z_{t+1}(\epsilon^t))}{P_t(\epsilon^t)} = \sum_{s=0}^{\infty} \beta^s \mathbb{E}_t \left[\frac{u_z(c_{t+s}(\epsilon^{t+s}), z_{t+s+1}(\epsilon^{t+s}))}{P_{t+s}(\epsilon^{t+s})} \right].$$

ii. Suppose that the marginal utility of real balances is strictly positive until period $k-1 > t$ and equals zero thereafter. Show that $P_k(\epsilon^k)$ is infinite. Does this imply that $P_t(\epsilon^t)$ is infinite?

[147] Consider the model with money in the utility function. Suppose that

$$u(c, z) = \frac{c^{1-\sigma}}{1-\sigma} + \gamma \frac{z^{1-\rho}}{1-\rho}.$$

Derive the elasticity of money demand with respect to i/I and with respect to consumption.

[148] Consider the model with a cash-in-advance constraint, $M_{t+1}(\epsilon^t)/P_t(\epsilon^t) \geq c_t(\epsilon^t)$. Derive condition (9.6) in the textbook.

[149] Consider the model with a cash-in-advance constraint subject to the modified timing convention, $M_t(\epsilon^{t-1})/P_t(\epsilon^t) \geq c_t(\epsilon^t)$.

i. Derive the condition given in the text,

$$\beta \mathbb{E}_t \left[\frac{\xi_{t+1}(\epsilon^{t+1})}{\Pi_{t+1}(\epsilon^{t+1}) \lambda_t(\epsilon^t)} \right] = \frac{i_{t+1}(\epsilon^t)}{I_{t+1}(\epsilon^t)}.$$

ii. Derive the asset pricing representation of the price of money, $1/P_t(\epsilon^t)$. How does the modified timing assumption affect the asset price?

9.4 The Price of Money

[150] Suppose a central bank issues non-redeemable, non-interest bearing fiat money at a strictly positive price in exchange for dividend yielding assets. The price is strictly positive because the money generates liquidity benefits; that is, it lubricates trade. Describe the central bank's balance sheet if the central bank values the outstanding money (i) at market prices or (ii) at cost. What changes when the central bank issues the money as a transfer ("helicopter money")?

10 Price Setting and Price Rigidity

10.1 Price Setting

[151] Derive the demand curve of a final good producer,

$$y_t(j,\epsilon^t) = \left(\frac{P_t(j,\epsilon^t)}{P_t(\epsilon^t)}\right)^{-\varepsilon} y_t(\epsilon^t).$$

[152] Derive the final good price index,

$$P_t(\epsilon^t) = \left(\int_0^1 P_t(j,\epsilon^t)^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}.$$

[153] Derive the first-order condition in the dynamic price setting problem of an intermediate good producer,

$$P_t(j,\epsilon^t) = \psi_t(\epsilon^t) P_t(\epsilon^t) \frac{\varepsilon}{\varepsilon - 1},$$

and linearize it about the steady state.

[154] Show that equilibrium production is inefficiently low, reflecting a markup.

10.2 Staggered Price Setting

[155] Consider a firm that changes its price with probability $1-\theta$ in each period. What is the expected duration of a newly set price?

[156] Show that $\Delta_t(\epsilon^t) \equiv \int_0^1 \left(\frac{P_t(j,\epsilon^t)}{P_t(\epsilon^t)}\right)^{-\varepsilon} dj \geq 1$ up to a second-order approximation. (Hint: Use the definition of the aggregate price index, $1 = \int_0^1 \left(\frac{P_t(j,\epsilon^t)}{P_t(\epsilon^t)}\right)^{1-\varepsilon} dj$.)

[157] (Staggered Wage Setting) Consider a household with a continuum of members indexed by $i \in [0,1]$. Member i supplies a specific differentiated labor service, $\ell_t^i(\epsilon^t)$. The household shares the labor income (all members consume the same) and maximizes utility,

$$U(c_t(\epsilon^t), \{\ell_t^i(\epsilon^t)\}_{i=0}^1) = \frac{c_t(\epsilon^t)^{1-\sigma} - 1}{1-\sigma} - \int_0^1 \frac{\ell_t^i(\epsilon^t)^{1+\phi}}{1+\phi} di.$$

Each member sets the wage for its labor service and faces price setting frictions akin to the frictions faced by intermediate good producers in the model analyzed in the textbook: With probability θ the member cannot reset the wage in the subsequent period and at all times, the member is committed to supplying whatever labor is demanded at the set wage. When member i resets its nominal wage at date t then the wage choice, $W_t(i,\epsilon^t)$, maximizes

$$\mathbb{E}_t \sum_{h=0}^{\infty} \theta^h \beta^h \left(c_{t+h}(\epsilon^{t+h})^{-\sigma} \frac{W_t(i,\epsilon^t)}{P_{t+h}(\epsilon^{t+h})} \ell^i_{t+h}(\epsilon^{t+h}) - \frac{\ell^i_{t+h}(\epsilon^{t+h})^{1+\phi}}{1+\phi} \right)$$

subject to the labor demand

$$\ell^i_{t+h}(\epsilon^{t+h}) = \left(\frac{W_t(i,\epsilon^t)}{W_{t+h}(\epsilon^{t+h})} \right)^{-\varepsilon} \ell_{t+h}(\epsilon^{t+h}).$$

Here, $W_t(\epsilon^t)$ and $\ell_t(\epsilon^t)$ denote the aggregate wage and labor supply, respectively, at date t.

i. Explain this program.
ii. Derive the first-order condition.
iii. What wage does the household member set when wages can be readjusted in every period, $\theta = 0$?

[158] (Price Adjustment Costs (Rotemberg, 1982)) Suppose that unlike in the model analyzed in the textbook, intermediate good producers can adjust their prices in every period but price adjustment is costly: Changing the price at date t from $P_{t-1}(j)$ to $P_t(j)$ costs producer j the resources

$$\frac{\phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 y_t,$$

where $\phi > 0$ and y_t denotes aggregate production. (To improve legibility we suppress histories throughout.) Demand for producer j's variety is given by

$$y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon} y_t,$$

where P_t denotes the aggregate price index.

i. Let $n_{t,t+k}$ denote the nominal stochastic discount factor of intermediate good firms between dates t and $t+k$ and let $\Psi(y_{t+k}(j))$ denote the real cost of producer j to produce $y_{t+k}(j)$ at date $t+k$. Show that the objective function of producer j at date t can be written as

$$\mathbb{E}_t \sum_{k=0}^{\infty} n_{t,t+k} \left\{ P_{t+k}(j) y_{t+k}(j) - P_{t+k}\Psi(y_{t+k}(j)) - P_{t+k}\frac{\phi}{2} \left(\frac{P_{t+k}(j)}{P_{t+k-1}(j)} - 1 \right)^2 y_{t+k} \right\}$$

subject to the demand function.

ii. Derive the first-order condition with respect to $P_t(j)$. Explain the condition.

iii. In equilibrium all producers behave in the same way. Use this fact to simplify the first-order condition to

$$(1-\varepsilon)y_t + \varepsilon y_t \psi(y_t) - \phi\Pi_t(\Pi_t - 1)y_t + \mathbb{E}_t\left[n_{t,t+1}\Pi_{t+1}^2\phi(\Pi_{t+1}-1)y_{t+1}\right] = 0$$

with $\Pi_t \equiv P_t/P_{t-1}$ and ψ the marginal cost function.

iv. Linearize this first-order condition around the zero inflation steady state to derive the Phillips curve relationship

$$\pi_t = \frac{\varepsilon-1}{\phi}\hat{\psi}_t + \beta\mathbb{E}_t[\pi_{t+1}].$$

v. The production function is given by $y_t = a_t l_t^\alpha$ where a_t denotes productivity and l_t labor supply. Use the production function, the resource constraint, and the first-order condition of households that relates labor supply to the wage and (the marginal utility of) consumption, $\eta\hat{l}_t = \hat{w}_t - \hat{c}_t$, to show that $\hat{\psi}_t = (\eta+1)(\hat{y}_t - \hat{a}_t)/\alpha$.

vi. Conclude that the Phillips curve can be written as

$$\pi_t = \frac{\varepsilon-1}{\phi}\frac{\eta+1}{\alpha}(\hat{y}_t - \hat{y}_t^n) + \beta\mathbb{E}_t[\pi_{t+1}],$$

where y_t^n denotes the natural level of output.

10.3 Price Rigidity in General Equilibrium

[159] (Sticky-Information Phillips Curve (Mankiw and Reis, 2002)) Suppose that intermediate good producers choose a price path rather than setting a fixed price as considered in the textbook. That is, a producer who adjusts at date t chooses the variable price path $\{P_{t,t+j}(\epsilon^t)\}_{j=0}^{\infty}$. With probability $1-\theta$ a producer revises this path in any given period; with probability θ the producer does not revise the plan. On average, a plan therefore is revised after $(1-\theta)^{-1}$ periods. Suppose that producer i sets (the logarithm of) its price j periods ahead, $p_{t,t+j}^i(\epsilon^t)$, equal to a markup over the expected marginal cost:

$$p_{t,t+j}^i(\epsilon^t) = \mu + \mathbb{E}_t[p_{t+j}(\epsilon^{t+j}) + \psi_{t+j}(\epsilon^{t+j})].$$

The aggregate price level (in logs) satisfies

$$p_t(\epsilon^t) = (1-\theta)\sum_{j=0}^{\infty}\theta^j p_{t-j,t}(\epsilon^{t-j}) = (1-\theta)\sum_{j=0}^{\infty}\theta^j\left(\mu + \mathbb{E}_{t-j}[p_t(\epsilon^t) + \psi_t(\epsilon^t)]\right),$$

where $p_{t-j,t}(\epsilon^{t-j})$ denotes the price at date t preset by firms j periods ago. (All producers who revised their price paths at that time chose the same path.)

i. Show that the last equation can be expressed in terms of inflation as

$$\pi_t(\epsilon^t) = \frac{1-\theta}{\theta}(\psi_t(\epsilon^t) + \mu) + (1-\theta)\sum_{j=0}^{\infty}\theta^j\mathbb{E}_{t-j-1}\left(\Delta\psi_t(\epsilon^t) + \pi_t(\epsilon^t)\right).$$

ii. Mankiw and Reis (2002) refer to this condition as the "sticky-information Phillips curve." Interpret the condition.

iii. Suppose that marginal cost reflects the output gap with output responding to real balances (or their deviation from the steady state value, all in logs), $\psi_t(\epsilon^t) + \mu = m_t(\epsilon^t) - p_t(\epsilon^t) - y^n_t(\epsilon^t)$; the natural output, $y^n_t(\epsilon^t)$, follows an exogenous process. Show that

$$\pi_t(\epsilon^t) = \frac{1-\theta}{\theta}(m_t(\epsilon^t) - p_t(\epsilon^t)) + (1-\theta)\sum_{j=0}^{\infty}\theta^j \mathbb{E}_{t-j-1}[\Delta m_t(\epsilon^t)] + \xi_t(\epsilon^t)$$

with

$$\xi_t(\epsilon^t) \equiv \frac{1-\theta}{\theta}\left(-y^n_t(\epsilon^t) - \theta\sum_{j=0}^{\infty}\theta^j \mathbb{E}_{t-j-1}[\Delta y^n_t(\epsilon^t)]\right).$$

iv. Suppose that nominal balances follow a random walk. Determine the dynamic response of inflation and output ($m_t(\epsilon^t) - p_t(\epsilon^t)$) to a money supply shock at date t.

[160] Consider the New Keynesian model analyzed in the textbook,

$$\begin{aligned} \pi_t &= \kappa\chi_t + \beta\mathbb{E}_t[\pi_{t+1}], \\ \chi_t &= \mathbb{E}_t[\chi_{t+1}] - \frac{1}{\sigma}\mathbb{E}_t\left[i_{t+1} - \pi_{t+1} - r^n_{t+1}\right], \end{aligned}$$

where we suppress histories to improve legibility. Suppose that at date $t = 0$ the agents in the model expect the nominal interest rate to follow a bounded, deterministic exogenous path, $\{i_{t+1}\}_{t\geq 0}$. They expect no shocks (or only mean zero shocks) to impinge on the system.

i. Argue that the model exhibits indeterminacy: The equilibrium values π_0 and χ_0 are only determined up to one degree of freedom.

ii. Characterize π_0 and χ_0.

11 The Government

11.1 Taxation and Government Consumption

[161] Consider the model analyzed in the textbook. What tax rate on net capital income is equivalent to a given tax rate on gross capital income?

[162] Consider the model analyzed in the textbook. What is the effect of a change in the labor income tax rate at date t on household wealth? Does the effect depend on capital income tax rates?

[163] Consider the model discussed in the textbook and assume that capital income tax rates equal zero; that is, the government only levies labor income taxes to finance the exogenous government consumption. Show that the equilibrium allocation coincides with the allocation in a Robinson Crusoe economy with the same "government" consumption. Conclude that the equilibrium allocation is Pareto inefficient when the government collects capital income taxes.

[164] Consider the model analyzed in the textbook and assume that capital income tax rates equal zero; that is, the government only levies labor income taxes to finance government consumption. Using a phase diagram, sketch the response to (i) a permanent and (ii) a temporary increase in government consumption by Δ when the initial capital stock equals the capital stock in steady state with zero government consumption.

[165] Consider the model discussed in the textbook and assume that labor income tax rates equal zero; that is, the government only levies capital income taxes to finance strictly positive government consumption. Households optimize subject to the dynamic budget constraint

$$a_{t+1} = a_t R_t (1 - \tau_t) + w_t - c_t.$$

Accordingly, they equalize the intertemporal marginal rate of substitution with the after-tax gross interest rate, $R_{t+1}(1-\tau_{t+1})$. How would behavior change if the household internalized the government budget constraint and perceived its own choices to be representative?

11.2 Government Debt and Social Security

[166] Consider the model in the textbook. Derive the government's intertemporal budget constraint.

[167] Consider the representative agent model with government debt. Show by means of simple examples that a change of government financing policy affects the allocation if households and the government cannot save or borrow at the same interest rates, or if taxes are distorting.

[168] Consider the overlapping generations model with government debt and assume Cobb-Douglas technology, full depreciation, and logarithmic preferences over consumption. Saving of a young household at date t is given by $sw_t(1-\tau_t)$ where the savings rate satisfies $s = \beta/(1+\beta)$. Let μ_t denote the fraction of savings invested in capital such that $k_{t+1}/b_{t+1} = \mu_t/(1-\mu_t)$. Derive the conditions characterizing equilibrium. Show that $c_{2,t+1} = \alpha k_{t+1}^{\alpha}\mu_t^{-1}\nu$.

[169] Consider the overlapping generations model with pay-as-you-go social security and assume Cobb-Douglas technology, full depreciation, and logarithmic preferences over consumption. Show that saving of a young household at date t is given by $s_t w_t(1-\tau_t^s)$ where the savings rate satisfies

$$s_t = \frac{\beta}{1+\beta} - \frac{T_{t+1}\nu}{R_{t+1}(1+\beta)w_t(1-\tau_t^s)}.$$

Derive the conditions characterizing equilibrium.

[170] Consider an overlapping generations economy without government consumption. Show that if a social security policy balances the government budget in each period then the equivalent tax-and-debt policy satisfies the dynamic government budget constraint in each period as well.

[171] Consider an overlapping generations economy without government consumption. With a social security system, the government budget constraint along a balanced growth path reads $\tau^s w = T$. Derive the corresponding condition for a tax-and-debt policy. What tax rate τ under the equivalent policy corresponds to the social security tax rate $\tau^s > 0$ when the economy is inefficient?

[172] Consider an efficient overlapping generations economy with a pay-as-you-go social security system. Discuss the following proposal to abolish the system: The current old should receive the regular social security benefits but current and future young should no longer pay social security contributions nor receive social security benefits. The benefits of the current old should be financed by contemporaneous taxes and debt issuance (i.e., taxes paid by future young) instead.

11.3 Equivalence of Policies

[173] (Bassetto and Kocherlakota (2004)) Consider a two-period economy with a representative household with preferences $u(c_0, \ell_0)+\beta u(c_1, \ell_1)$ where c_t and ℓ_t denotes consumption and labor supply. Production is linear and one-to-one in labor. The government consumes g units of the good at date $t = 1$ and raises revenues through taxes on labor income. Taxes at date t are a nonlinear function of labor income at date t, $f_t(\ell_t)$. What is government debt at the end of date $t = 0$? How can taxes be adjusted to support zero government debt and the same equilibrium allocation?

[174] Consider the intertemporal budget constraint of a household that pays proportional taxes on labor income, consumption expenditures, and gross capital income. Suppose that $a_0 = 0$. Derive the equivalent (from the perspective of the household) tax system when tax rates on (i) labor income, (ii) consumption expenditures, or (iii) gross capital income are constrained to equal zero.

11.4 Fiscal-Monetary Policy Interaction

[175] Consider a stochastic environment with a given sequence of stochastic discount factors. Derive the intertemporal budget constraint of the consolidated government (suppressing histories),

$$b_t R_t + \frac{B_t I_t}{P_t} = \sum_{s=0}^{\infty} \mathbb{E}_t \left[(m_{t+1} \cdots m_{t+s}) \left(\tau_{t+s} - g_{t+s} + \frac{M_{t+1+s} - M_{t+s}}{P_{t+s}} \right) \right].$$

[176] Consider a stochastic environment with a given sequence of stochastic discount factors. Derive the intertemporal budget constraint of the consolidated government (suppressing histories),

$$b_t R_t + \frac{B_t I_t + M_t}{P_t} = \sum_{s=0}^{\infty} \mathbb{E}_t \left[(m_{t+1} \cdots m_{t+s}) \left(\tau_{t+s} - g_{t+s} + \frac{m_{t+1+s} M_{t+1+s} i_{t+1+s}}{P_{t+1+s}} \right) \right].$$

[177] Show that if the central bank pays the bond interest rate on the money it issues then the intertemporal budget constraint of the consolidated government does not contain a seignorage term.

[178] (Inflation Laffer Curve in the Turnpike Model) Consider the turnpike model in which households alternate between having a high or low endowment. Nonconstrained households buy the bubble $a = M_{t+1}/P_t$ and sell it at value M_{t+1}/P_{t+1} in the subsequent period. The government consumes g per period but does not raise taxes. Assuming CIES preferences, compute the bubble demand as a function of the constant inflation rate. For which values of government consumption does the model exhibit one, two, or zero equilibria?

[179] (Inflation Effects of Government Financing when Money Serves as Store of Value (Sargent, 1987, 8)) Consider an economy with two-period lived overlapping generations. Young households receive an endowment which they consume or save in the form of capital or money. Capital depreciates after a period and has the exogenous history-contingent gross return, $R(\epsilon^{t+1})$. Money serves as a store of value. Total capital is the sum of capital held by households and the capital held by the government, $k_{t+1}(\epsilon^t) \equiv k^p_{t+1}(\epsilon^t) + k^g_{t+1}(\epsilon^t)$. The government budget constraint reads

$$k^g_{t+1}(\epsilon^t) = \tau^o_t(\epsilon^t) + \tau^y_t(\epsilon^t) - g_t(\epsilon^t) + k^g_t(\epsilon^{t-1})R(\epsilon^t) + \frac{M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})}{P_t(\epsilon^t)},$$

where τ^y_t and τ^o_t denote nondistorting taxes on young and old households, respectively. Suppose that a policy, $p = \{\tau^o_t, \tau^y_t, g_t, k^g_{t+1}, M_{t+1}\}_{t\geq 0}$, and an initial state, (M_0, k^p_0, k^g_0), support an equilibrium with private capital and money holdings; the endogenous variables include $\{c^y_t(\epsilon^t), c^o_t(\epsilon^t), k^p_{t+1}(\epsilon^t), P_t(\epsilon^t)\}_{t\geq 0}$.

i. Argue that in equilibrium the stochastic discount factor and inflation must satisfy the following condition:

$$\mathbb{E}_t[m_{t+1}(\epsilon^{t+1})\Pi^{-1}_{t+1}(\epsilon^{t+1})] = \mathbb{E}_t[m_{t+1}(\epsilon^{t+1})R(\epsilon^{t+1})].$$

ii. Consider policy changes that have no effect on the equilibrium allocation. A new policy, $\hat{p}$, then must support the same young- and old-age consumption, total capital stock, government consumption, and stochastic discount factor processes. State the implications of these requirements for the new price level, government capital stock, and taxes.

iii. Establish a Ricardian equivalence result: A change in the timing of taxes that are collected from one and the same cohort does not change the allocation. Does the policy change affect inflation?

iv. Establish a result due to Wallace (1981): Money issuance that finances government purchases of capital does not alter the equilibrium allocation if old-age taxes in the subsequent period are reduced by an amount corresponding to the increase in the return on the government's portfolio. Does the policy change affect inflation?

v. Establish a result due to Chamley and Polemarchakis (1984): Money issuance that finances government purchases of capital does not alter the equilibrium allocation even if taxes are not adjusted. Does the policy change affect inflation?

[180] (Eigenvalues in Model with Active and Passive Policy Rules) Show that the eigenvalues of a triangular 2×2 matrix are equal to the elements on the diagonal.

11.5 Determinate Inflation and Output

[181] Suppose that output, inflation, and the nominal interest rate satisfy

$$y_t(\epsilon^t) = \mathbb{E}_t[y_{t+1}(\epsilon^{t+1})] - \frac{1}{\sigma}(i_{t+1}(\epsilon^t) - \mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})] - r^n),$$

where output follows an autoregressive process with parameter $\rho \in [0,1)$. Derive an interest rate rule that is linear in inflation and output and that stabilizes inflation at the level $\pi^\star$ at all times (if we rule out explosive paths). What happens if the rule's coefficient on inflation is smaller than unity in absolute value?

11.6 Real Effects of Monetary Policy

[182] Consider the model analyzed in the textbook:

$$\begin{aligned}
\pi_t(\epsilon^t) &= \kappa\chi_t(\epsilon^t) + \beta\mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})], \\
\chi_t(\epsilon^t) &= \mathbb{E}_t[\chi_{t+1}(\epsilon^{t+1})] - \frac{1}{\sigma}\mathbb{E}_t[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1}) - r^n], \\
i_{t+1}(\epsilon^t) &= r^n + \phi_\pi\pi_t(\epsilon^t) + \phi_\chi\chi_t(\epsilon^t) + \zeta_t(\epsilon^t), \\
\zeta_t(\epsilon^t) &= \rho\zeta_{t-1}(\epsilon^{t-1}) + u_t(\epsilon^t)
\end{aligned}$$

with $\rho \in [0,1)$ and $u_t(\epsilon^t)$ white noise. Use the method of undetermined coefficients to express inflation and the output gap as functions of the monetary policy shock, $\zeta_t(\epsilon^t)$. Distinguish between the cases of fully flexible and completely rigid prices.

[183] Consider the model

$$\begin{aligned}
\pi_t(\epsilon^t) &= \kappa\chi_t(\epsilon^t) + \beta\mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})], \\
\chi_t(\epsilon^t) &= \mathbb{E}_t[\chi_{t+1}(\epsilon^{t+1})] - \frac{1}{\sigma}\mathbb{E}_t[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1}) - r^n], \\
i_{t+1}(\epsilon^t) &= r^n + \phi_\pi\left(\pi_t(\epsilon^t) + \zeta_t(\epsilon^t)\right) + \phi_\chi\chi_t(\epsilon^t), \\
\zeta_t(\epsilon^t) &= \rho\zeta_{t-1}(\epsilon^{t-1}) + u_t(\epsilon^t)
\end{aligned}$$

with $\rho \in [0,1)$ and $u_t(\epsilon^t)$ white noise. Unlike in the previous exercise the exogenous driver $\zeta_t(\epsilon^t)$ now represents measurement error (the monetary authority observes inflation with an error), not a policy shock. Use the method of undetermined coefficients to express inflation and the output gap as functions of the measurement error. Compare your results to those from the previous exercise.

[184] Consider the basic New Keynesian model,

$$\begin{aligned}
y_t(\epsilon^t) &= \mathbb{E}_t[y_{t+1}(\epsilon^{t+1})] - \frac{1}{\sigma}\mathbb{E}_t\left[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1}) - r^n_{t+1}(\epsilon^{t+1})\right], \\
\pi_t(\epsilon^t) &= \beta\mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})] + \kappa y_t(\epsilon^t),
\end{aligned}$$

where $y_t(\epsilon^t)$ denotes output, $i_{t+1}(\epsilon^t)$ is the nominal interest rate, and $\pi_t(\epsilon^t)$ is inflation. Let $r^n_{t+1}(\epsilon^{t+1}) = \rho$ (the natural rate of interest is constant). The natural level of output is constant and normalized to zero. The central bank follows the interest rate rule $i_{t+1}(\epsilon^t) = \rho + \phi_\pi\pi_t(\epsilon^t)$ with $\phi_\pi > 1$.

i. Suppose that the economy is in steady state, $y_t(\epsilon^t) = \pi_t(\epsilon^t) = 0$ and $i_{t+1}(\epsilon^t) = \rho$. Unexpectedly, the central bank announces at date $t = 0$ that it will deviate from its rule in one period, namely at date $t = T > 0$, when it will set the interest rate according to $i_{T+1}(\epsilon^T) = \rho + u > \rho$. Determine the equilibrium response of output and inflation at date $t = 0, 1, \ldots, T, T + 1$.

ii. Determine the equilibrium response of output and inflation at date $t = 0, 1, \ldots, T, T+1$ when prices are completely rigid ($\kappa = 0$).

[185] (Fiscal Policy in the New Keynesian Model) Consider the basic New Keynesian model analyzed in the textbook, augmented by fiscal policy. The aggregate resource constraint is $y_t = c_t + \tilde{g}_t$ where $\tilde{g}_t$ denotes government consumption which is financed by lump-sum taxes. Suppose that $\tilde{g}_t = \xi_t y_t$ where ξ_t is an exogenous fiscal shock. Production of intermediate goods is linear in labor, $y_t(i) = l_t(i)$. The representative household's felicity function is given by $u(c_t, l_t) = \ln(c_t) - \frac{l_t^{1+\eta}}{1+\eta}$ with $\eta > 0$. (We suppress histories throughout.)

i. Derive the goods market clearing condition, $\hat{y}_t = \hat{c}_t + \hat{g}_t$, where $g_t \equiv (1 - \xi_t)^{-1}$ is a monotonic transformation of the fiscal shock.

ii. Derive the New Keynesian Phillips curve relating inflation, expected inflation, and the output gap (output relative to its flexible-price level).

iii. Suppose that g_t follows an autoregressive process, $g_t = \rho_g g_{t-1} + \epsilon_t^g$ where $|\rho_g| < 1$ and ϵ_t^g is i.i.d. with mean zero. Derive the dynamic IS equation relating output gaps, expected inflation, the interest rate, and the natural rate of interest.

iv. Suppose the central bank follows the interest rate rule $i_{t+1} = -\ln(\beta) + \phi\pi_t$ with $\phi > 1$. Use the method of undetermined coefficients to solve for the equilibrium output gap and equilibrium inflation as functions of $\hat{g}_t$.

v. Does the goal to stabilize inflation conflict with the goal to stabilize the output gap?

vi. Compute the fiscal multiplier, $d\hat{y}_t/d\hat{g}_t$. How does ϕ affects its size?

12 Optimal Policy

12.1 Tax Smoothing

[186] Consider a representative household economy without capital. The household is endowed with one unit of time per period which can be transformed into $w_t(\epsilon^t)$ units of the good. Household preferences over consumption, c, and leisure, x, are represented by the utility function $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_0[u(c_t(\epsilon^t), x_t(\epsilon^t))]$, where u is strictly concave and increasing and β denotes the discount factor. There is a given stream of government consumption, $\{g_t(\epsilon^t)\}_{t\geq 0}$. Derive the first-order conditions of the social planner's problem.

[187] Consider the complete markets Ramsey program analyzed in the textbook. Suppose that the economy is deterministic, productivity is constant, and g_t equals zero except at date $t = T$ when it is strictly positive. Why does the government optimally raise taxes after date $t = T$? How does the government spend that tax revenue?

[188] Consider the same environment as in the previous exercise except that government consumption at date $t = T$ is stochastic; it may take a high or a low value. Prove that the value of government debt at date $t = T$ under the Ramsey policy is contingent on g_T.

[189] Consider the environment in which the government is constrained to only issue short-term, risk-free debt. Show that any implementable government primary surplus, $\tau_t(\epsilon^t)(1 - x_t(\epsilon^t)) - g_t(\epsilon^t)$, can be expressed as a function of $c_t(\epsilon^t)$ and $g_t(\epsilon^t)$ only.

[190] Consider the environment in which the government is constrained to only issue short-term, risk-free debt. Derive the implementability constraint (omitting histories to improve legibility),

$$u_{c,0} b_0 = \sum_{t=0}^{\infty} \int \beta^t u_{c,t} s_t dH_t(\epsilon^t).$$

[191] Consider the environment in which the government is constrained to only issue short-term, risk-free debt. Show that the government's Lagrangian simplifies to the Lagrangian in the complete markets case if the measurability and borrowing constraints are dropped.

[192] Consider the environment in which the government is constrained to only issue short-term, risk-free debt. Show that the government's first-order conditions simplify to those from the complete markets program when the measurability and borrowing constraints are dropped.

[193] Consider the environment in which the government is constrained to only issue short-term, risk-free debt. Suppose that preferences are quasilinear, $u(c, x) = c + G(x)$ where G is increasing and strictly concave. Show that the deadweight loss is a strictly convex function of tax revenue (on the rising segment of the Laffer curve) and reaches a minimum when no taxes are collected.

[194] Consider the same environment as in the previous question. Show that the first-order condition with respect to taxes can be written as $-\mathcal{D}'(\rho_t) = \nu_t$.

[195] (Tax Collection Costs (Barro, 1979)) Consider a small, open, two-period economy. The gross world interest rate equals R. Households and the government can borrow and lend at this rate. The government needs to finance government consumption g_t at date t. Taxes at date t, $\tau_t w_t$, reduce the exogenous pre-tax household income, w_t, to $w_t(1 - \tau_t) - h(\tau_t, w_t)$ where $h(\tau_t, w_t)$ denotes "tax collection costs." Function h is strictly convex in the tax rate and reaches a minimum of zero at $\tau_t = 0$. The household values consumption.

i. State the intertemporal budget constraints of the government and the household.
ii. Argue that household wealth is a sufficient statistic for lifetime household utility. Specify the Ramsey program.
iii. Show that tax collection costs drive a wedge between the shadow value of public and private funds. Characterize the optimal policy.
iv. Show that government debt allows to minimize tax collection costs.
v. Suppose that $h(\tau, w) = \tau w f(\tau)$ for some increasing function f. Derive the optimal tax policy.

[196] (Alternative Approach to Solving the Barro (1979) model) Consider the same environment as in the previous exercise, except that the economy is infinitely lived. The representative household (and the government) maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$.

i. Specify the constraints of the Ramsey government including the household equilibrium conditions and the resource constraint.
ii. Solve the Ramsey program and show that the Euler equation is not a binding constraint.

[197] Consider the problem of a Ramsey government whose portfolio includes a broad set of assets and liabilities. As shown in the textbook the first-order condition with respect to security i reads

$$\nu_-(\epsilon_-)u_c(\epsilon_-) = \beta\mathbb{E}[\nu_\circ(\epsilon_\circ)u_c(\epsilon_\circ)R^i(\epsilon_\circ)|\epsilon_-]$$

and the stochastic discount factor of the government is given by

$$\beta\frac{\nu_\circ(\epsilon_\circ)u_c(\epsilon_\circ) + \mu_\circ(\epsilon_\circ)}{\nu_-(\epsilon_-)u_c(\epsilon_-) + \mu_-(\epsilon_-)}.$$

Show that if markets are complete the multiplier ν is constant over time and across histories. What does this imply for the government's stochastic discount factor?

[198] (Tax Collection Costs and Financial Instruments (Bohn, 1990)) Consider a closed economy with a risk neutral representative household that values consumption. Households and the government have access to a set of securities with arbitrary returns (as long as households are willing to hold the securities). The government needs to finance stochastic government consumption $g_t(\epsilon^t)$ at date t. Taxes at date t, $\tau_t(\epsilon^t)w_t(\epsilon^t)$, reduce the stochastic pre-tax household income, $w_t(\epsilon^t)$, to $w_t(\epsilon^t)(1 - \tau_t(\epsilon^t)) - h(\tau_t(\epsilon^t), w_t(\epsilon^t))$ where $h(\tau, w)$ denotes "tax collection costs." Function h is strictly convex in the tax rate and reaches a minimum of zero at $\tau_t = 0$.

i. Specify the Ramsey program.
ii. Derive the optimality conditions for tax rates and the government portfolio. Characterize the correlation between asset returns and the shadow value of public funds. Explain.
iii. What happens when markets are complete?

[199] Consider a variant of the model in the textbook in which the zero capital income taxation result is derived: Rather than one, let there be two groups of households whose labor productivity may differ. Show that the zero capital income taxation result is robust if the Ramsey government has access to household specific labor income tax rates (and uniform capital income tax rates) but not if labor income tax rates must be uniform across groups.

[200] Consider a two-period model with two groups of households indexed by $i = 0, 1$. Each household chooses whether to supply zero or one unit of labor in both periods, $\ell = 0$ or $\ell = 1$. Households in group 0 have the wage profile $(w_0^0, w_1^0) = (w, 0)$, households in group 1 have the profile $(w_0^1, w_1^1) = (0, w)$. Preferences are given by $c_0 + c_1 - \gamma\ell$ where $\gamma < w$. The budget constraint of a household in group i is given by $\ell(w_0^i(1 - \tau_0) + w_1^i(1 - \tau_1)) + z = c_0 + c_1$ with $z > 0$ (exogenous income). The interest rate equals zero.

i. Derive the equilibrium labor supply of households in the two groups.
ii. What is the maximum tax revenue the government can collect?
iii. Suppose the government needs to collect revenue $w - \gamma$. Show that the government can impose the tax burden fully on either group.
iv. Is debt policy neutral?

12.2 Social Insurance and Saving Taxation

[201] Consider the two-period model discussed in the textbook.

i. Characterize the allocation implemented by a social planner that observes productivity realizations.

ii. Argue that this allocation cannot be implemented when the social planner does not observe productivity realizations but instead relies on reports by households about their productivity realizations.
iii. Suppose that an incentive compatible arrangement assigns households with high self-reported labor productivity more work and rewards these households with more consumption. Argue that the reward has to be more pronounced (consumption cannot be as well insured) if households can save between the first and the second period and the planner does not observe this saving. Conclude that the constrained optimal planner intervention imposes an (implicit) tax on saving.

12.3 Monetary Policy

[202] (Cost Push Shocks and Stabilization Poliy) Consider the New Keynesian model with the following Phillips curve and dynamic IS equation (we suppress histories to improve legibility):

$$\begin{aligned} \pi_t &= \kappa\chi_t + \beta\mathbb{E}_t[\pi_{t+1}] + \upsilon_t, \\ \chi_t &= \mathbb{E}_t\left[\chi_{t+1}\right] - \frac{1}{\sigma}\mathbb{E}_t\left[i_{t+1} - \pi_{t+1} - r^e\right]. \end{aligned}$$

The output gap is the relative deviation of output from its efficient level, the efficient rate of interest is constant, and υ_t denotes an i.i.d. cost push shock with variance σ_υ^2. Suppose that the central bank follows a Taylor rule of the form $i_{t+1} = r^e + \phi\pi_t$ with $\phi > 0$.

i. Using the method of undetermined coefficients solve for the equilibrium output gap and equilibrium inflation as functions of υ_t.
ii. Assume that $\phi > 1$. Show that there is a tradeoff between stabilizing inflation and stabilizing the output gap.
iii. Determine the value of the coefficient ϕ that minimizes the loss function $\alpha\mathbb{V}\text{ar}(\chi_t) + \mathbb{V}\text{ar}(\pi_t)$.

13 Time Consistent Policy

13.1 Time Consistency and the Role of State Variables

[203] Consider a two-period economy. At date $t = 0$ the government announces a policy, τ^a, to be implemented at date $t = 1$. Thereafter, the private sector forms expectations about the policy that the government will actually implement, τ^e, and takes an action, $s(\tau^e) = 1 - \tau^e$. At date $t = 1$, the government chooses the policy, τ. When the government is committed it must set $\tau = \tau^a$. The government's objective function is given by $s - \alpha(\tau - z)^2$ where $\alpha z > 0.5$, $z \in [0, 1]$. The choice of τ must satisfy $\tau \in [0, 1]$.

i. Suppose the private sector believes the government's policy announcement, $\tau^e = \tau^a$. Derive the optimal policy announcement and policy choice, (τ^a, τ). Is this outcome consistent with rational expectations?
ii. Derive the rational expectations equilibrium when the government cannot commit. Compute the equilibrium value of the government's objective.
iii. Derive the rational expectations equilibrium when the government can commit. Compute the equilibrium value of the government's objective. When is commitment valuable?
iv. Suppose the government cannot commit but can choose a state variable, k, at date $t = 0$ at no cost which makes it extremely costly for the government at date $t = 1$ to set $\tau \neq k$. Solve for the equilibrium.

13.2 Credible Tax Policy

[204] Consider the model analyzed in the textbook. The government chooses the maturity structure of public debt to render the Ramsey policy time consistent. Suppose that from date t, history ϵ^t to date $t + 1$, history ϵ^{t+1} the composition of the outstanding debt does not change, $\{_t b_s(\epsilon^s)\}_{s \geq t+1, \epsilon^s} = \{_{t+1} b_s(\epsilon^s)\}_{s \geq t+1, \epsilon^s}$. What does this imply about the tightness of the implementability constraint in the two nodes?

13.3 Capital Income Taxation

[205] Consider the sustainable equilibrium analyzed in the textbook. Show that the optimal plan in the infinite horizon environment cannot be implemented when the horizon is finite.

13.4 Sovereign Debt and Default

[206] (Insurance) Consider the static model of insurance analyzed in the textbook. Suppose that $L(\epsilon^1)$ is constant across states at value L. Characterize $T(\epsilon^1)$ and $c(\epsilon^1)$ (i) under the assumption of commitment and (ii) under the assumption that the incentive compatibility constraint is binding in some states. (Assume for simplicity that $w(\epsilon^1)$ is uniformly distributed between $\underline{w}$ and $\bar{w}$.)
[207] Consider the same setup as in the previous question except that $L(\epsilon^1) = \max[0, a + w(\epsilon^1)]$. For which values of a can the commitment outcome be implemented?
[208] (Borrowing with Noncontingent Debt) Consider the model with noncontingent debt. Suppose that at date $t = 1$ there are two states, x and y, with associated endowments and default costs $w(i), L(i), i = x, y$. The endowment at date $t = 0$ is much smaller than either $w(x)$ or $w(y)$ such that the borrower wants to issue debt. Characterize the equilibrium debt level.
[209] (Dilution) Suppose that a zero-coupon bonds are outstanding from previous periods; each of these bonds promises payment of one unit of the good in the subsequent period. The government issues additional b bonds which also come due in the subsequent period. All debt is treated equally in the subsequent period and the risk-free gross interest rate equals one. The probability of repayment in full equals $1 - H(a + b)$ where H denotes the cumulative distribution function of default costs in the subsequent period. What is the effect of the issuance of b on the market price of outstanding bonds? What is the effect on the revenue raised by the government? Distinguish between the case with and without renegotiation.
[210] Derive condition (13.2) in the textbook.
[211] Suppose that debt cannot be renegotiated. Show that a sufficient condition for a unique maximum of the debt Laffer curve is that the hazard rate $H'(d)/(1 - H(d))$ be nondecreasing.
[212] (Debt Overhang) Consider the same environment as in the previous question. Suppose that there is debt overhang such that $1 - H(d) - H'(d)d < 0$. Show that both the borrowing country and investors benefit from a debt-writedown. Does this also hold true when debt can be renegotiated?

13.5 Redistribution in Politico-Economic Equilibrium

[213] Consider the model of redistribution analyzed in the textbook. Show that the equilibrium policy function, $\mathcal{T}(v_t)$, is the unique equilibrium policy function in a finite horizon economy. (In the last period, the policy function is different, but also unique.)

[214] Consider the model of redistribution analyzed in the textbook. Recall that the marginal effect of a tax change on the political objective function is given by

$$\frac{\omega(1-\alpha)}{\alpha+(1-\alpha)\tau_t} - \nu_t \frac{1+\alpha\beta}{1-\tau_t}.$$

Show that this effect can be decomposed into two components, one reflecting the direct distributive effect, the other the general equilibrium implications for workers.

13.6 Monetary Policy

[215] Consider the New Keynesian model analyzed in the textbook, augmented by a distorting labor income tax. The aggregate resource constraint is $y_t = c_t$. Production of intermediate goods is linear in labor, $y_t(i) = l_t(i)$. The representative household's felicity function is given by $u(c_t, l_t) = \ln(c_t) - \frac{l_t^{1+\eta}}{1+\eta}$ with $\eta > 0$. Labor income is taxed at rate τ_t and rebated lump sum. (We suppress histories throughout.)

i. Derive the logarithm of the natural (flexible price) level of output, $\ln(y_t^n)$, and of the efficient (undistorted) level of output, $\ln(y_t^e)$. (Use the approximation $\ln(1-\tau_t) \approx -\tau_t$.)
ii. Let $x_t \equiv \ln(y_t) - \ln(y_t^e)$ denote the welfare relevant output gap. Explain why the presence of distorting taxes and markups generates a tradeoff between the welfare relevant output gap and inflation.
iii. Write the New Keynesian Phillips curve as

$$\pi_t = \kappa\chi_t + \beta\mathbb{E}_t[\pi_{t+1}],$$

where $\chi_t \equiv \ln(y_t) - \ln(y_t^n)$ denotes the output gap. Express the Phillips curve in terms of the welfare relevant output gap, x_t.
iv. Consider a monetary policy maker acting under discretion who is constrained by the Phillips curve and takes tax policy as well as expectations of future inflation as given. Suppose the policy maker minimizes the loss function $\alpha x_t^2 + \pi_t^2$. Derive the equilibrium relationship between x_t and π_t.
v. Using the previous result and the Phillips curve derive equilibrium inflation under the assumption that $\mathbb{E}_t[\tau_{t+i}] = \rho^i\tau_t$ where $\rho \in (0, 1)$.
vi. Show that the discretionary policy gives rise to an inflation bias.

[216] (Forward Guidance, Commitment, and the Zero Lower Bound) Consider the basic New Keynesian model,

$$\begin{aligned} y_t &= y_{t+1} - \frac{1}{\sigma}(i_{t+1} - \pi_{t+1} - r_{t+1}^n), \\ \pi_t &= \beta\pi_{t+1} + \kappa y_t, \end{aligned}$$

where y_t denotes output, i_{t+1} is the nominal interest rate, and π_t is inflation. The natural level of output is constant and normalized to zero. The natural rate of interest is constant as well, $r_t^n = \rho$, except in the initial period when it equals $r_1^n = \rho - z < 0$; this is known at date $t = 0$. In response to the temporary change of the natural rate, the central bank stabilizes output and inflation by adjusting nominal interest rates subject to the zero lower bound (ZLB) constraint, $i_{t+1} \geq 0$.

i. Show that full stabilization is not possible in the presence of the ZLB.
ii. Suppose that the central bank minimizes the loss function $\sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \xi y_t^2)$, where ξ denotes the weight attached to output deviations. Characterize the time consistent policy at date $t = 0$ and date $t = 1$ under the assumption that the economy is in steady state after date $t = 1$.
iii. Characterize the Ramsey policy under the assumption that the economy is in steady state after date $t = 1$. Show that it is optimal for the central bank to keep the interest rate lower for longer after the temporary change of the natural rate, compared to the discretionary case. Relate this result to "forward guidance."

[217] Consider the trigger strategy equilibrium in the model with an inflation bias. Can this equilibrium be sustained when the time horizon is finite? What about the equilibrium with delegation?

Bibliography

Barro, Robert J. 1979. On the determination of the public debt. *Journal of Political Economy* 87 (5): 940–971. (Cited on p. 44.)

Bassetto, Marco, and Narayana Kocherlakota. 2004. On the irrelevance of government debt when taxes are distortionary. *Journal of Monetary Economics* 51 (2): 299–304. (Cited on p. 39.)

Bohn, Henning. 1990. Tax smoothing with financial instruments. *American Economic Review* 80 (5): 1217–1230. (Cited on p. 45.)

Caballero, Ricardo J. 1990. Consumption puzzles and precautionary savings. *Journal of Monetary Economics* 25 (1): 113–136. (Cited on p. 11.)

Chamley, Christophe, and Herakles Polemarchakis. 1984. Assets, general equilibrium and the neutrality of money. *Review of Economic Studies* 51 (1): 129–138. (Cited on p. 40.)

Frenkel, Jacob A. 1976. A monetary approach to the exchange rate: Doctrinal aspects and empirical evidence. *Scandinavian Journal of Economics* 78 (2): 200–224. (Cited on p. 30.)

Hall, Robert E. 1978. Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence. *Journal of Political Economy* 86 (6): 971–987. (Cited on p. 12.)

Hiraguchi, Ryoji. 2012. Multiple equilibrium in the overlapping generations model revisited. *Economics Letters* 115 (1): 70–72. (Cited on p. 9.)

Ljungqvist, Lars, and Thomas J. Sargent. 2018. *Recursive macroeconomic theory*, 4. edn. Cambridge, Massachusetts: MIT Press. (Cited on p. 17.)

Lucas, Robert E. 1978. Asset prices in an exchange economy. *Econometrica* 46 (6): 1429–1445. (Cited on pp. 17 and 18.)

Mankiw, N. Gregory, and Ricardo Reis. 2002. Sticky information versus sticky prices: A proposal to replace the New Keynesian Phillips curve. *Quarterly Journal of Economics* 117 (4): 1295–1328. (Cited on pp. 35 and 36.)

McCall, J. J. 1970. Economics of information and job search. *Quarterly Journal of Economics* 84 (1): 113–126. (Cited on p. 26.)

Mussa, Michael. 1976. The exchange rate, the balance of payments and monetary and fiscal policy under a regime of controlled floating. *Scandinavian Journal of Economics* 78 (2): 229–248. (Cited on p. 30.)

Prescott, Edward C., and Rajnish Mehra. 1980. Recursive competitive equilibrium: The case of homogeneous households. *Econometrica* 48 (6): 1365–1379. (Cited on p. 18.)

Rotemberg, Julio J. 1982. Sticky prices in the United States. *Journal of Political Economy* 90 (6): 1187–1211. (Cited on p. 34.)

Sargent, Thomas J. 1987. *Dynamic macroeconomic theory*. Cambridge, Massachusetts: Harvard University Press. (Cited on p. 39.)

Solow, Robert M. 1956. A contribution to the theory of economic growth. *Quarterly Journal of Economics* 70 (1): 65–94. (Cited on p. 20.)

Wallace, Neil. 1981. A Modigliani-Miller theorem for open-market operations. *American Economic Review* 71 (3): 267–274. (Cited on p. 40.)