Solutions to Odd-Numbered Problems in

Lectures on Microeconomics: The Big Questions Approach

Romans Pancs

Chapter 1

Problem 1.1

1. Agent i's demand for good l is

$$x_l^i = \alpha_l \left(e_l^i + \frac{p_{-l} e_{-l}^i}{p_l} \right),$$

where $-l \in \mathcal{L}$ is such that $-l \neq l$. By inspection of the display above, x_l^i is increasing in p_{-l}. As a result, $\sum_{i \in \mathcal{I}} x_l^i$, the aggregate demand, is also increasing in p_{-l}. Hence, the economy has the gross substitutes property.

The market clearing condition implies equilibrium prices $p_l = \alpha_l / \bar{e}_l$. The equilibrium allocations are given by

$$x_l^i = \bar{e}_l \left(\sum_{l' \in \mathcal{L}} \frac{\alpha_{l'} e_{l'}^i}{\bar{e}_{l'}} \right).$$

2. The aggregate excess demand for good 1 is given by

$$z_1 \left(p_1, \frac{\alpha_2}{\bar{e}_2} \right) \equiv \sum_{i \in \mathcal{I}} \left(x_1^i - e_1^i \right) = \alpha_1 \alpha_2 \left(\frac{1}{p_1} - \frac{\bar{e}_1}{\alpha_1} \right)$$

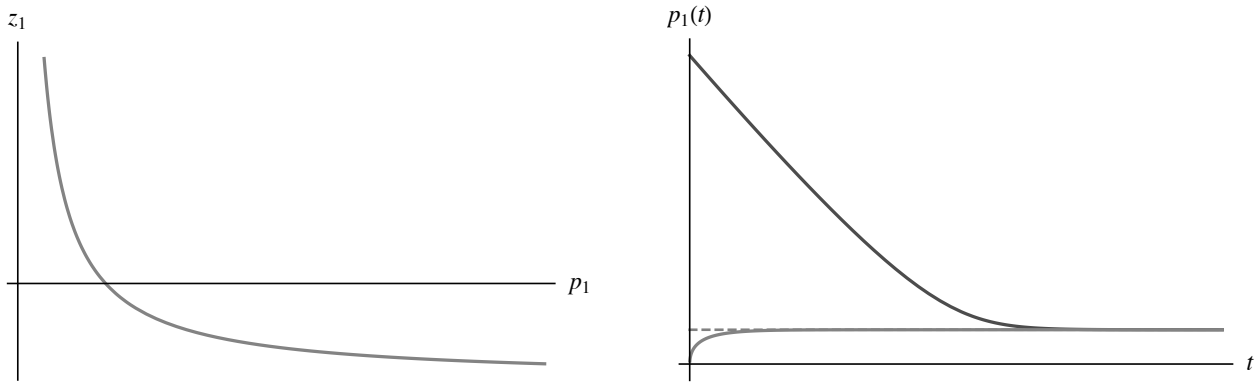
and is plotted in figure SO.1a. Tâtonnement $\mathrm{d}p_1(t)/\mathrm{d}t = z_1(p(t), \alpha_2/\bar{e}_2)$ yields the equation of motion for the price of good 1 (the price of good 2 is fixed)

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \alpha_1 \alpha_2 \left(\frac{1}{p_1(t)} - \frac{\bar{e}_1}{\alpha_1} \right).$$

By inspection of the display above, the stationary point—the Walrasian equilibrium price—has $p_1 = p_1^* \equiv \alpha_1 / \bar{e}_1$. Moreover, the sign of $\mathrm{d}p_1(t)/\mathrm{d}t$ is the sign of $p_1^* - p_1(t)$, and, hence, tâtonnement converges to the Walrasian equilibrium price from any initial price $p_1(0)$, as figure SO.1b illustrates.

3. Graphically, an allocation is Pareto efficient if the agents' indifference curves are tangent at this allocation. The slope of agent i's indifference curve at consumption bundle x^i is $-\alpha_2 x_1^i / \alpha_1 x_2^i$. Equating the slopes of the agents' indifference curves gives $x_2^i / x_1^i = \bar{e}_2 / \bar{e}_1$. That is, the Pareto set is the line segment that passes through the agents' origins in the Edgeworth box. Formally, the Pareto set is

$$\left\{ \left(x^1, x^2 \right) \in \mathbb{R}_+^4 \mid x^1 = \omega^1(\bar{e}_1, \bar{e}_2), x^2 = \omega^2(\bar{e}_1, \bar{e}_2), \omega^1 + \omega^2 = 1, \left(\omega^1, \omega^2 \right) \in [0,1]^2 \right\}.$$



(a) The excess demand function for good 1.

(b) The tâtonnement convergence of the price of good 1 to its equilibrium value (the dashed line). The top curve starts at an above-equilibrium price, whereas the bottom curve starts at a below-equilibrium price.

Figure SO.1: (Problem 1.1)

4. The solution to the planner's problem is $x_l^i = \omega^i \bar{e}_l$, which immediately yields the Pareto set as one varies ω^1. The planner's problem chooses the Walrasian equilibrium allocation if

$$\omega^i = \sum_{l \in \mathcal{L}} \frac{\alpha_l e_l^i}{\bar{e}_l}, \qquad i \in \mathcal{I}.$$

Problem 1.3

1. Both Alice and Bob are weakly better off in AB than in autarky, by WFT.

2. Let $\left(p_1^{ABC}, p_2^{ABC}\right)$ be the equilibrium price vector in ABC. For $i \in \{Alice, Bob\}$, let $\left(x_1^{AB,i}, x_2^{AB,i}\right)$ be agent i's equilibrium allocation in AB. Suppose, by contradiction, that Alice is strictly, and Bob is weakly, worse off in ABC than in AB. Then, by the argument in the proof of the FWT, Alice cannot, and Bob at most barely can, afford her or his bundle in AB at the prices prevailing in ABC:

$$\begin{aligned} p_1^{ABC}\left(x_1^{AB,Alice} - e_1^{Alice}\right) + p_2^{ABC}\left(x_2^{AB,Alice} - e_2^{Alice}\right) &> 0 \\ p_1^{ABC}\left(x_1^{AB,Bob} - e_1^{Bob}\right) + p_2^{ABC}\left(x_2^{AB,Bob} - e_2^{Bob}\right) &\geq 0. \end{aligned}$$

Adding up the inequalities above gives

$$\begin{aligned} &p_1^{ABC}\left(x_1^{AB,Alice} + x_1^{AB,Bob} - e_1^{Alice} - e_1^{Bob}\right) \\ &\qquad + p_2^{ABC}\left(x_2^{AB,Alice} + x_2^{AB,Bob} - e_2^{Alice} - e_2^{Bob}\right) > 0. \end{aligned}$$

Because the agents' utility functions are increasing, it must be that $p_1^{ABC} > 0$ and $p_2^{ABC} > 0$. Then, the display above implies that, at least for one of the goods, the term in the parenthesis

is positive, thereby contradicting market clearing for that good in AB. This contradiction implies, as required, that Alice cannot be strictly worse off in ABC without Bob being strictly better off.

3. By symmetry, Alice's consumption bundle in AB can be guessed and verified to be $x^{AB,Alice} = \left(\frac{3}{2}, \frac{3}{2}\right)$. Alice's consumption of good $l \in \{1, 2\}$ in ABC is

$$x_l^{ABC,Alice} = \frac{p_1^{ABC} e_1^{Alice} + p_2^{ABC} e_2^{Alice}}{2p_l^{ABC}},$$

where the market-clearing prices can be verified to be

$$p_1^{ABC} = 1 \qquad \text{and} \qquad p_2^{ABC} = \frac{9}{6+2y}.$$

Alice's implied consumption bundle in ABC is

$$x^{ABC,Alice} = \left(\frac{1}{2} + \frac{9}{6+2y}, 1 + \frac{3+y}{9}\right).$$

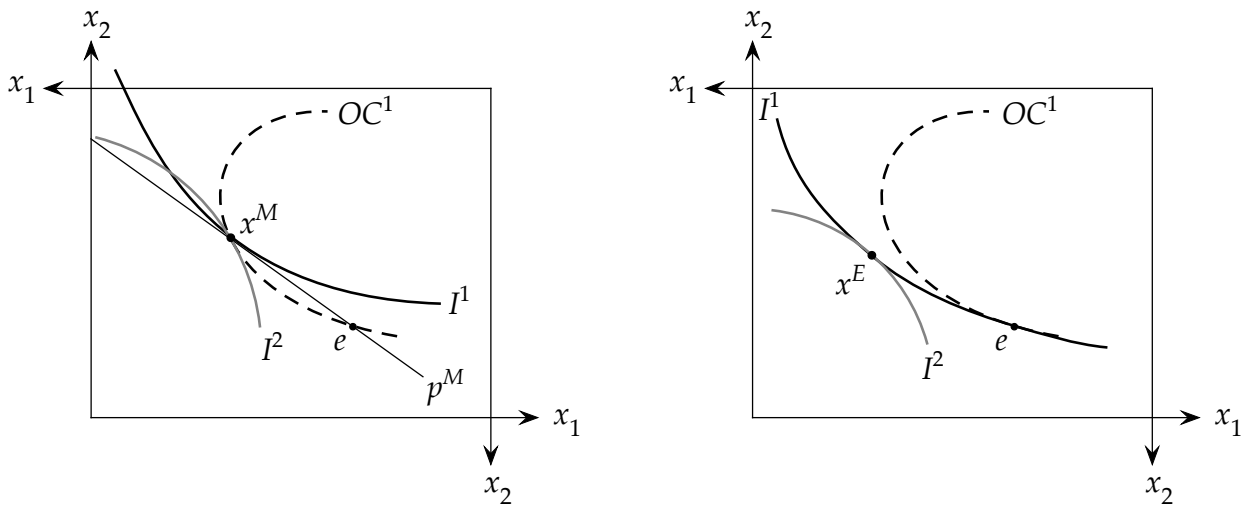
One can verify (numerically or analytically) that

$$u^{Alice}\left(x^{AB,Alice}\right) > u^{Alice}\left(x^{ABC,Alice}\right) \iff y \in \left(\frac{3}{2}, 15\right).$$

Analytically, $\mathrm{d}u^{Alice}\left(x^{ABC,Alice}\right)/\mathrm{d}y = (y-6)/((3+y)(12+y))$. So, in particular, at $y = 6$, $u^{Alice}\left(x^{ABC,Alice}\right)$ reaches its minimal value, which is approximately 0.69, less than $u^{Alice}\left(x^{AB,Alice}\right) \approx 0.81$.
Intuitively, when $y = \frac{3}{2}$, there are no gains from trade between Alice and Bob on the one hand and Carol on the other hand; Alice's consumption bundle and utility are the same in AB and ABC. When $y < \frac{3}{2}$, Alice's is relatively well endowed with the good that is relatively scarce, good 2; the price for good 2 is then higher in ABC than AB, and Alice is better off. By contrast, when $y > \frac{3}{2}$, good 2 is cheaper in ABC than in AB, which hurts Alice unless good 2 is so ample—and, hence, cheap—that (in ABC relative to AB) Alice's increased consumption of good 2 compensates for the reduction in the consumption of good 1.

4. Let Alice's and Bob's new endowments be their consumption bundles in AB at the Walrasian equilibrium from their initial endowments. Then, trivially, there exists an equilibrium in AB from the new endowments, such that Alice and Bob both consume their endowments. Furthermore, by refusing to trade, each can obtain at least as good a bundle in ABC as his endowment, which is also his bundle in AB.

5. Suppose Alice, Bob, and Carol are three distinct countries. Should one's goal be to draw these countries into a free-trade agreement, one should draw them at the same time, rather than sequentially. Doing so will ensure the unanimous support of all three countries.



(a) Monopoly outcome with constant prices.

(b) Monopoly outcome with nonlinear prices.

Figure SO.2: (Problem 1.5) In the Edgeworth boxes, OC^1 is agent 1's offer curve; I^1 and I^2 are the agents' indifference curves; x^M is the monopoly allocation, and p^M is the monopoly price; e is the endowment.

Suppose Alice and Bob are two types of consumers in country 1, and Carol is the representative consumer in country 2. Suppose also that trade is free within country 1; Alice and Bob are already in AB; there is no way around it. Then, because country 1's population is heterogeneous, opening up to trade—moving to state ABC—may cause some of the country's population lose. One way to ensure that free trade is Pareto-improving for country 1 is to redistribute endowments before trade liberalization. In practice, in the world that is dynamic, the redistribution would have to occur repeatedly; one's endowment can be thought of as one's ability to work, and this ability persists over time and cannot be redistributed one-off, right before the liberalization.

Problem 1.5

1. Figure SO.2a exhibits the unique monopoly price vector p^M and the associated monopoly allocation $x^M \equiv \left(x^1\left(p^M\right), x^2\left(p^M\right)\right)$.

2. In figure SO.2a, the monopoly outcome is not a Walrasian equilibrium. At the monopoly allocation x^M, agent 2's indifference curve is not tangent to the price line p^M. That is, agent 2 does not maximize his utility taking the price p^M as given.

3. In figure SO.2a, the monopoly outcome is not Pareto efficient. The intersection of the agents' upper contour sets at allocation x^M—the areas above the corresponding indifference curves—is nonempty.

4. Agent 2 is better off proposing a different allocation, which is denoted by x^E (where the subscript stands for the full extraction of agent 1's surplus) in figure SO.2b. This allocation makes agent 1 indifferent between consuming his endowment and the proposed bundle. The proposed allocation is Pareto efficient; the agents' indifference curves are tangent at x^E. Agent 2 can still be interpreted as a monopolist, but one whose offers are not restricted to take-it-or-leave-it offers of constant prices.

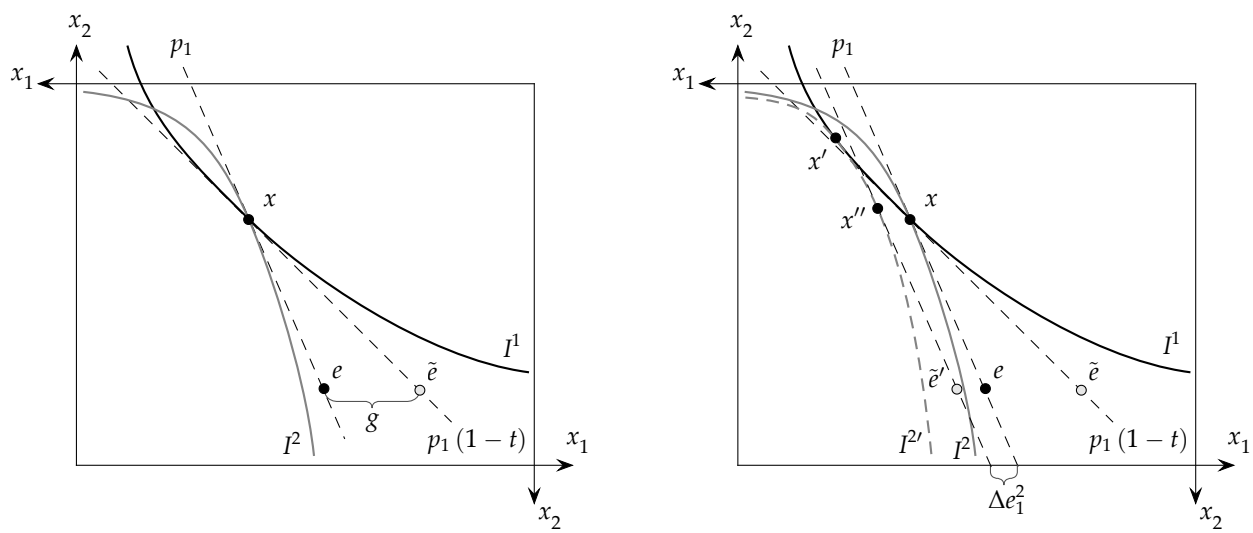
Problem 1.7

The point of this problem is to illustrate that there is efficiency loss to taxation even when taxation is redistributive, meaning that the tax receipts are rebated back to one of the agents.

1. Agent 2's budget constraint (1.12) is the same as (1.9); it reflects the fact that the price the agent receives as a seller is reduced by the government's tax. Agent 1's budget constraint (1.12) modifies (1.9) to reflect the fact that he receives from the government an unconditional transfer of good 1 in the amount g. The agent knows that he is entitled to g and, therefore, treats it as his endowment, which he can consume or sell. (The agent also knows that the transfer is not taxed; what is taxed is the agent's trade away from his new, post-transfer, endowment.) The market clearing conditions (1.13) contain no government expenditure because g is redistributed to agent 1, instead of being altogether denied to the agents. The government's budget constraint (1.14) reflects the fact that exactly one agent will be a seller of good 1 (which can be verified by contradiction). In addition, the constraint reflects the fact that agent 1 treats the transfer g as a component of his endowment, and so any sale of g is also taxed.

2. Figure SO.3a illustrates the construction of a Walrasian equilibrium. Normalize $p_2 = 1$. Even though the endowment is $e = (e^1, e^2)$, agent 1 behaves as if the endowment is $\tilde{e} \equiv (\tilde{e}^1, e^2)$, where $\tilde{e}^1 \equiv (e_1^1 + g, e_1^2)$, as he anticipates the transfer g from the government. At the equilibrium, agent 1 is the seller of good 1; he sells his entire transfer g plus some of his initial endowment e_1^1. The price he faces is $p_1(1-t)$. Agent 2 is the buyer of good 1. The price he faces is p_1. Because the markets must clear and each agent must select a bundle on his budget line, the equilibrium allocation must lie at the point of the intersection of the budget line $p_1(1-t)$ passing through $\tilde{e}$ and budget line p_1 passing through e. The equilibrium is found by varying p_1 until the intersection point becomes the tangency point (relative to the relevant budget line) for the agents' indifference curves. The equilibrium point is allocation x, with the corresponding indifference curves being I^1 and I^2.

3. The Walrasian equilibrium allocation x is inefficient; the agents' better-than-x sets (i.e., upper contour sets for the indifference curves passing though x) have a nonempty intersection. Figure SO.3b quantifies the inefficiency from the tax in a sequence of steps: (i) find the allocation (denoted by x') that is feasible and is most preferred by agent 2 subject to keeping agent 1 at least as happy as he is at the equilibrium allocation x, (ii) identify the additional



(a) Walrasian equilibrium. The seller, agent 1, faces the budget line labeled $p_1(1-t)$, which passes through his perceived endowment point $\tilde{e}$. The buyer, agent 2, faces the price line labeled p_1, which passes through the endowment point e. The Walrasian equilibrium is allocation x, which is the point at which each agent's indifference curve is tangent to his budget line (utility maximization), and at which the budget lines intersect (market clearing).

(b) Welfare loss. The Walrasian equilibrium x is not Pareto efficient, because the agents' better-than-x sets have a nonempty intersection.

Figure SO.3: (Problem 1.7) Walrasian equilibrium with redistributive taxation.

amount Δe_1^2 of good 1 that would make agent 2 as well off at price p_1 as he is with allocation x' (in this case, agent 2 perceives his endowment to be $\tilde{e}^2 \equiv (e_1^2 + \Delta e_1^2, e_2^2)$ and chooses allocation x'' as he maximizes his utility subject to the price vector $(p_1, 1)$). Call Δe_1^2 "welfare loss from taxation." This is the amount of good 1 that agent 2 would demand in order to forgo moving away from the equilibrium allocation x and toward the Pareto efficient allocation x'. (Agent 1 would not demand anything; he is indifferent between x and x'. Also, it is understood that, instead of consuming $x + (\Delta e_1^2, 0)$, agent 2 would use the extra endowment of good 1 to trade toward allocation x''.)

Problem 1.9

The answer to each question is best pondered in an Edgeworth box.

1. Yes, it is possible that one agent envies the bundle of the other at a Walrasian equilibrium. Figure SO.4 is an example that satisfies u-continuity, u-monotonicity, and u-concavity. Intuitively, if one agent is very rich, and the other one is very poor, the poor will envy the rich at equilibrium.

2. No, it cannot be that both agents envy each other. The conclusion requires none of the assumptions of u-continuity, u-monotonicity, and u-concavity.
   By contradiction, suppose that both agents envy each other. The fact that agent 1 envies agent 2 means that agent 1 cannot afford the bundle of agent 2: $p \cdot x^2 > p \cdot e^1$. Similarly for agent 2: $p \cdot x^1 > p \cdot e^2$. Of course, each agent can afford his own equilibrium bundle: $p \cdot x^1 \leq p \cdot e^1$ and $p \cdot x^2 \leq p \cdot e^2$. Combining the four inequalities leads to the contradiction:

   $$p \cdot x^2 > p \cdot e^1 \geq p \cdot x^1 > p \cdot e^2 \geq p \cdot x^2.$$

   To arrive at the result indirectly, recall that any Walrasian equilibrium is Pareto efficient if u-monotonicity holds. If two agents envied each other, there would exist a Pareto improvement; the agents would simply have to swap their equilibrium bundles.
   (The generalization of the result to more than two agents is that there exists no cycle $i_1, i_2, .., i_k, .., i_{K+1}$ with $i_{K+1} = i_1$ of $K \geq 2$ agents such that agent $i_k \in \mathcal{I}$ with $k = 1, 2, \ldots, K$ would be envied by agent $k + 1$.)

3. No, at a Walrasian equilibrium from equal endowments, no agent can envy the other, regardless of whether u-continuity, u-monotonicity, and u-concavity hold. By contradiction, suppose that agent 1 envies agent 2. Then,

   $$p \cdot x^2 > p \cdot e^1 = p \cdot e^2 \geq p \cdot x^2,$$

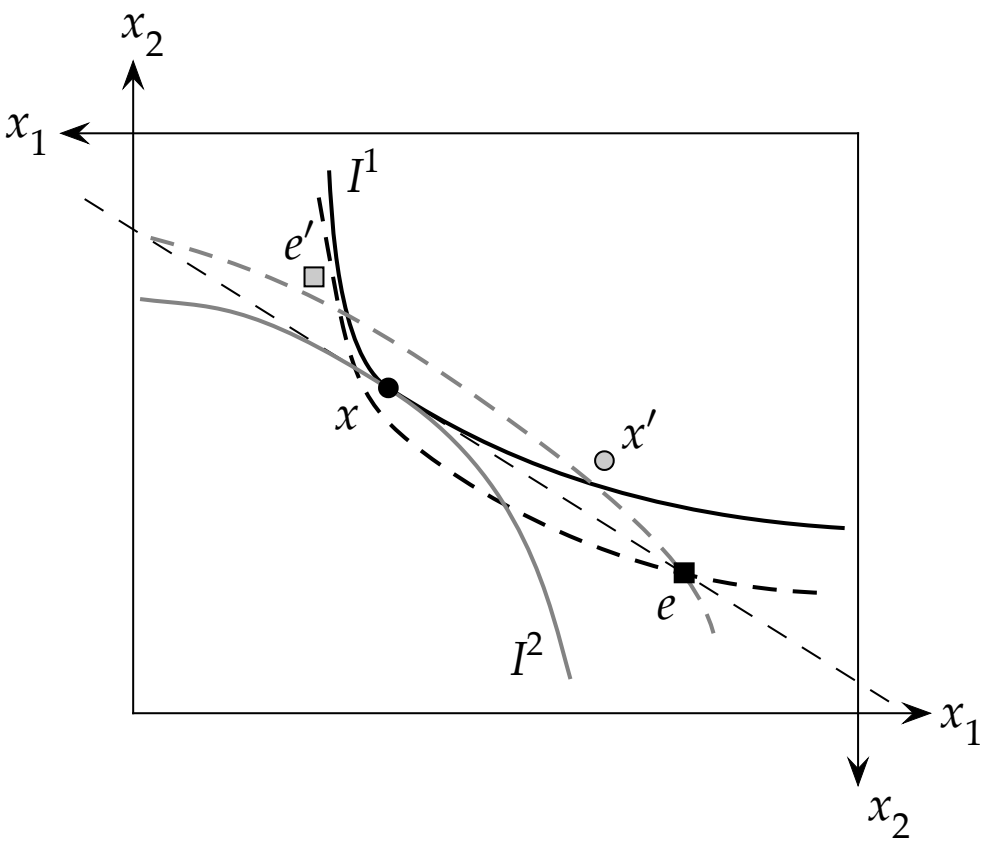


Figure SO.4: (Problem 1.9) An equilibrium in which agent 1 envies agent 2. The endowment bundle profile e is denoted by the solid square. The equilibrium budget line is the dashed line. Agent 1 views the equilibrium bundle profile $x \equiv (x^1, x^2)$, denoted by the solid dot, as inferior to its permuted counterpart $x' \equiv (x^2, x^1)$, denoted by the shaded dot. Note that bundle x' is the point of reflection of x with respect to the center of the Edgeworth box (not marked). The agents' indifference curves passing through x are the solid curves labeled I^1 (for agent 1) and I^2 (for agent 2).

which is a contradiction.

The Walrasian equilibrium from equal endowments suggests a compelling allocation rule, which is Pareto efficient (if u-monotonicity holds) and in which no agent envies another.

4. Figure SO.4 illustrates the possibility of a non-envy-free equilibrium starting from envy-free endowments. The endowment bundle profile $e \equiv (e^1, e^2)$ is denoted by the solid square and is preferred by each agent to the permuted profile $e' \equiv (e^2, e^1)$, denoted by the shaded square, as the dashed indifference curves of agents 1 and 2 indicate. The equilibrium budget line is the dashed line. Agent 1 views the equilibrium bundle profile $x \equiv (x^1, x^2)$, denoted by the solid dot, as inferior to its permuted counterpart $x' \equiv (x^2, x^1)$, denoted by the shaded dot.

5. Call an allocation (x^1, x^2) envy-free if

$$u^1\left(x^1, x^2\right) \geq u^1\left(x^2, x^1\right) \qquad \text{and} \qquad u^2\left(x^2, x^1\right) \geq u^2\left(x^1, x^2\right).$$

Problem 1.11

1. See figure SO.5a.

2. Geometrically, the ex-ante payoff profile at any sunspot equilibrium is a convex combination of the payoff profiles at the three Walrasian equilibria. See figure SO.5b.

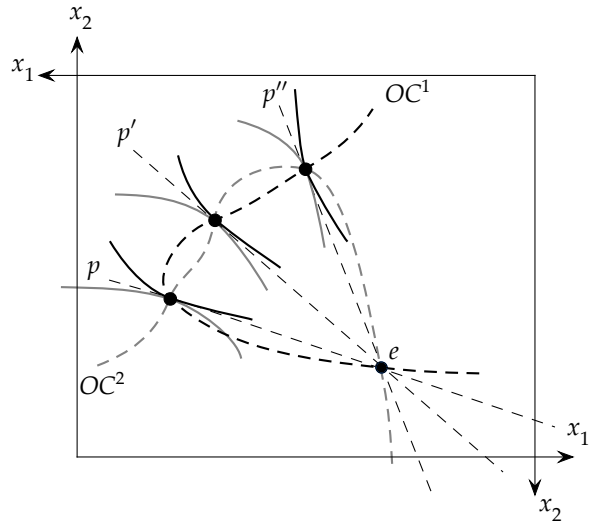
3. A sunspot equilibrium may ex-ante Pareto dominate a Walrasian equilibrium. Indeed, there are many sunspot equilibria that dominate the Walrasian equilibrium $(p', x(p'))$. These sunspot equilibria are depicted in figure SO.5c.

4. The answer does not contradict the FWT, which specifies when Walrasian equilibria are (ex-post) Pareto efficient. In the statement of the FWT, Pareto efficiency is concerned with domination by deterministic allocations, not lotteries over allocations, which are employed in the definition of ex-ante Pareto efficiency.

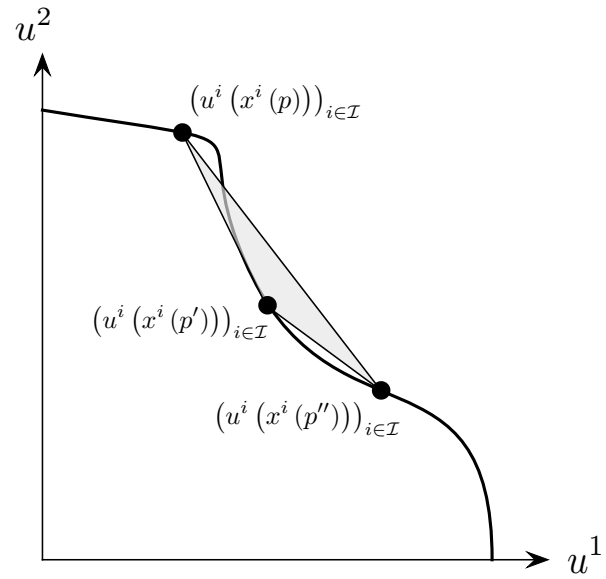
5. No, even if ex-ante Pareto undominated by another sunspot equilibrium, a sunspot equilibrium need not be on the ex-ante Pareto-efficient frontier. The reason for equilibrium inefficiency is that the support of an ex-ante Pareto dominant lottery need not be restricted to Walrasian equilibrium allocations; any feasible allocations would do. Figure SO.5d illustrates. In the figure, the stochastic allocation that induces the sunspot-equilibrium payoff profile A is ex-ante Pareto dominated by a deterministic allocation that induces the payoff profile A'. The stochastic allocation that induces the sunspot-equilibrium payoff profile B is ex-ante Pareto dominated by a stochastic allocation that induces the payoff profile B'.

Problem 1.13

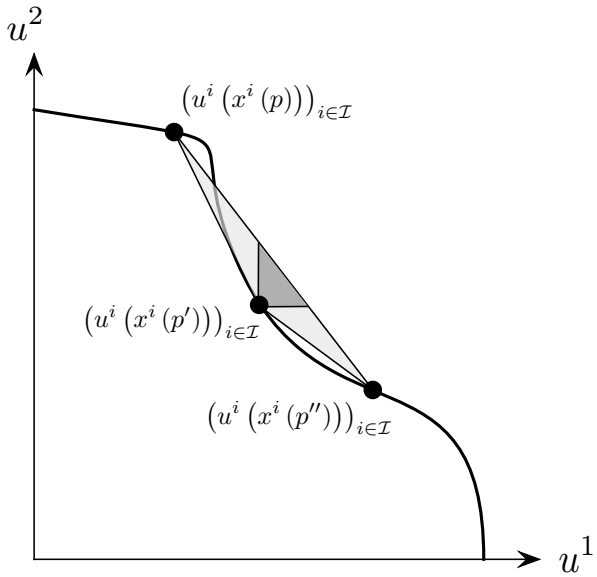
1. Figure SO.6 illustrates the construction.



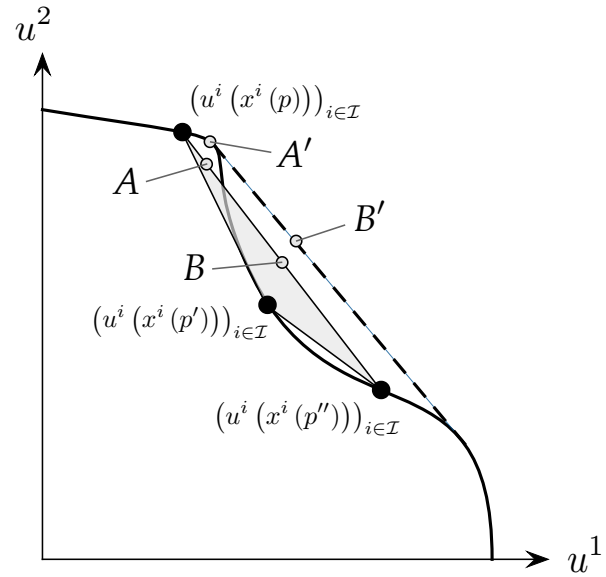
(a) The figure (a clone of figure 1.4a) illustrates three Walrasian equilibria, with the supporting prices p, p', and p'' labeled to agree with figure 1.20.



(b) The set of sunspot-equilibrium ex-ante payoffs is the shaded triangle with the vertices at the three Walrasian equilibrium payoffs.



(c) Every point north-east of the Walrasian equilibrium payoff vector corresponding to price p' is a sunspot-equilibrium payoff that ex-ante Pareto dominates the Walrasian equilibrium.



(d) The ex-ante Pareto-efficient frontier is the concave envelope of the utility possibility frontier. The dashed line is a component of this envelope. Every sunspot equilibrium except the one that induces Walrasian equilibrium price vector p with probability one is below the concave envelope and, hence, is not ex-ante Pareto efficient. For instance, A' and B' ex-ante Pareto dominate, respectively, A and B.

Figure SO.5: (Problem 1.11) Sunspot equilibria.

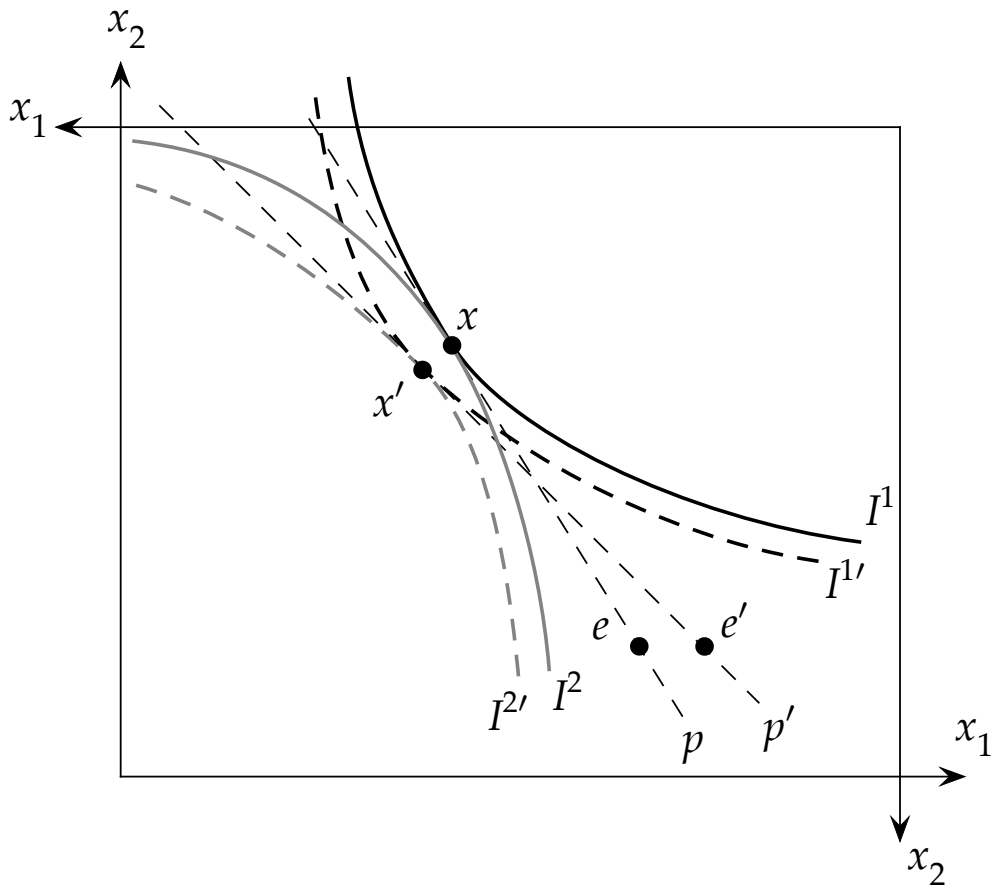


Figure SO.6: (Problem 1.13) The transfer paradox. Even though agent 1's endowment of good 1 increases as the endowment point moves from e to e', the relative price of good 1 falls, from p to p'. Because agent 1 is the seller of good 1, this price change is against his interest. As a result, agent 1 is worse off and agent 2 is better off at the new equilibrium allocation x' compared to the old equilibrium allocation x.

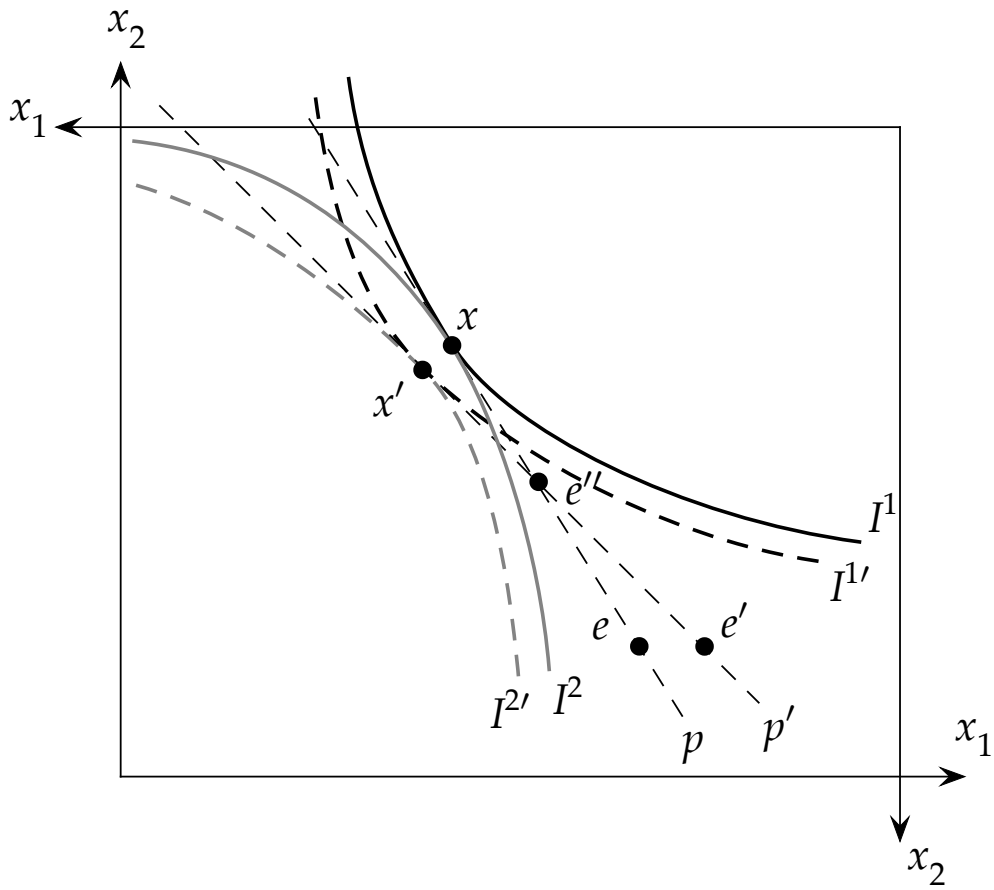


Figure SO.7: (Problem 1.13) The transfer paradox and equilibrium multiplicity. Consider two economies that differ only in endowments, e and e', and that illustrate the transfer paradox. Suppose that the supporting price lines p and p' intersect at an allocation denoted by e''. The third economy that differs from the first two only in the endowment e'' has at least two equilibria, (p, x) and (p, x').

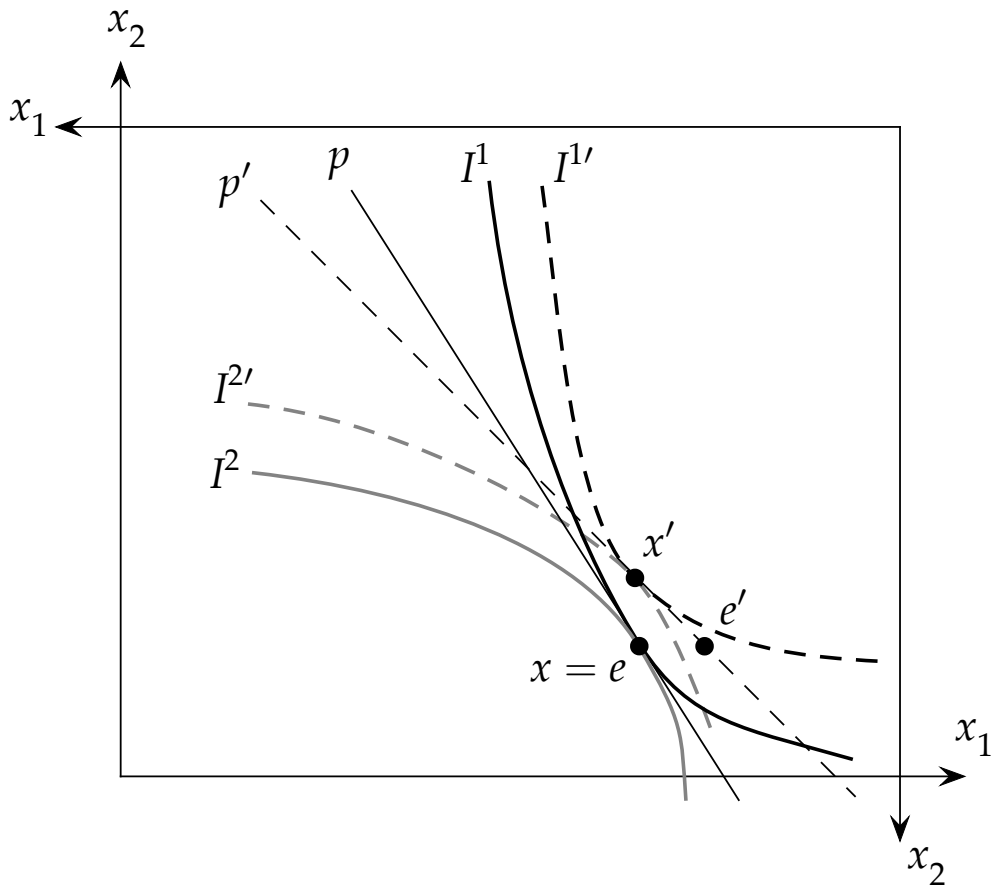
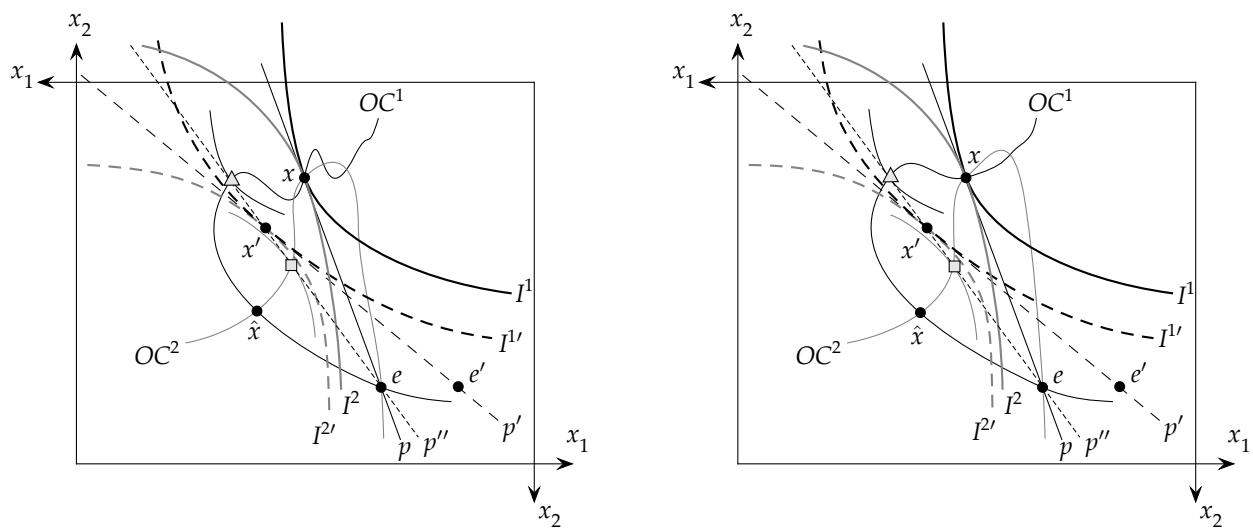


Figure SO.8: (Problem 1.13) The transfer paradox. The initial endowment is e. Because e is Pareto efficient, and u-concavity holds, e is also the unique Walrasian equilibrium allocation. Agent 1 cannot become worse off when his endowment increases from e to e', because, at all prices, he can afford to consume his initial endowment, which is also his initial equilibrium allocation.

2. Figure SO.7 illustrates the multiplicity of equilibria implied by the transfer paradox.

3. No. If agent 1's initial endowment is Pareto efficient, u-concavity holds, and agent 1 receives some amount of good 1 from agent 2, then agent 1 can only be better off, and agent 2 can only be worse off. Figure SO.8 explains why this is so; the initial endowment is also the initial Walrasian equilibrium allocation, and so agent 1's initial equilibrium bundle remains affordable as his endowment rises. The analogous argument does not go through without Pareto efficiency of the initial endowment, in which case agent 1's initial equilibrium bundle may be unaffordable to him from the new endowment at new equilibrium prices.
   In applications, the economy is likely to be at a Walrasian equilibrium before a transfer is administered. Then, if the conditions of the FWT hold, the initial allocation is Pareto efficient. Hence, in applications, the transfer paradox is unlikely.

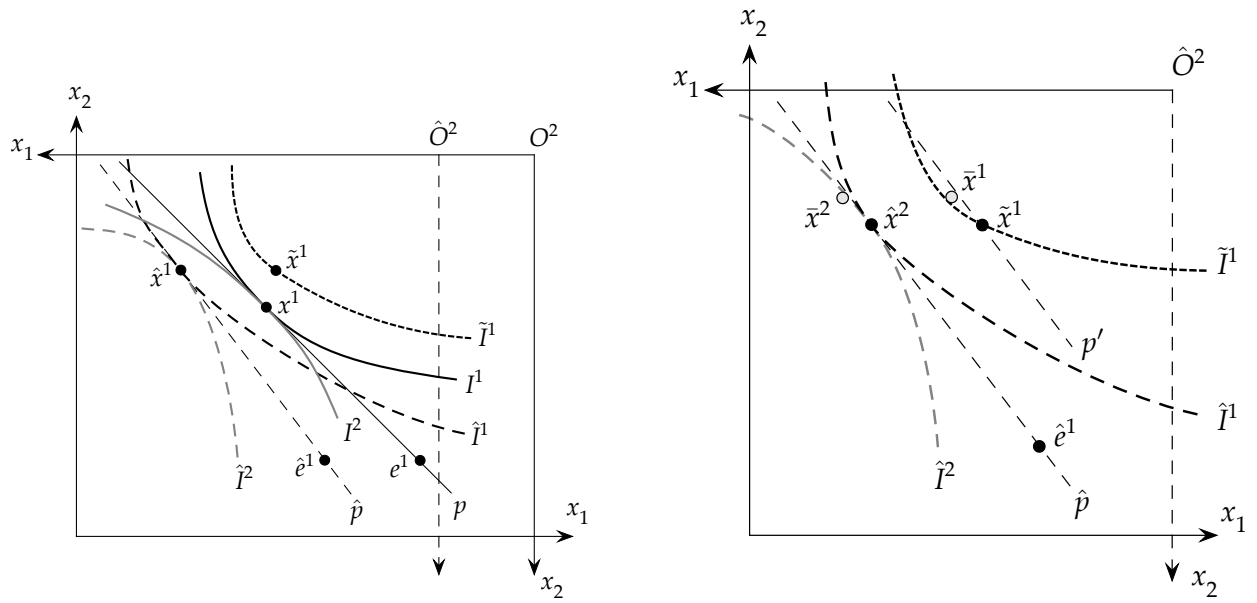
4. Figure SO.9 illustrates the argument.



(a) The initial equilibrium, x, is odd-numbered and, hence, stable.

(b) The initial equilibrium, x, is even numbered and, hence, unstable.

Figure SO.9: (Problem 1.13) The transfer paradox and equilibrium multiplicity and stability. Each panel considers two economies, which differ only in endowments: e, which we call "initial endowment," and e', which we call "new endowment." We would like to partially reconstruct the agents' offer curves, OC^1 and OC^2, for the economy with endowment e. These curves intersect at x, the initial equilibrium from e. Furthermore, OC^1 passes through the point marked by the triangle northwest of x'; indeed, because the line (denoted by p'') that passes through e and x' is steeper than p', its tangency point with agent 1's indifference curve must be northwest of x'. Analogously, OC^2 passes through the point marked by the rectangle southeast of x'. Because the triangle lies northwest of the rectangle, and OC^2 cannot remain above OC^1. As the price line going through e becomes flatter, OC^1 and OC^2 must intersect at least once more, at $\hat{x}$. Furthermore, at $\hat{x}$, agent 1 is worse off than at x'. Hence, the transfer paradox relies on not picking $\hat{x}$ as the initial equilibrium from e.



(a) Profitable endowment withholding. Alice chooses to withhold amount $e^1 - \hat{e}^1$. Her Walrasian bundle induced by this withholding is $\hat{x}^1$. Without withholding, Alice's Walrasian bundle would have been x^1, which she prefers to $\hat{x}^1$. However, Alice consumes amount $\tilde{x}^1 \equiv \hat{x}^1 + e^1 - \hat{e}^1$, which she prefers to x^1.

(b) Pareto inefficiency of withholding. If the agents' indifference curves have different slopes at the post-withholding equilibrium allocation $\left(\tilde{x}^1, \hat{x}^2\right)$ (graphically, if p', constructed to have the same slope as $\hat{p}$, crosses $\tilde{I}^1$ at $\tilde{x}^1$), then one can construct an alternative allocation $\left(\bar{x}^1, \bar{x}^2\right)$ that is feasible and that Pareto improves on $\left(\tilde{x}^1, \hat{x}^2\right)$. The axes are as seen by the planner conditional on Alice's withholding.

Figure SO.10: (Problem 1.15) Endowment withholding. The economy is described by the Edgeworth box with agent 2's origin at O^2. If Alice misreports her endowment as being $\hat{e}^1$, instead of e^1, then the planner believes that the economy is described by the Edgeworth box with agent 2's origin at $\hat{O}^2$, relative to which agent 2's depicted allocations and indifference curves are measured.

Problem 1.15

The endowment-withholding problem resembles the transfer and immiserizing-growth paradoxes (problems 1.13 and 1.14). Instead of transferring some of her endowment to the other agent (as in the transfer paradox) or destroying it (as in the immiserizing-growth paradox), Alice fails to declare her endowment at first, obtains and consumes the equilibrium bundle with respect to her declared endowment, and then consumes her undeclared, withheld, endowment. The existence of the immiserizing growth paradox implies the occasional profitability of endowment withholding. Indeed, if Alice gains by destroying her endowment, then she gains even more by withholding the same amount, obtaining the induced Walrasian allocation, and then consuming the withheld amount.

1. Figure SO.10a illustrates the case in which Alice would have lost from destroying amount $e^1 - \hat{e}^1$ of her endowment (so the figure does not illustrate immiserizing growth) but benefits from withholding that amount.

2. The outcome need not be Pareto efficient, as figure SO.10b illustrates. There is no a priori reason for Alice's indifference curve at her bundle $\tilde{x}^1$ to have the same slope as agent 2's indifference curve at that agent's bundle $\hat{x}^2$. If the two slopes differ, then there exists a trade, away from $(\tilde{x}^1, \hat{x}^2)$ and toward some new allocation, say, $(\bar{x}^1, \bar{x}^2)$, that makes no agent worse off and makes at least one agent better off.

Chapter 2

Problem 2.1

It is optimal for Alice to choose $y = v$, her true valuation. It would have been optimal for Alice to choose $y = v$ even if she knew x, and so it remains optimal to do so when she does not know x. Indeed, suppose that Alice knows x. If $v > x$, choosing $y > x$ yields Alice the payoff of zero, whereas choosing $y \leq x$, yields her the payoff $x - v$, which is negative. Thus, $y = v$ is optimal when $v > x$. If $v \leq x$, choosing $y \leq x$ yields Alice the payoff of $v - x$, which is nonnegative, whereas choosing $y > x$ yields her the payoff of zero. Thus, $y = v$ is optimal when $v \leq x$.

Problem 2.3

1. The probability that exactly $(I-1)/2$ individuals vote in favor and $(I-1)/2$ vote against is
$$\frac{(I-1)!}{\left(\left(\frac{I-1}{2}\right)!\right)^2} p^{\frac{I-1}{2}} (1-p)^{\frac{I-1}{2}},$$
where the coefficient multiplying $p^{\frac{I-1}{2}} (1-p)^{\frac{I-1}{2}}$ is the number of ways in which one can choose $(I-1)/2$ individuals out of I to vote in favor (and the remaining one's vote against). This coefficient is called "combination" or "binomial coefficient," and would sometimes be denoted by $C(I, (I-1)/2)$.

2. The implicit idea is that the expression in part 1 is a measure of responsibility of an individual voter. So if the responsibility is, in some informal sense shared, as it is in voting, does the aggregate responsibility add up to 1? Is it increasing or decreasing in the number of individuals? The sum of the probabilities of being pivotal is
$$\frac{I!}{\left(\left(\frac{I-1}{2}\right)!\right)^2} p^{\frac{I-1}{2}} (1-p)^{\frac{I-1}{2}}.$$
Suppose $p = 1/2$. Then, the above display becomes
$$\frac{I!}{\left(\left(\frac{I-1}{2}\right)!\right)^2} \left(\frac{1}{2}\right)^{I-1} \approx \sqrt{\frac{2I}{\pi}},$$
where the approximate equality, valid for large I, is by Stirling's approximation of the factorial.[SO.1] Thus, the aggregate responsibility is increasing in the number of voters. Figure il-

[SO.1] Stirling's approximation is $I! \approx \sqrt{2\pi I}\,(I/e)^I$, where I is "large." Stirling's approximation is a more precise version of the rough approximation obtained by observing $\ln(I!) = \sum_{x=1}^{I} \ln x \approx \int_1^I \ln x \mathrm{d}x = [x \ln x - x]_1^I = I \ln I - I + 1 \approx I \ln I - I$, and exponentiating both sides of the resulting approximation $\ln(I!) \approx I \ln I - I$ to obtain $I! \approx (I/e)^I$, which differs from Stirling's approximation in that the multiplicative correction $\sqrt{2\pi I}$ is absent.

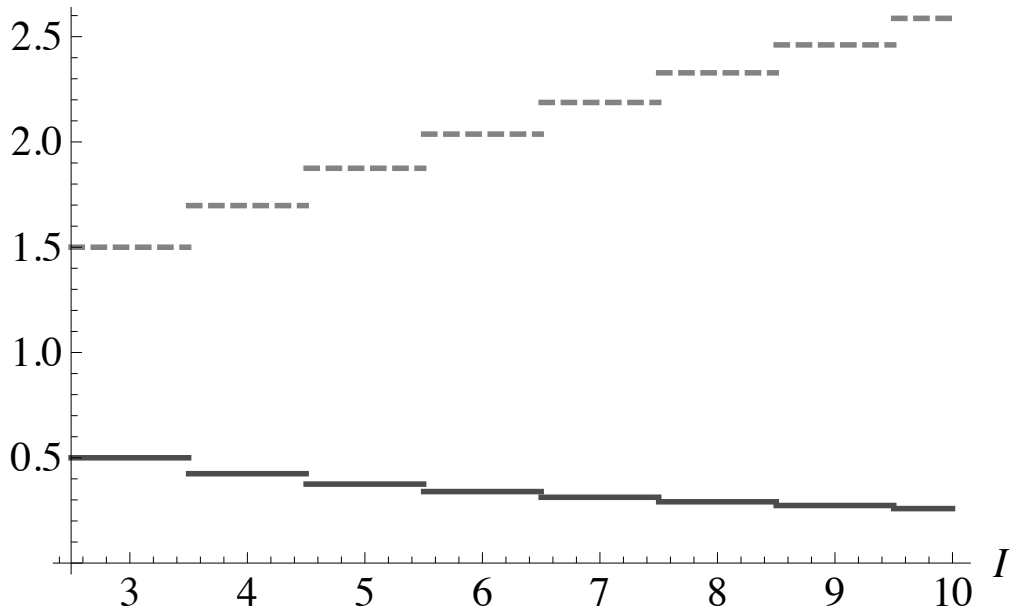


Figure SO.11: (Problem 2.3) Individual and collective responsibilities. The individual responsibility, the probability of being pivotal when $p = 1/2$, is the solid schedule, decreasing in I, the number of voters. The collective responsibility, the sum of probabilities of being pivotal when $p = 1/2$, is the dashed schedule, increasing in I.

lustrates how the probability of being pivotal and the sum of these probabilities varies with the number of voters.

3. Supposing $p = 1/2$ and using Stirling's approximation, the probability in part 1 approximately equals

$$\sqrt{\frac{2}{\pi I}},$$

which is decreasing in I. Thus, the individual responsibility is decreasing in the number of voters, but at a rate smaller than the number of voters.

4. It is impossible to judge which operationalization of the concept of responsibility is best without deciding first what this operationalization will be used for. Tentatively, the property of being ex-ante pivotal is an appealing operationalization of responsibility in that this property illustrates the dilution of responsibility without insisting that the collective responsibility (the sum of individual responsibilities) remain constant as the number of individuals in the group increases. More generally, it is not obvious that one should identify collective responsibility with the sum of individual responsibilities, and that the individual responsibility should be decreasing in the number of participants.

Chapter 3

Problem 3.1

Let π_t denote the probability that the adjustment process has arrived at the jungle equilibrium (SD allocation) by time t. The adjustment process is said to converge if $\lim_{t\to\infty} \pi_t = 1$.

To prove convergence of the described procedure, set $t = In$, where n is a positive integer number of periods, and I is the number of agents. Wait for agent 1 to get the chance to expropriate his favorite house during the first block of n periods. If his turn comes, if he encounters the agent who holds his favorite house, and if that agent happens to be less powerful, then he will choose his favorite house and will never part with it. More generally, wait for agent i to get his chance to expropriate his SD house during the ith block. Agent i will choose his most preferred house among the houses not chosen by agents $1, 2, \ldots, i-1$, the more powerful agents, and will never part with it. If each agent i expropriates during his designated i^{th} block, then no agent will ever be expropriated after his designated block, nor will he expropriate anyone else.

Formally, the probability that agent i gets a chance to expropriate his SD house during his designated block is $1 - \left(1 - \frac{1}{I}\frac{1}{I-1}\frac{I-i}{I-1}\right)^n$, where $1/I$ is the probability that agent i is chosen to expropriate in a given period, $1/(I-1)$ is the probability that he is matched with an agent who holds his house, and $(I-i)/(I-1)$ is the probability that agent i is more powerful than the agent he is matched with. The probability that each agent i gets a chance to expropriate during i^{th} block is at least $\prod_{i=1}^{I}\left(1 - \left(1 - \frac{1}{I}\frac{1}{I-1}\frac{I-i}{I-1}\right)^n\right)$, whose limit is 1 as $n \to \infty$. Thus, the described scenario alone is guaranteed to lead to convergence with probability that approaches 1 as $n \to \infty$. And the described convergence scenario is not the only one. Thus, the adjustment process converges.

Problem 3.3

Following the hint given in the problem's statement, execute TTC to obtain the Walrasian equilibrium allocation. Define the power relation so that the agent to whom TTC allocates a house at an earlier round is more powerful. Let the power ranking for the agents who get their houses at the same round be arbitrary. It will be shown that the jungle equilibrium of the economy induced by the constructed power relation selects the TTC allocation.

The jungle equilibrium outcome is SD outcome. The agents to whom TTC allocated at the first round (sequentially) choose first in SD and face no conflict. Each of them chooses his most preferred house, as in TTC. For an argument by induction, suppose that everyone who is allocated at round r in TTC gets the same house in SD. The agents to whom TTC allocated at round $r+1$ choose in SD among the best available houses and face no conflict in their choices, just as in TTC. Thus, TTC and SD with the derived power relation lead to identical allocations.

Problem 3.5

1. It suffices to construct an example in which there are no bilateral mutually beneficial house exchanges, but there are three-way or more complex exchanges. Example 3.2 accomplishes just that.

2. First, note that allocation x is pairwise efficient. Any two agents can agree on exactly one good in common that each of them values. Each agent in the pair holds some of that good and some of another good, which he values, but the other agent does not. Hence, there is no room for a profitable bilateral exchange. Nevertheless, x is not Pareto efficient. It is dominated by feasible allocation $\hat{x}$ with $\hat{x}^1 = (2, 0, 0)$, $\hat{x}^2 = (0, 2, 0)$, and $\hat{x}^3 = (0, 0, 2)$.

3. Both P and P^{ij} for each $i, j \in \mathcal{I}$ are problems with concave objective functions (u-concavity) and convex feasible sets described by the constraints (u-concavity ensures that all upper contour sets are convex, whose intersection with each other and the half-space induced by the linear feasibility constraint is also convex). Thus, the Lagrangian approach characterizes solutions. The Lagrangian for P^{ij} is

$$\mathcal{L}^{ij} = u^i\left(y^i\right) + \gamma\left(u^i\left(y^i\right) - u^j\left(x^j\right)\right) + \sum_{l\in\mathcal{L}} \lambda_l\left(x_l^i + x_l^j - y_l^i - y_l^j\right).$$

The corresponding optimality conditions, which are satisfied by the interior x by hypothesis, are

$$u_l^i\left(x^i\right) = \lambda_l \qquad \text{and} \qquad \gamma u_l^j\left(x^j\right) = \lambda_l, \qquad l \in \mathcal{L},$$

where $\lambda_l > 0$ for all $l \in \mathcal{L}$ and $\gamma > 0$ by u-monotonicity. Taking the ratios of the optimality conditions in each P^{ij} gives

$$\frac{u_l^i\left(x^i\right)}{u_{l'}^i\left(x^i\right)} = \frac{u_l^j\left(x^j\right)}{u_{l'}^j\left(x^j\right)}, \qquad i, j \in \mathcal{I} \quad \text{and} \quad l, l' \in \mathcal{L}.$$

That is, pairwise efficiency equates the marginal rates of substitution between any two goods for any two agents, which is the same condition that characterizes Pareto efficiency. Indeed, the Lagrangian for P is

$$\mathcal{L} = u^1\left(y^1\right) + \sum_{i\in\mathcal{I}\setminus\{1\}} \gamma^i\left(u^i\left(y^i\right) - u^i\left(x^i\right)\right) + \sum_{l\in\mathcal{L}} \lambda_l\left(\bar{e}_l - \sum_{i\in\mathcal{I}} y_l^i\right).$$

By u-monotonicity, $\lambda_l > 0$ for all $l \in \mathcal{L}$, and $\gamma^i > 0$ for all $i \in \mathcal{I}$, and hence the corresponding sufficient optimality conditions for an interior solution y can be rewritten (after computing appropriate ratios to remove λs and γs) as

$$\frac{u_l^i\left(y^i\right)}{u_{l'}^i\left(y^i\right)} = \frac{u_l^j\left(y^j\right)}{u_{l'}^j\left(y^j\right)}, \qquad i, j \in \mathcal{I} \quad \text{and} \quad l, l' \in \mathcal{L}$$

combined with all the constraints in P holding as equalities. Note that the interior pairwise efficient allocation x satisfies each of these constraints.

Problem 3.7

1. By choosing his report, no agent can affect the order in which the agents will be called upon to choose and hence also the goods available to him when he is called upon to choose. Conditional on being called upon to choose, the agent can do no better than be truthful, thereby securing his most preferred bundle among the remaining bundles.

2. The argument is identical to that for SD.

3. The agent who gets his house in round 1 gets his first choice and hence has no incentive to lie. Suppose that no agent who gets his house at round r or earlier has any incentive to lie, regardless of others' reports. Whatever he reports, an agent who gets his house at round $r+1$—call this agent "Bob"—cannot secure a house allocated at an earlier round. Indeed, to get a house allocated at an earlier round, Bob must become a part of a cycle that includes that house's owner—call her Alice—and the house itself. For this to occur, Alice must have pointed at Bob's house at one of the earlier rounds. If she did so once, she must have continued doing so all the way until round $r+1$ because Bob's house remained available, in contradiction of the assumption that Alice and her house were allocated at an earlier round. Hence, Alice did not point at Bob's house before round $r+1$, and hence no report of Bob can make him a part of a cycle that includes Alice's house.
   Of the houses allocated at round $r+1$, Bob secures his most preferred house by being truthful. Hence, he cannot benefit from lying regardless of others' reports.

Problem 3.9

1. In the example, each man gets his first choice. So, trivially, no man has any incentive to lie.

2. The conclusion follows from the fact that man-proposing DA selects the man-optimal match, and woman-proposing DA selects the man-pessimal match. If the two matches coincide, than each man must have the same match in each stable match. That is, the stable match must be unique.

3. In SD, the most popular man gets his first choice, the second popular man gets his first choice among the remaining women, etc.
   In any stable match, the most popular man must get his first-choice woman. If this were not so, the most popular man and his first-choice would have formed a blocking pair. So, remove the matched couple and consider the remaining agents. The problem is then analogous to the one initially considered. Among the remaining men, the most popular one is guaranteed to get his first choice among the remaining women. And so on.

4. By the symmetry of the problem, it does not matter which agent drops out. Assume m'' drops out. Then, the man-proposing DA and the woman-proposing DA both select the match $\{\{m, w''\}, \{m', w\}, \{w', w'\}\}$. As a result, the stable match is unique by the result in part 2. Note that each man gets his first choice, which would also be the outcome of SD with women objectified and men granted an arbitrary priority order.

5. An easy upper bound is n^2. Suppose that, in the man-proposing DA, all men are rejected one by one, by each woman. It will take n^2 rounds for all men to be rejected, so that no further proposals can be made.

Problem 3.11

1. No. In a direct mechanism, there is only one time at which an agent may contemplate deviating, which is the time at which everyone reports his preference ranking. Suppose that Bob has the lowest priority in SD (and he knows it). The worst he can do by being truthful is to receive his last choice if that choice is the last remaining one when the mechanism chooses for him. The best Bob can do by lying is to receive his first choice. Indeed, even if Bob, say, reports his first choice as his last, his first choice may nevertheless be all that remains when it is his turn to be assigned an object. Thus, one can conceive of others' preferences for which Bob's lie would lead to a better outcome for Bob than truth-telling would for some other constellation of others' preferences.

2. No. The same reasoning as above applies, with the irrelevant for the argument's validity qualification that an agent may not know the realization of the priority order.

3. No. The worst one can do by being truthful is to remain with one's own endowment, if, for instance, everyone else prefers his own endowment to any other object. The best one can do by lying is to name one's second choice and receive it. Hence, with at least two agents and two objects, TTC is not obviously strategy-proof.

4. Yes. The only time an agent may contemplate a deviation is when he is asked to choose from the set of remaining objects. The worst the agent can do by being truthful is to get is preferred remaining object. The best the agent can do by lying is to get his second choice from the remaining objects.

5. Yes. The argument is the same as in the case of the sequential SD.

6. No. Consider the first round. If Alice is truthful, the worst she can get is to remain with her endowment. If she lies, the best she can get is her first choice. She will not get it in the first round because she lied, meaning that she failed to declare her first choice. Suppose she passes onto the second round, as does Bob, who is endowed with Alice's first choice and whose first choice has been allocated to someone else in the first round. In the second round, Alice truthfully declares her first choice (Bob's house), and Bob declares his first choice among the

remaining houses, which is assumed to be Alice's house. Thus, Alice and Bob swap their houses, and Alice ends up with her first choice.

Problem 3.13

1. At first, the agents' preferences are

$$m' \succ^{w} m, \qquad m \succ^{w'} m', \qquad w' \succ^{m} w, \qquad w' \succ^{m'} w.$$

The man-proposing DA selects the match $\{(w, m'), (w', m)\}$.
Then man m's preference changes; he flips his ranking of the two women:

$$w \succ^{m} w'.$$

The man-proposing DA selects the match $\{(w, m), (w', m')\}$. Thus, woman w is worse off in spite of having moved up in a man's ranking. Woman w' is also worse off.
The paradox is due to a "general equilibrium" effect. Man m, whom woman w does not care much about, suddenly prefers her to woman w'. As a result, there is less competition for woman w', and, so, man m' manages to get accepted by woman w'. Not being rejected by woman w', man m' does not come to propose to woman w in the following round.
To summarize, as in the transfer paradox in the Walrasian model, the recipient of the "transfer," woman w, is worse off. By contrast to the transfer paradox in the Walrasian model, however, the payer of the "transfer," woman w', is worse off.
One practical advice based on the example is this. If, say, Alice and Carol come to a party together, Alice should immediately do something that would make her less desirable to the men whom she does not quite fancy. (For instance, she could intimidate short men by wearing high heels, even if she does not care at all about the man's height per se; she just knows that the particular man she is after, Bob, is tall.) Doing so will increase competition for Carol, who will have to reject all men but one. Some of the men rejected by Carol—including, say, Bob—would then approach Alice, to Alice's advantage.

2. In general, for the marriage problem with any number of agents, suppose that man m is matched to woman w in the match x selected by the man-proposing DA. If w or some woman whom m likes less than w raises him in her ranking, match x is still selected. If some woman w' whom m likes more than w raises him in her ranking, one of two outcomes is possible. When m proposes to w', w' may tentatively accept him and then never reject him, thereby leading to a better outcome for him. Alternatively, when m proposes to w', w' may nevertheless eventually reject him, in which case m will be eventually retained by w.

Chapter 4

Problem 4.1

1. This case is analyzed in part 1 of theorem 4.1. Each worker consumes γ.

2. There are even more workers now; part 1 of theorem 4.1 still applies, and each worker consumes γ. PEAs have nothing to donate.

3. Assume that Hulac workers anticipate the transfers. Then, each worker weakly prefers working for the capitalist to remaining on the farm, at any nonnegative wage, for the top-up in the amount of γ comes from PEAs. Analogously to the argument in part 1 of theorem 4.1, conclude that the equilibrium wage is zero. Each worker consumes γ. (The capitalist's profit rises by $\gamma Kb/a$, the amount of PEAs' subsidy to the employed workers.)

4. Assume that Hulac workers anticipate the transfers. If the transfer exceeds γ, a worker may wish to work for the capitalist even if the wage is negative, to claim the transfer. Hence, one should make a stance on whether wages are allowed to be negative. If not, the capitalist would have to ration employment. But auctioning employment (by collecting bribes) seems a more natural modeling assumption and agrees with the definition of Walrasian equilibrium. Then, the equilibrium wage (positive or negative) will be such that the worker is indifferent between staying on the farm and working for the capitalist:

$$w + \frac{P}{K+P}\left[\frac{K+P}{a} - (K+P) - \frac{b\,(K+P)}{a}w\right]\frac{a}{b\,(K+P)} = \gamma,$$

where the bracketed expression is the capitalists' profit, fraction $P/\,(P+K)$ of which is claimed by PEAs and transferred to the employed, of which there are $(K+P)\,b/a$. The display above can be rewritten as

$$\frac{P}{K+P}\frac{1}{\lambda} + \frac{K}{K+P}w = \gamma.$$

The implied wage is

$$w = \gamma - \frac{P}{K}\left(\frac{1}{\lambda} - \gamma\right),$$

where the term in parenthesis is positive by $\gamma < 1/\lambda$. Thus, in particular, $w < \gamma$.

5. Now, PEAs make labor scarce relative to capital. By part 2 of theorem 4.1, the capitalists' profit is zero, and, so, there is no profit to transfer. Nevertheless, all workers are the proletariat and better off than without PEAs because their equilibrium wage exceeds their value of staying on the farm: $w = 1/\lambda > \gamma$.

6. Scenario D with $W/b < (K+P)\,/a$ is the only one in which PEAs benefit workers. Under this scenario, PEAs benefit workers by improving the workers' wages by making workers

scarce relative to capital. So, Peter Singer's advice in *The Most Good You Can Do: How Effective Altruism Is Changing Ideas About Living Ethically* (Yale University Press, 2015) to his student not to pursue doctoral studies in philosophy but rather work on the Wall Street and donate his earnings to charity has merit (although, according to this problem, there will be no earnings left to donate if philosophy majors join the Wall Street en masse).

Problem 4.3

1. The agents' problems are independent across periods. Hence, one can normalize the prices of both stuff and capital to 1 in each period. At $t = 1$, the market clearing wage is $w_1 = \gamma$, which keeps each worker indifferent between working on the farm and working for the capitalist. Measure bK_1/a of workers work for the capitalist, whereas the remaining $W - bK_1/a$ workers stay on the farm. The capitalist's payoff at $t = 1$ and, hence, also the capital at $t = 2$ is

$$K_2 = \frac{K_1}{a} - \gamma \frac{bK_1}{a} = \frac{K_1 (1 - \gamma b)}{a}.$$

At $t = 2$, two cases are possible: $W/b > K_2/a$ and $W/b < K_2/a$. (Neglect the knife-edge case $W/b = K_2/a$.) In the former case, when

$$W > \left(\frac{b}{a}\right)^2 K_1 \left(\frac{1}{b} - \gamma\right),$$

$w_2 = \gamma$, and bK_2/a workers work for the capitalist. In the latter case, when

$$W < \left(\frac{b}{a}\right)^2 K_1 \left(\frac{1}{b} - \gamma\right),$$

$w_2 = 1/\lambda$, and all workers work for the capitalist.

2. With scarce labor, $w_1 = 1/\lambda$, $b(K_1 - \varepsilon)/a$ workers work for the capitalist, and $W - b(K_1 - \varepsilon)/a$ workers stay on the farm. The capitalist's payoff, and, so, the capital at $t = 2$, is

$$K_2 = \max_{k \in [0, K_1]} \left\{ \min \left\{ \frac{K_1 - \varepsilon}{a}, \frac{k}{a} \right\} - \frac{1}{\lambda} \frac{b(K_1 - \varepsilon)}{a} - k \right\} + K_1 = K_1.$$

In other words, with scarce labor, the equilibrium wage w_1 is so high that the capitalist is left with no surplus and, hence, no new capital to carry over into $t = 2$.
At $t = 2$, because $W/b > K_2/a$ (by $K_2 = K_1$), $w_2 = \gamma$, and bK_2/a workers work for the capitalist. Thus, with the labor union at $t = 1$, the workers are guaranteed to be down to their subsistence consumption level at $t = 2$.

3. The workers are better off at $t = 2$ without labor unions at $t = 1$ if and only if

$$W < \left(\frac{b}{a}\right)^2 K_1 \left(\frac{1}{b} - \gamma\right).$$

Thus, if $a < (1 - \gamma b)$, then one can find a W such that both the condition above holds and $W/b > K_1/a$.

4. Without labor unions at $t = 1$, the workers' aggregate wage bill over the two periods is

$$\gamma W + \frac{1}{\lambda} W$$

if and only if

$$W < \left(\frac{b}{a}\right)^2 K_1 \left(\frac{1}{b} - \gamma\right).$$

With labor unions at $t = 1$, the workers' aggregate wage bill over the two periods is

$$\left(\frac{1}{\lambda} \frac{b\,(K_1 - \varepsilon)}{a} + \gamma \left(W - \frac{b\,(K_1 - \varepsilon)}{a}\right)\right) + \gamma W,$$

which is lower than the highest payoff without labor unions reported above. Thus, the requisite condition is the same as in part 3.

5. With the labor union present in both periods, the workers' aggregate wage bill is

$$2\left(\frac{1}{\lambda} \frac{b\,(K_1 - \varepsilon)}{a} + \gamma \left(W - \frac{b\,(K_1 - \varepsilon)}{a}\right)\right).$$

This wage bill is smaller than the maximal aggregate wage bill without the labor union if

$$W > \frac{2b\,(K_1 - \varepsilon)}{a},$$

which implies $W/b > K_1/a$ (recall that $\varepsilon > 0$ is arbitrarily small). Thus, the two requisite conditions are

$$W < \left(\frac{b}{a}\right)^2 K_1 \left(\frac{1}{b} - \gamma\right) \qquad \text{and} \qquad W > \frac{2b\,(K_1 - \varepsilon)}{a}.$$

Combining the two conditions in the display above, one concludes that one can find a W and an ε for which the labor union leads to a lower aggregate wage bill if

$$2a < 1 - \gamma b.$$

The condition essentially requires that the Leontief technology be sufficiently productive (i.e., both a and b be small) so that, without the labor union, production at $t = 1$ leads to a high profit at $t = 1$, which translates into abundant capital at $t = 2$, which renders labor relatively scarce at $t = 2$. It also helps if the workers' subsistence consumption is low, so that getting

a high wage from the capitalist at $t = 2$ makes a lot of a difference to the workers relative to remaining on the farm.

6. If permitted, the capitalist may wish to consume at $t = 1$. Then, capital would be less abundant at $t = 2$, and, so, the workers' bargaining power at $t = 2$ may not improve without the labor union.

Problem 4.5

1. Optimization occurs twice. The first, outer, maximization commits the capitalist to a capital mix before the capitalist learns which policy is implemented. The second, inner, maximization selects employment mix when the implemented policy is already known.

2. The capitalist will optimally hire as much labor as is necessary to fully employ the skill-specific capital in which he has invested: $h_s = b_s k_s / a_s$. Thus, the capitalist's problem becomes

$$\max_{(k_s)_{s\in S}\in\mathbb{R}_+} \left\{ \sum_{s\in\mathcal{S}} k_s \eta_s \mid k_1 + k_2 \leq K \right\}, \qquad \text{where } \eta_s \equiv \frac{p_s\left(1 - w_s b_s\right)}{a_s} - 1.$$

The problem in the display above is linear in (k_1, k_2). Then, because $W_s > b_s K / a_s$, the capitalist invests all his generic capital in the skill-specific capital whose corresponding η_s is the highest as long as it is nonnegative. If each η_s is negative, the capitalist does not invest in any skill-specific capital.

3. The equilibrium wage is $w_s = \gamma_s$, $s \in S$. When p contains $p_s = 1$, the capitalist's profit is

$$\Pi\left(p\right) = K\left(\frac{1 - \gamma_s b_s}{a_s} - 1\right).$$

4. The requisite condition is

$$\max_{s\in S}\left\{p_s \frac{1 - \gamma_s b_s}{a_s}\right\} < \min_{s\in S}\left\{\frac{1 - \gamma_s b_s}{a_s}\right\}.$$

The probability vector p that minimizes the left-hand side of the inequality above has

$$p_1 = \frac{1}{1 + \frac{a_2(1-\gamma_1 b_1)}{a_1(1-\gamma_2 b_2)}}, \qquad p_2 = 1 - p_1.$$

Substituting the display above in the preceding inequality implies that one can find an uncertain immigration policy that is worse than the worst deterministic immigration policy if

$$\frac{1}{\frac{a_1}{1-\gamma_1 b_1} + \frac{a_2}{1-b_2\gamma_2}} < \min_{s\in S}\left\{\frac{1 - \gamma_s b_s}{a_s}\right\}.$$

The obtained inequality is equivalent to

$$\min_{s \in S} \left\{ \frac{\frac{a_s(1-\gamma_{-s}b_{-s})}{a_{-s}(1-\gamma_s b_s)}}{\frac{a_1}{1-\gamma_1 b_1} + \frac{a_2}{1-b_2\gamma_2}} \right\} > 0,$$

which always holds.

5. Yes. The welfare of immigrant labor is unaffected by NonCanada's immigration policy. At equilibrium, each worker earns his outside option. Hence, all Pareto comparisons coincide with the comparisons of the capitalist's welfare.

Problem 4.7

1. The Pareto efficient allocations can be identified by brute force. One can check each of $3! = 6$ possible allocations and verify whether a Pareto improvement exists. One will then be left with three Pareto efficient allocations: $\{\{1, A\}, \{2, B\}, \{3, C\}\}$, $\{\{1, C\}, \{2, B\}, \{3, A\}\}$, and $\{\{1, B\}, \{2, C\}, \{3, A\}\}$.
   A systematic way for identifying Pareto efficient allocations is to note that, not only any SD outcome is Pareto efficient, but also any Pareto efficient allocation is delivered by SD for some power ordering. Thus, by trying all possible power orders and executing the associated SD, one can identify all Pareto efficient allocations. (Multiple power orderings may induce the same allocation.) This observation is nontrivial and requires a proof.
   Here is the proof (due to Lina Lukyantseva). First, a claim: in any Pareto efficient allocation, at least one agent gets his first choice. By contradiction, consider a Pareto efficient allocation in which no agent gets his first choice. Draw a directed graph whose nodes are agents. Each agent i points at an arbitrary agent who has been assigned a house that agent i prefers to his own assignment. Because, by hypothesis, no agent gets his first choice, each agent points to some other agent. The graph is finite, and, so, a cycle exists. Exchanging the houses among the agents in a cycle Pareto improves on the original allocation, which contradicts the original allocation's Pareto efficiency.
   Fix a Pareto efficient allocation. Pick an agent who gets his first choice in this allocation (such an agent exists by the claim), and make him the most powerful one in the power ordering to be used in SD. Then remove this agent and his house from consideration. Among the remaining agents, at least one gets his first choice among the remaining houses. Designate one of such agents to be the second most powerful one, and remove him along with his house. And so on. SD with the power ordering so constructed induces the Pareto efficient allocation the power ordering has been constructed from.

2. TTC selects the unique core allocation, which, here, is $\{\{1, C\}, \{2, B\}, \{3, A\}\}$. Indeed, each agent who is allocated a house at the first round gets his first choice and, hence, cannot be a member of a blocking coalition. Without these agents, no agent allocated at the second

round can improve on his TTC allocation and, hence, will not be a member of a blocking coalition. Continuing in this vein, conclude that no agent can be a member of a blocking coalition. Hence, the TTC allocation is in the core.

To see that the core is unique, note that the house assignments to the agents allocated in the first round are necessary in any core, or else these agents can form a blocking coalition and exchange the endowments among themselves. Given the assignments of the first-round agents are necessary, the assignments to the second-round agents also necessary, too, or else the second-round agents would form a blocking coalition of their own. Continuing in this vein, conclude that the TTC allocation is the unique core allocation.

Problem 4.9

1. The condition in (4.6) says that, whenever an agent finds the other agent's trade feasible, he prefers the allocation induced by his own trade to the allocation induced by that other agent's trade.

2. To show that envy-freeness in bundles does not imply envy-freeness in trades, assume that Alice's endowment and Bob's endowment are such that neither envies the other. Assume further that trade occurs according to the (non-Walrasian) rule that instructs Alice to give a little of every good to Bob. Assume that the final consumption bundles are still sufficiently close to endowments, so that no agent envies the other. However, Alice envies Bob's trade.

   To show that envy-freeness in trades does not imply envy-freeness in bundles, suppose that Alice's endowment and Bob's endowment are such that Alice envies Bob. Then trade occurs according to the (possibly Walrasian) rule that instructs Alice and Bob not to trade at all. The final consumption bundles coincide with the endowments, and, so, Alice still envies Bob. No agent envies the other's (zero) trade, though.

3. It is tempting to say that envy is a hurtful emotion, that it reduces an agent's enjoyment of his consumption bundle, and, hence, should be avoided by careful design whenever possible. This justification of the envy-freeness in trades is facile. For one thing, agents consume bundles, not trades. What ought to matter is whether individuals envy each other's consumption bundles, not trades. For another thing, if individuals envy trades, these trades must enter utilities (i.e., agent i's utility function must be $u^i\left(x^i, t^i, t^{-i}\right)$, not $u^i\left(x^i\right)$), and, so, the present model is misspecified.[SO.2]

   If one insists on a normative justification of envy-freeness in trades, it is better to deemphasize the private sentiment of envy and to seek a disembodied, extrinsic interpretation of the condition. One such interpretation is the equality of opportunity. Each agent has access to the same opportunity set of trades and picks the trade he likes most.

[SO.2] One may be further tempted to dismiss such other-regarding preferences as promoting what Deirdre McCloskey calls (in a book review) the "narrow ethic of envy," but this dismissal would be unjustified. One cannot banish an offensive preference out of existence.

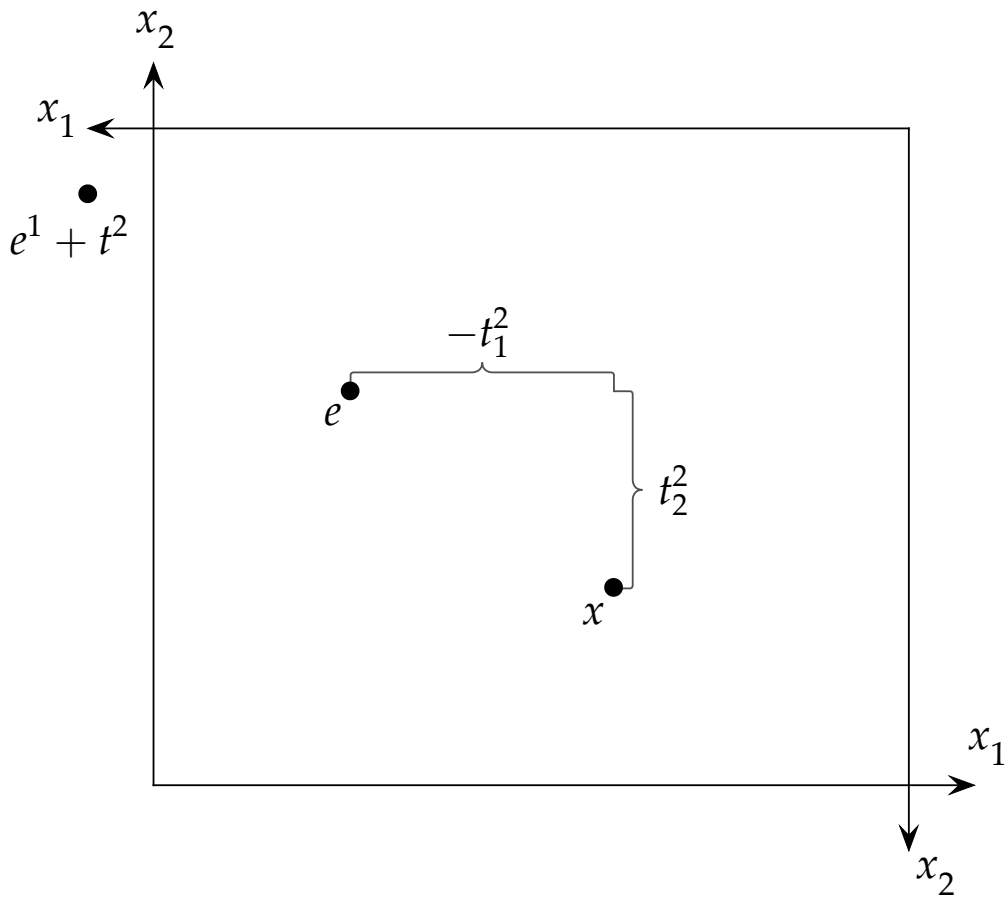


Figure SO.12: (Problem 4.9) For an agent, executing another agent's trade may be infeasible. Agent 1 is endowed with very little of good 1, whereas agent 2 is endowed with a lot of it and sells a lot of it.

A positive justification of envy-freeness in trades acknowledges that it captures aspects of incentive compatibility. Suppose that a planner knows that the agents' utilities and endowments are (u^1, e^1) and (u^2, e^2), and would like to implement a Walrasian equilibrium outcome (x^1, x^2). The planner does not know, however, which of the two agents, Alice or Bob, is the (u^1, e^1)-agent, and which one is the (u^2, e^2)-agent. He can ask, but Alice and Bob can lie if they expect to benefit from lying. Envy-freeness in trades implies that the planner can devise an incentive-compatible way to elicit Alice's and Bob's types. In particular, the planner computes equilibrium trades (t^1, t^2) and asks, say, Alice to pick her preferred trade. If executing Bob's trade is infeasible, Alice cannot possibly lie. If executing Bob's trade is feasible, the envy-freeness in trades ensures that Alice has no incentive to lie. The same argument applies to Bob. (The heroic assumption here is that the planner knows the utility-endowment pairs, and only does not know which agent is characterized by which.)

4. Figure SO.12 illustrates.

5. The conclusion follows by revealed preference. Because the value of all equilibrium trades is the same and zero, an agent can always afford the other agent's trade but chooses his own trade. Hence, he cannot envy the other agent's trade. Formally, let (p, x) be a Walrasian equilibrium. Then, $p \cdot x^i \leq p \cdot e^i$ (i.e., each agent can afford his own trade), or, equivalently, $p \cdot (x^i - e^i) = p \cdot t^i \leq 0$ (i.e., the value of each trade is nonpositive), for all $i \in \mathcal{I}$. As a result, agent i can also afford the other agent's trade: $p \cdot (e^i + t^{-i}) \leq p \cdot e^i$.

Problem 4.11

1. Consider the core as an operationalization of the outcomes of free contracting. Then, free contracting leads to decision making that is better in the sense of being Pareto efficient. Free contracting is at the heart of how market economies work in the sense that, in large economies, under appropriate conditions, Walrasian equilibrium outcomes and the core are essentially equivalent (theorems 4.2 and 4.3).

2. One may be unable to avoid exposure to free speech, which may be offensive or incendiary, but one can avoid being a party to an offensive contract. Contracting is voluntary. Being a victim of insults and misinformation is not.
   Of course, Alice's contracting with Bob may hurt Carol. With free contracting, however, Carol would bribe Alice and Bob---that is, contract with Alice and Bob---to avoid being hurt. Or she would not do so; Carol may be unable to amass enough resources to bribe, in which case subjecting Carol to the externality would be efficient. That is, the same principle that says that Alice and Bob are free to contract in a way that would hurt Carol guarantees that Carol is free to bribe Alice and Bob to negotiate a Pareto efficient amount of the externality. By contrast, free speech does not quite work that way. If Alice calls Carol "idiot," calling Alice "idiot" in return (or calling out Alice's assertion as factually inaccurate) will not undo the damage that the insult has inflicted on Alice and will probably not lead to a Pareto efficient outcome.

3. Suppose the core is empty. Then, (among other things) any Pareto efficient allocation can be blocked by some coalition of agents. Restrictions on contracting that rule out the formation of such blocking coalitions would promote efficient outcomes.

4. Political correctness makes socially unacceptable some speech that is not illegal. In contracting, analogous is the opprobrium toward some exchanges that are not illegal, such buying fur coats, paying one's kids for good grades, or hiring one's mother to mow the lawn. Political correctness also segments individuals into safe spaces. The analog of safe spaces in an exchange economy is the segmentation of goods into submarkets such that goods can be exchanged within, but not across, submarkets. (Section 3.4 discusses the value of such submarkets.)

Problem 4.13

1. At equilibrium, $w < A$ cannot be; if $w < A$, the firm's demand for labor is unbounded, thereby crashing any hope of labor-market clearing. Nor can it be that $w > A$; if $w > A$, the firm demands no labor, whereas the supply of labor is positive ($x_2^i = 1 - s^i$), thereby contradicting labor-market clearing. Thus, $w = A$ is the only candidate for an equilibrium wage.

2. The Walrasian equilibrium has $w = A$, $x_1^i = s^i A$, and $x_2^i = 1 - s^i$. The equilibrium is unique because the candidate for an equilibrium wage is unique, and each agent's consumption choice is unique given the wage.

3. Agent i's equilibrium utility function is $\ln s^i + \ln A - 1$, which depends on s^i.

4. The Walrasian social choice function (WSCF) is not socialist because it does not equalize the agents' utilities, thereby contradicting the implication of socialism in theorem 4.4.

5. Even if one is a socialist (of definition 4.2), one shall recognize that the WSCF fares quite well if one is willing to restrict the domain of the socialist conditions to environments with linear production functions of the form $f(h) = Ah$. Indeed, by the answer in part 3, the WSCF satisfies equal treatment of equals (condition 4.2), limited private ownership (condition 4.3), solidarity (4.4), and collective ownership (condition 4.5)—all on the restricted domain. Pareto efficiency (condition 4.1) on the restricted domain holds as well because the equilibrium allocation can be verified to solve

$$\begin{aligned} &\max_x \sum_{i \in \mathcal{I}} u\left(s^i, x^i\right) \\ &\text{s.t.} \sum_{i \in \mathcal{I}} x_1^i \leq f\left(\sum_{i \in \mathcal{I}} \left(1 - x_2^i\right)\right), \end{aligned}$$

   implying that there is no way to feasibly make one agent better off without making the other agent worse off.

   Thus, on the restricted domain, the WSCF is socialist. This result highlights the importance of the implicit unrestricted domain assumption for the conclusion of theorem 4.4. Thus, on the restricted domain at least, WSCF is compelling.

Chapter 5

Problem 5.1

Denote $\bar{\alpha} = \alpha_1 + \alpha_2$.

1. The equilibrium prices are $(p_1, p_2) = (\alpha_1, \alpha_2)$. The induced demands are identical for both goods:
$$x_1^i = x_2^i = \sum_{l \in \mathcal{L}} \frac{\alpha_l e_l^i}{\bar{\alpha}}.$$
The induced equilibrium utility is
$$V\left(e^i\right) = \left(\sum_{l \in \mathcal{L}} \frac{\alpha_l e_l^i}{\bar{\alpha}}\right)^{\bar{\alpha}}.$$

2. Agent i's utility from consuming $\mathbb{E}\left[x^i\right]$ is $\mathbb{E}\left[x_1^i\right]^{\alpha_1} \mathbb{E}\left[x_2^i\right]^{\alpha_2}$. His expected utility from consuming x^i is
$$\mathbb{E}\left[u\left(x^i\right)\right] = \mathbb{E}\left[\left(x_1^i\right)^{\alpha_1}\left(x_2^i\right)^{\alpha_2}\right] = \mathbb{E}\left[\left(x_1^i\right)^{\alpha_1}\right]\mathbb{E}\left[\left(x_2^i\right)^{\alpha_2}\right] \leq \mathbb{E}\left[x_1^i\right]^{\alpha_1}\mathbb{E}\left[x_2^i\right]^{\alpha_2},$$
where the second equality follows by the independence of the components of the bundle x^i, and the inequality follows by Jensen's inequality because u is concave in each of its arguments (by $\alpha_l \in (0,1)$). Thus, the agent weakly prefers $\mathbb{E}\left[x^i\right]$ to x^i, and strictly so if lottery x^i is nondegenerate.

3. With the aggregate endowment fixed at $(\bar{e}_1, \bar{e}_2) = (1,1)$, each agent's equilibrium utility is $V\left(e^i\right)$, computed in part 1. The agent's expected utility, before he knows the realization of his endowment, satisfies
$$\mathbb{E}\left[V\left(e^i\right)\right] = \mathbb{E}\left[\left(\sum_{l \in \mathcal{L}} \frac{\alpha_l e_l^i}{\bar{\alpha}}\right)^{\bar{\alpha}}\right] \begin{cases} \geq \left(\sum_{l \in \mathcal{L}} \frac{\alpha_l \mathbb{E}\left[e_l^i\right]}{\bar{\alpha}}\right)^{\bar{\alpha}} & \text{if } \bar{\alpha} > 1 \\ \leq \left(\sum_{l \in \mathcal{L}} \frac{\alpha_l \mathbb{E}\left[e_l^i\right]}{\bar{\alpha}}\right)^{\bar{\alpha}} & \text{if } \bar{\alpha} < 1, \end{cases}$$
where the inequalities are by Jensen's inequality. When $\bar{\alpha} > 1$, the agent's equilibrium utility is convex in his wealth, and, so, he likes lotteries in his wealth or, equivalently, in his endowment.

4. In part 2, the components of the agent's consumption bundle are distributed independently, so one can assess the risk independently, component by component, and conclude that the agent dislikes this risk because he is risk averse in each component. By contrast, in part 3, the components of the agent's equilibrium consumption bundle are perfectly correlated; at equilibrium, the agent spreads his wealth so as to equalize his consumption of each good. The product structure of the utility function reveals "complementarity" in the sense that,

other things being equal, the agent prefers a lottery with positively correlated components of the consumption bundle. (See figure SO.13.) Hence, the agent prefers an endowment lottery to a deterministic endowment fixed at the lottery's mean if he is not too risk averse in each good individually. By inspection of the agent's equilibrium utility in part 1, in the present problem, an agent likes a mean-zero lottery endowment lottery if and only if he likes a mean-zero lottery in wealth.

Problem 5.3

The planner chooses the thresholds y_1 and y_2, and the matching function m so as to maximize the aggregate output

$$\max_{y_1,y_2,m}\left\{\int_0^{y_1} F\left(m\left(a\right)\right)g\left(a\right)\mathrm{d}s+\int_{y_1}^{y_2} F\left(a\right)\mathrm{d}a\right\}$$

subject to the market clearing condition (5.6) on m, and the boundary conditions $m\left(0\right)=y_2$ and $m\left(y_1\right)=1$.

Under the conditions of example 5.1, combining the market clearing condition (5.6) with $m\left(0\right)=y_2$ gives the matching function (5.7):

$$m\left(a\right)=y_2+c\left(a-\frac{a^2}{2}\right),\qquad a\in\left[0,y_1\right].$$

The planner's problem becomes

$$\max_{y_1,y_2}\left\{\int_0^{y_1}\left(y_2+c\left(a-\frac{a^2}{2}\right)\right)\mathrm{d}a+\int_{y_1}^{y_2} a\mathrm{d}s\right\}$$

subject to the constraint

$$y_2=1-c\left(y_1-\frac{y_1^2}{2}\right),$$

which has been obtained from $m\left(y_1\right)=1$. Substituting the constraint into the objective function and computing the first-order condition with respect to y_1 gives[SO.3]

$$y_1=1-\frac{1-\sqrt{\left(3-c\right)\left(1-c\right)}}{c}\qquad\text{and}\qquad y_2=\frac{2-\sqrt{\left(3-c\right)\left(1-c\right)}}{c}-1.$$

Thus, the planner's solution $\left(y_1,y_2\right)$ coincides with the equilibrium thresholds in (5.10). If it did not, the planner would be able to induce a greater output than equilibrium does, for the planner's overt objective in choosing $\left(y_1,y_2\right)$ is to maximize output. If that were so, the planner would be able to distribute the surplus output to Pareto improve on the equilibrium outcome. But the planner cannot improve on the equilibrium output. Thus, what we have shown here is a limited version

[SO.3] Even though, after the substitution, the planner's objective function is not concave in y_1, it can nevertheless be shown that the optimal threshold y_1 is characterized by the first-order condition.

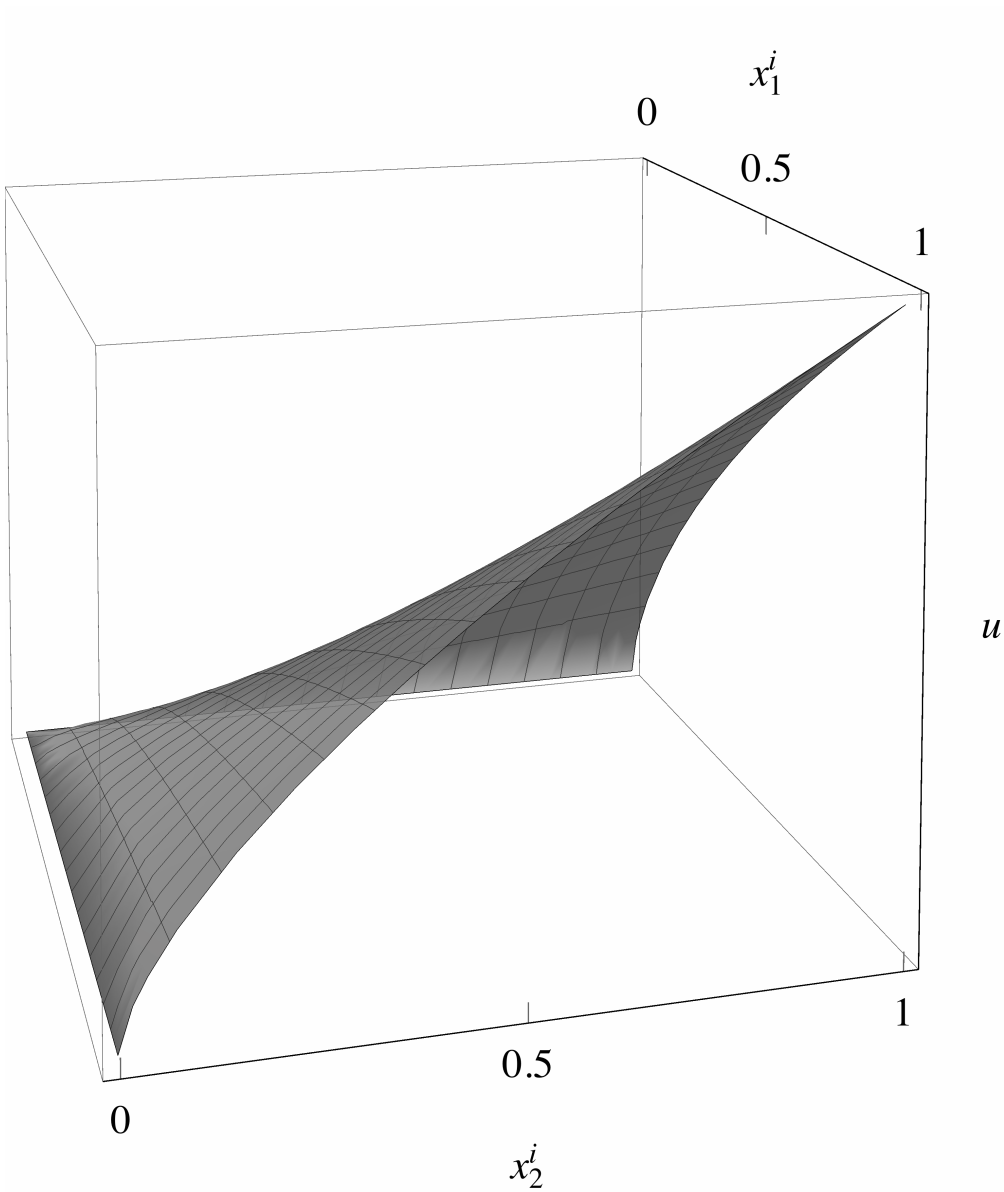


Figure SO.13: (Problem 5.1) The utility $u\left(x^i\right) = \left(x_1^i\right)^{\frac{2}{3}}\left(x_2^i\right)^{\frac{2}{3}}$ from a deterministic bundle x^i is concave in each component of the bundle (while the other component is held fixed) but is convex in $z = x_1^i = x_2^i$ (as both components vary at the same time).

of the FWT: by varying occupational thresholds, the planner cannot improve on the equilibrium outcome.

Problem 5.5

1. In autarky, when Nature draws a problem of complexity, say, p (according to c.d.f. F), this problem is equally likely to end up with any agent and, hence, is solved with probability

$$\int_0^1 g(a)\,\mathbf{1}_{\{p\leq a\}}\mathrm{d}a = 1 - G(p).$$

As a result, the probability density for solved problems is

$$h^a(p) = \frac{f(p)(1-G(p))}{\int_0^1 f(s)(1-G(s))\,\mathrm{d}s},$$

where the superscript "a" stands for "autarky." The fraction in the display above is the "share" of the problems that happen to be of complexity p and are solved in the total pool of problems, of any complexity, that happen to be solved.[SO.4] When G and F are uniform,

$$h^a(p) = 2(1-p).$$

2. At the occupational choice equilibrium, when Nature draws a problem of complexity, say p, this problem gets solved with probability

$$\begin{aligned}&\int_0^{z_1} g(a)\,\mathbf{1}_{\{p\leq m(a)\}}\mathrm{d}a + \int_{z_1}^{z_2} g(a)\,\mathbf{1}_{\{p\leq a\}}\mathrm{d}a\\ &= \mathbf{1}_{\{p\leq z_1\}}G(z_2) + 1_{\{p\in(z_1,z_2)\}}\left(G(z_1) + (G(z_2)-G(p))\right) + 1_{\{p\geq z_2\}}\left(G(z_1) - G\left(m^{-1}(p)\right)\right).\end{aligned}$$

The left-hand side is the sum of two components: the probability that a worker-manager team (indexed by the ability of a worker) receives and solves the problem and the probability that a self-employed agent receives and solves the problem. The right-hand side evaluates the integrals on the left-hand side, case by case, depending on the value of p. In particular, when $p \leq z_1$, the problem is solved if it arrives either to a worker (who may or may not pass it to his manager) or a self-employed agent, but not to a manager, who spends no time codifying problems. When $p \in (z_1, z_2)$, the problem is solved if it arrives either to a worker or a self-employed agent with ability at least p. Finally, when $p \geq z_2$, the problem is solved if it arrives to a worker who works for a manager with ability at least p; the ability of such a worker is at least $m^{-1}(p)$, where m^{-1} is the inverse of m.

[SO.4] The scary quotes are warranted because, strictly speaking, the share is zero. Any complexity has probability zero of being drawn from a continuum of complexities. Instead, the expression is the probability density, obtained by Bayes's rule.

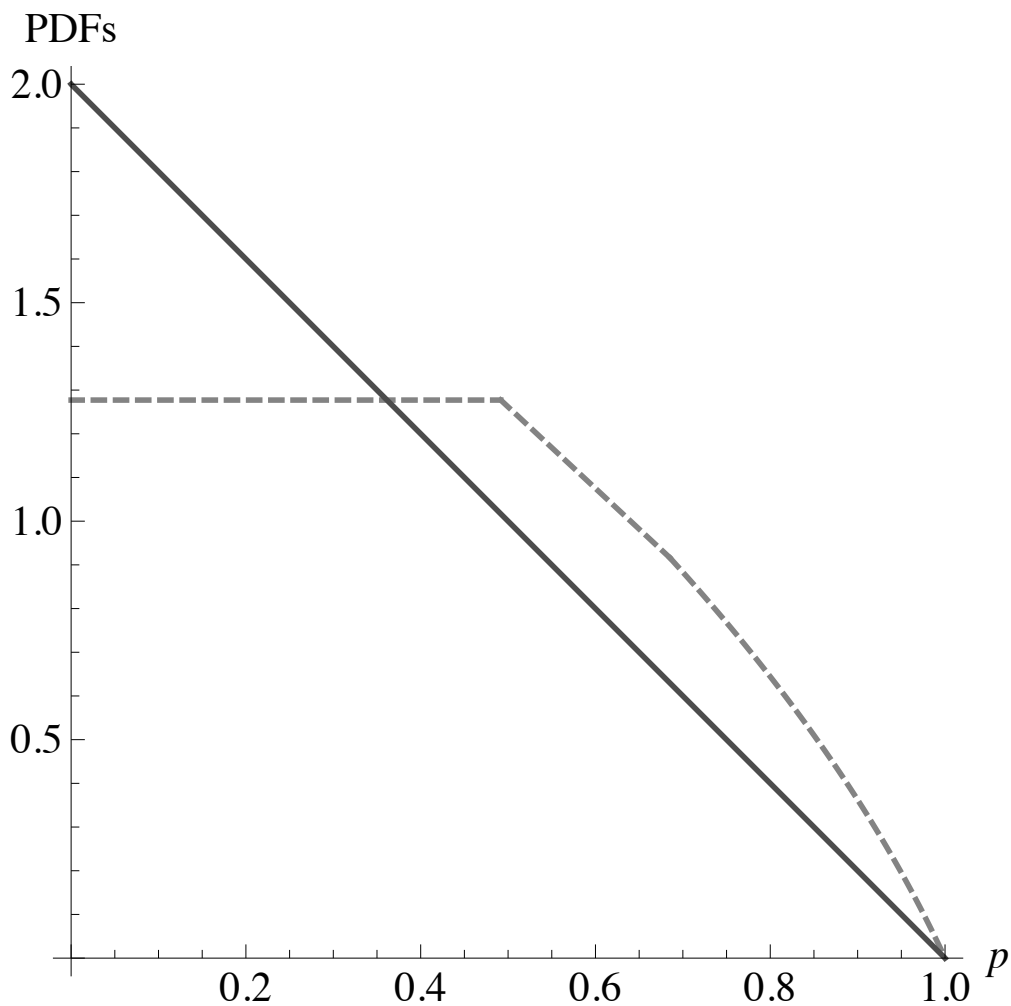


Figure SO.14: (Problem 5.5) Two p.d.f.s for the complexity of solved problems. The p.d.f. for autarky, h^a (the solid line), crosses the p.d.f. for markets, h^m (the dashed line), from above. That is, with markets, simple problems are underrepresented and complex problems are overrepresented relative to autarky.

The probability density for solved problems is given by

$$h^m(p) = \frac{f(p)\left(\int_0^{z_1} g(a)\,\mathbf{1}_{\{p\le m(a)\}}\mathrm{d}a + \int_{z_1}^{z_2} g(a)\,\mathbf{1}_{\{p\le a\}}\mathrm{d}a\right)}{\int_0^1 f(s)\left(\int_0^{z_1} g(a)\,\mathbf{1}_{\{s\le m(a)\}}\mathrm{d}a + \int_{z_1}^{z_2} g(a)\,\mathbf{1}_{\{s\le a\}}\mathrm{d}a\right)\mathrm{d}s},$$

where the superscript "m" stands for "markets." When G and F are uniform, the display above becomes

$$h^m(p) = \frac{\mathbf{1}_{\{p\le z_1\}}z_2 + 1_{\{p\in(z_1,z_2)\}}(z_1+z_2-p) + 1_{\{p\ge z_2\}}(z_1 - m^{-1}(p))}{\int_0^1\left(\mathbf{1}_{\{s\le z_1\}}z_2 + 1_{\{s\in(z_1,z_2)\}}(z_1+z_2-s) + 1_{\{s\ge z_2\}}(z_1 - m^{-1}(s))\right)\mathrm{d}s}.$$

3. Figure SO.14 compares the two p.d.f.s. With markets, highly able agents specialize in solving

more complex problems, which are referred to them by workers. Equivalently, markets enable better artists to exploit their comparative advantage: the production of finer art. Fewer simple problems are solved because, with trade, only workers and the self-employed have a chance to tackle them; managers receive only sufficiently complex problems to tackle.

Problem 5.7

1. At equilibrium, robot-manager and self-employed robots must make the same amount of money:

$$n - nw(a) = 1, \qquad a \in [\hat{z}, 1],$$

where n satisfies the manager's budget constraint $n[c(1-a)/a](1-a) = 1$. Hence,

$$w(a) = 1 - c\frac{(1-a)^2}{a}, \qquad a \in [\hat{z}, 1].$$

If ability-$\hat{z}$ agent is indifferent between being self-employed and being a worker, then $w(\hat{z}) = \hat{z}$ and, hence,

$$\hat{z} = \frac{c}{1+c}.$$

Note also that, the amount of time a manager needs to spend in order to receive measure one of codified problems from an ability-$\hat{z}$ agent is $c(1-\hat{z})/\hat{z} = 1$. If the worker ability were lower, then the corresponding amount of time would exceeds 1, and the manger would not want to hire this worker. Finally, set the function to be

$$w(a) = a, \qquad a \in [0, \hat{z}),$$

so that no self-employed is better off if employed at $w(a)$. As a result,

$$w(a) = \max\left\{a, 1 - c\frac{(1-a)^2}{a}\right\}, \qquad a \in [0, 1],$$

which is continuous and increasing in a and satisfies $w(0) = 0$, $w(1) = 1$, and (as can be shown) $w(a) \geq a$ for all $a \in [0, 1]$.

We now verify the occupational-choice conditions. The verification is routine and follows the steps analogous to those performed in sections 5.3 and 5.4.

First, note that robots are indifferent among all three occupations. By construction, being self-employed and managing humans gives each robot the same payoff of 1, and $w(1) = 1$. Hence, we assume that just enough robots are managers to manage all humans, and the remaining robots are self-employed. No robot is a worker.

Because $w(a) \geq a$, no worker prefers being self-employed. No self-employed prefers being a worker because $w(a) = a$ for all $a \in [0, \hat{z})$, by construction. It remains to verify that no worker and no self-employed agent would rather be a manager.

We shall first show that no human prefers being a manager subject to hiring workers with abilities in $[\hat{z}, 1]$, and then show that no human prefers being a manager subject to hiring workers with abilities in $[0, \hat{z})$, to conclude that no human prefers being a manager, full stop. The profit of a type-a' manager who is restricted to hire workers with abilities in $[\hat{z}, 1]$ is

$$\max_{a \in [\hat{z}, 1]} \frac{(a' - w(a))\, a}{(1-a)^2 c}.$$

The maximand in the display above is weakly decreasing in a and, so, is maximized at $a = \hat{z}$. The induced profit, $a' - c(1 - a')$, is less than what a self-employed agent with ability in $[0, \hat{z})$ earns and is weakly less than what a worker with ability in $[\hat{z}, 1]$ earns:

$$a' - c(1 - a') \leq w(a'), \qquad a' \in [\hat{z}, 1],$$

as can be verified. So, no human would prefer to be a manager if restricted to hire from $[\hat{z}, 1]$. The profit of a type-a' manager who is restricted to hire workers with abilities in $[\hat{z}, 1]$ is

$$\max_{a \in [0, \hat{z}]} \frac{(a' - a)\, a}{(1-a)^2 c},$$

where the wage function is $w(a) = a$. When evaluated at the extremes $a = 0$ or $a = \hat{z}$, the maximand in the display above does not exceed a', as can be verified by substitution. When evaluated at the other extreme, which is the unique a that solves the first-order condition (i.e., $a = a'/(2 - a') \in [0, \hat{z}]$), the difference between the maximand and a'—the payoff of the self-employed, with $a' \in [0, \hat{z})$—is

$$-\frac{a'}{4}\left(4 - \frac{a'}{c(1-a')}\right) \leq 0,$$

where the inequality follows from $4 - a'/(c(1 - a')) > 3$, by $a' < \hat{z}$. Further, when evaluated at the (same as above) unique a that solves the first-order condition, the difference between the maximand and $w(a')$—the payoff of a worker with ability $a' \in [\hat{z}, 1]$—is

$$\frac{(a')^2}{4c(1-a')} + \left(a' + \frac{1}{a'} - 2\right)c - 1 < \frac{\left(\frac{2\hat{z}}{1+\hat{z}}\right)^2}{4c\left(1 - \frac{2\hat{z}}{1+\hat{z}}\right)} + \left(\hat{z} + \frac{1}{\hat{z}} - 2\right)c - 1 = -\frac{c^2}{1 + 3c + 2c^2} < 0,$$

where the first inequality uses the facts that $a' + 1/a'$ is decreasing, that $a' \geq \hat{z}$ (because we are looking at workers), and that $a'/(2 - a') < \hat{z} \iff a' < 2\hat{z}/(1 + \hat{z})$ (because the first-order condition identifies an extreme point only if it delivers an interior solution). So, no human would prefer to be a manager if restricted to hire from $[0, \hat{z}]$.

We thus conclude that no human (neither the self-employed nor a worker) would rather be a manager.

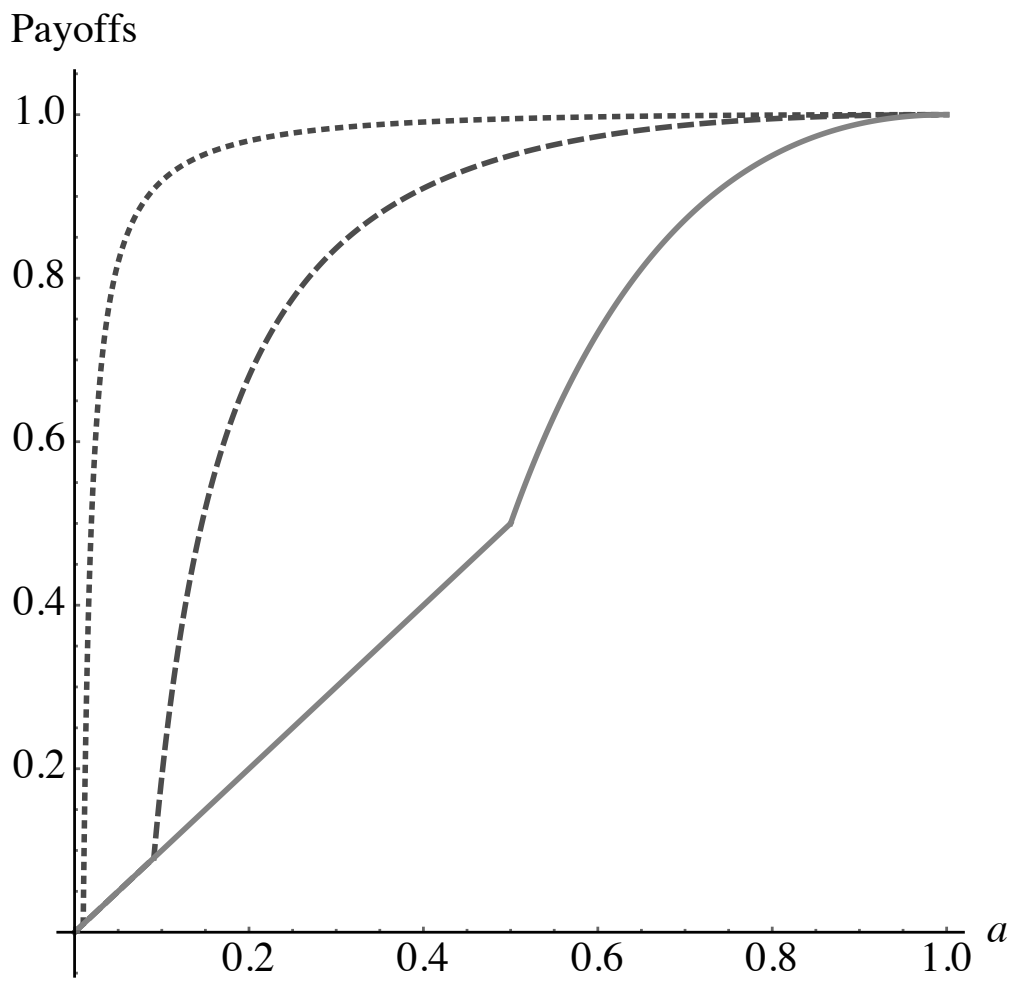


Figure SO.15: (Problem 5.7) Humans' equilibrium payoffs as functions of their abilities, for $c = 0.01$ (the dotted curve), $c = 0.1$ (the dashed curve), and $c = 1$ (the solid curve). The payoff of a worker with ability $a > 0$ approaches 1 as $c \to 0$.

2. One must have a sufficient measure $\hat{A}$ of robots to hire workers with abilities in $[\hat{z}, 1]$. This measure equals

$$\begin{aligned}\hat{A} &= \int_{\hat{z}}^{1} c\frac{1-a}{a}\left(1 - F\left(a\right)\right) g\left(a\right) \mathrm{d}a \\ &= c\int_{\hat{z}}^{1} \frac{\left(1-a\right)^2}{a}\mathrm{d}a = c\left(\ln\left(1+\frac{1}{c}\right) - \frac{3+2c}{2\left(1+c\right)^2}\right).\end{aligned}$$

3. Figure SO.15 illustrates the payoffs. For each ability $a > 0$, the payoff converges to one as c converges to zero. The convergence can be seen by observing that

$$\lim_{c\to 0} \hat{z} = \lim_{c\to 0} w\left(a\right) = 0, \qquad a \in \left[\hat{z}, 1\right].$$

The payoff is continuous in a, however, with the payoff of an ability-0 agent being zero. (Not only an ability-0 agent cannot solve any problem, but he also cannot communicate any of the problem he has codified to the manager.) Overall, as $c \to 0$, there is no immiseration (quite to the contrary), and there is sense (apparent by eyeballing figure SO.15) in which inequality falls (except all ability-0 agents are left behind).

Chapter 6

Problem 6.1

Once committing a crime becomes costlier, it becomes a better signaling device, to signal how tough and nonconformist a certain group is. So, the group's members commit more crime. As the cost of signaling continues to grow, however, in a two-type model of signaling, less crime suffices to distinguish the high type from the low type.

Formally, for some cost-of-crime parameter $\gamma > 0$, let the cost of effort $e \geq 0$ for type $\theta \in \{\theta_L, \theta_H\}$ ($\theta_H > \theta_L > 0$) be

$$c(e, \theta \mid \gamma) = \frac{e}{\max\{1/\gamma, \theta\}}.$$

When $\gamma \leq 1/\theta_H$, $c(e, \theta \mid \gamma) = \gamma e$ for each $\theta \in \{\theta_L, \theta_H\}$, and, so, the effort cannot distinguish the two types. As γ increases in the interval $(1/\theta_H, 1/\theta_L)$, the two types' costs of efforts start to diverge.

One can interpret γ as parameterizing the severity of sentencing and the police detection effort. As γ rises, at first, all types find committing crime costlier. As γ rises further, however, the two types separate; some are more skillful than others in evading police or are more reckless and less risk averse with respect to consequences.

Problem 6.3

When plastic bags are free, a consumer uses his abstention from the consumption of these bags to signal his environmental righteousness to others (and, perhaps, to himself). When plastic bags are no longer free, abstention is a poorer signal; the signal's receiver does not know whether the shopper's abstention signals thriftiness or righteousness. As abstention becomes a poorer signal, it is used less; more shoppers use plastic bags, even though they have to pay for them.

Formally, let us reinterpret the model of section 6.3. Bob can be of two types, θ_L and θ_H, with $\theta_H > \theta_L > 0$. Type-θ_L Bob cares about money but not environment. Type-θ_H Bob cares about environment but not money. Alice would like to spend more time with the Bob who cares about environment, and she does not care whether he cares about money. Bob's effort e is interpreted as the extent to which he refrains from using plastic bags.

Suppose that plastic bags are free. Then, the cost-of-effort function, say, $c(e, \theta) \equiv e/\theta$ ($\theta \in \{\theta_L, \theta_H\}$), indicates that type-$\theta_H$ finds it less costly to abstain from using the bags than type-θ_L does because of the warm glow type-θ_H experiences from doing the "right thing." At the best separating equilibrium, type-θ_L uses a lot of plastic bags, whereas type-θ_H uses fewer, thereby revealing his type to Alice. Note that type-θ_H Bob uses fewer bags exclusively to impress Alice; even though his concern for environment generates a warm glow when he declines a plastic bag, this warm glow by itself does not suffice to compensate for the inconvenience from rejecting the bag.

Now suppose that plastic bags have a fee. Type-θ_H does not care, whereas type-θ_L does and, hence, finds it easier to abstain from plastic bags. Furthermore, suppose that the cost of plastic bags happens to be such that type-θ_L's concern for the extra expense exactly equals type-θ_H's concern for environment: $c(e,\theta) = e/\theta_H$ ($\theta \in \{\theta_L, \theta_H\}$). With this cost function, the only signaling equilibrium is the pooling one (note that condition 6.1 is violated), in which both types of Bob exert zero effort in abstaining from plastic bags: $e_L = e_H$. Thus, type-θ_H's post-fee usage of plastic bags rises to type-θ_L's pre-fee usage, which is unchanged post-fee. The reason is that the rejection of plastic bags is no longer an effective signaling device.]

Problem 6.5

1. One can consider three agents, Alice, type-θ_L Bob, and type-θ_H Bob, to figure in Pareto comparisons. One can seek a signaling technology that would lead to a Pareto best pooling-equilibrium payoff profile, and separately, seek a signaling technology that would lead to a Pareto-best best-separating-equilibrium payoff profile.

2. The best pooling equilibrium has each type exert zero effort, at zero cost. As a result, any signaling technology considered in this chapter is best; at the best pooling equilibrium, each agent's payoff is independent of the signaling technology.
   The following signaling technology is best at the best separating equilibrium: $c(0,\theta_L) = c(0,\theta_H) = 0$, and $c(e,\theta_L) = \theta_H$ and $c(e,\theta_H) = 0$ for all $e > 0$. Alice's payoff is maximal feasible, type-θ_L Bob's payoff is θ_L (the maximal feasible at a separating equilibrium, as it is at any best separating equilibrium), and type-θ_H Bob's payoff is θ_H (the maximal feasible at a separating equilibrium).

3. As has been argued above, for the best pooling equilibrium, any value of γ is optimal. At the best separating equilibrium, only the payoff of type-θ_H Bob potentially depends on the cost-of-effort function. Type-θ_H exerts effort e_H, which makes type-θ_L Bob indifferent between adhering to its equilibrium zero effort and imitating type-θ_H Bob:

$$e_H = \frac{(\theta_H - \theta_L)\,\theta_L}{\gamma}.$$

   Type-θ_H's equilibrium payoff then is

$$\theta_H - \frac{\gamma e_H}{\theta_H} = \theta_H - \left(1 - \frac{\theta_L}{\theta_H}\right)\theta_L,$$

   which is independent of γ. When effort is costlier, type-θ_H Bob exerts less of it, and the two effects exactly cancel out. Hence, any $\gamma > 0$ is optimal. Type-θ_H's equilibrium payoff is smaller than its highest feasible payoff.

Problem 6.7

Let the type space be $\Theta = \{\theta_1 = \theta_L, \theta_2, \ldots, \theta_I = \theta_H\}$. Let $e(\theta)$ denote the effort of type $\theta \in \Theta$ at the best separating equilibrium. One can verify that $e(\theta_1) = 0$, and any type θ_i with $i \in \{2, 3, \ldots, I\}$ exerts the lowest effort that makes type θ_{i-1} just indifferent between exerting its equilibrium effort $e(\theta_{i-1})$ and imitating type θ_i by exerting effort $e(\theta_i)$:

$$\theta_{i-1} - \frac{e(\theta_{i-1})}{\theta_{i-1}} = \theta_i - \frac{e(\theta_i)}{\theta_{i-1}}.$$

Then, the equilibrium effort of each type $\theta_i \in \{\theta_2, \theta_3, \ldots, \theta_I\}$ is given by the recursive relationship:

$$e(\theta_i) = e(\theta_{i-1}) + \theta_{i-1}(\theta_i - \theta_{i-1}), \qquad e(\theta_1) = 0. \tag{SO.1}$$

One can verify (to begin with, draw a picture for $I = 3$) that with efforts so constructed, no type would gain from imitating any other type, not just the type immediately above it, which is true by construction. Because no type gains by imitating the lowest type, θ_L, with pessimistic beliefs—which stipulate that any off-equilibrium-path effort is attributed to θ_L—no type will gain from exerting an off-equilibrium-path effort.

Using the recursive formula above, one can answer:

1. When $\Theta = \{\theta_L, \theta_H\}$, $e(\theta_L) = 0$ and $e(\theta_H) = \theta_L(\theta_H - \theta_L)$.

2. When $\Theta = \{\theta_1, \theta_2, \ldots, \theta_I\}$, each type's equilibrium effort is given by (SO.1). In particular,

$$e(\theta_H) = \theta_L(\theta_H - \theta_L) + \sum_{i=2}^{I-1} (\theta_H - \theta_i)(\theta_i - \theta_{i-1}). \tag{SO.2}$$

3. There are two ways to approach the problem with a continuum of types. One is to assume that it is a limit of a discrete problem with I types as $I \to \infty$ and the maximal distance between adjacent types converges to zero. Then, one can rewrite (SO.1) as a differential equation

$$\frac{\mathrm{d}e(\theta)}{\mathrm{d}\theta} = \theta, \qquad \theta \in [\theta_L, \theta_H] \qquad \text{and} \qquad e(\theta_L) = 0, \tag{SO.3}$$

whose solution is

$$e(\theta) = \frac{\theta^2 - \theta_L^2}{2}, \qquad \theta \in [\theta_L, \theta_H].$$

An alternative way to approach the continuum problem is to contemplate the first-order optimality condition for the problem of type θ who choose a type $\hat{\theta}$ to imitate:

$$\max_{\hat{\theta} \in [\theta_L, \theta_H]} \left(\hat{\theta} - \frac{e\left(\hat{\theta}\right)}{\theta} \right), \qquad \theta \in (\theta_L, \theta_H).$$

At a separating equilibrium, no local deviation from $\hat{\theta} = \theta$ is optimal, and the corresponding first-order condition is the differential equation in (SO.3).

4. By inspection of (SO.2), the effort $e(\theta_H)$ is higher when there are $I \geq 3$ types than when there are just two types. By introducing more types, we are introducing types that find it less costly to exert effort and hence more eager to imitate type θ_H, who then must exert a higher effort to avoid imitation.
   The effort $e(\theta_H)$ is also higher with the continuum of types than with any finite number I of types:

$$\begin{aligned} e(\theta_H)|_{I\text{ types}} &= \theta_L(\theta_H - \theta_L) + \sum_{i=2}^{I-1}(\theta_H - \theta_i)(\theta_i - \theta_{i-1}) \\ &< \theta_L(\theta_H - \theta_L) + \sum_{i=2}^{I}\left(\theta_H - \theta_i + \frac{\theta_i - \theta_{i-1}}{2}\right)(\theta_i - \theta_{i-1}) \\ &= \theta_L(\theta_H - \theta_L) + \frac{(\theta_H - \theta_L)^2}{2} \\ &= \frac{\theta_H^2 - \theta_L^2}{2} = e(\theta_H)|_{\text{continuum of types}}, \end{aligned}$$

   where the inequality and the second equality are best seen graphically, by representing the sums of products as areas. The intuition remains the same. As the type space becomes more densely populated, each type has a closer adjacent lower type to differentiate himself from and raises his effort. As he raises his effort, his equilibrium payoff falls. He is therefore more eager to imitate a given effort of the higher adjacent type, who, in turn, would have to raise his effort to avoid being imitated. Thus, there is a direct and an indirect effects that cause the highest type to raise his effort as the type space becomes denser.

5. The answers are independent of probabilities. All that matters is whether a particular type has a positive or zero probability, not what this probability is. In particular, the equilibrium payoff of each type is unaffected by the inequality of the type distribution. Each type only wishes that there were fewer types lower than him.

Problem 6.9

1. The construction of the best separating equilibrium follows from part 1 in problem 6.7. In particular, one can find beliefs for Alice that would support efforts $e_L = 0$, $e_H = \theta_L(\theta_H - \theta_L)$, and $e_X = \theta_L(\theta_H - \theta_L) + \theta_H(\theta_X - \theta_H)$ for Bob's types, respectively, θ_L, θ_H, and θ_X.

2. Figure SO.16 illustrates the construction.
   To give the best chance to the desired effort profile to be equilibrium, assume that Alice's belief is pessimistic, meaning that, whenever she observes an off-equilibrium-path effort, she attributes this effort with probability one to the lowest Bob's type whom she deems to be possible conditional on Carol's message, "naughty" or "nice."

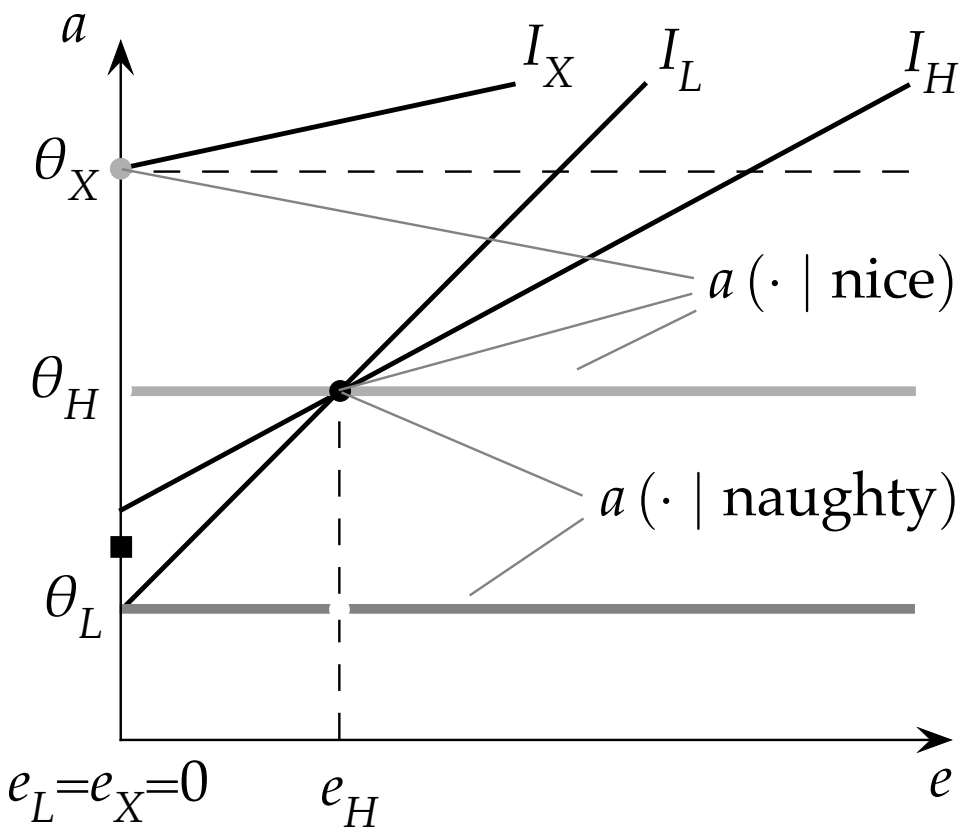


Figure SO.16: (Problem 6.9) The best separating countersignaling equilibrium. Schedule $a(\cdot \mid \text{naughty})$ is Alice's action given the pessimistic belief conditional on message "naughty." Schedule $a(\cdot \mid \text{nice})$ is Alice's action given the pessimistic belief conditional on message "nice." The indifference curves of the three types of Bob are the straight lines labeled I_L, I_H, and I_X. Types θ_L and θ_X exert effort $e_L = e_X = 0$. Type θ_H exerts effort $e_H > 0$, which makes type θ_L indifferent between imitating type θ_H and adhering to his equilibrium effort. The solid rectangle on the vertical axis is type θ_H's expected allocation (i.e., effort-payment pair) if he deviates to zero effort. If p (the probability of "naughty" conditional on θ_H) is sufficiently high, this rectangle lies below I_H.

We shall focus on the best countersignaling equilibrium, not because of its welfare properties, but, in this case, because it is most likely to exist, as will be explained. At this equilibrium, type θ_H's effort e_H makes type θ_L indifferent between exerting its equilibrium effort $e_L = 0$ and imitating type θ_H. As a result, $e_H = (\theta_H - \theta_L)\,\theta_L$.

Type θ_L, thus, does not benefit from imitating type θ_H by exerting effort e_H, by construction. Type θ_L cannot possibly imitate type θ_X by exerting effort $e_L = e_X = 0$ because type θ_L cannot ever trigger message "nice," which Alice must observe to believe that zero effort has come from type θ_X. Type θ_L does not benefit from exerting an off-equilibrium-path effort because Alice's is pessimistic.

Type θ_X cannot benefit from deviating because his payoff is θ_X, the highest feasible.

Because Alice's belief is pessimistic, type θ_H's most profitable deviation is to zero effort. So, to verify the equilibrium, it remains to check that type θ_H does not benefit from deviating from $e_H > 0$ to the effort $e_L = e_X = 0$:

$$p\theta_L + (1-p)\,\theta_X \leq \theta_H - \frac{e_H}{\theta_H} = \theta_H - \left(1 - \frac{\theta_L}{\theta_H}\right)\theta_L,$$

where the left-hand side of the inequality, type θ_H's payoff from the deviation, acknowledges his uncertainty about whether Alice has observed "naughty" (with probability p) or "nice" (with probability $1-p$). The inequality's right-hand side, decreasing in e_H, justifies the focus on the best (i.e., lowest-type-θ_H's-effort) countersignaling equilibrium, for this equilibrium has the best chance to exist by giving the best chance to the inequality to hold. The inequality holds if and only if $p \in [\hat{p}, 1]$, where

$$\hat{p} = 1 - \frac{\theta_H - \theta_L}{\theta_X - \theta_L}\left(1 - \frac{\theta_L}{\theta_H}\right).$$

That is, type θ_H does not gain from imitating type θ_L or θ_X by exerting zero effort if he believes that, with a sufficiently high probability, Alice has observed "naughty" and, hence, will take him for θ_L, not θ_X.

The equilibrium is called countersignaling because Bob's effort is nonmonotone in his type. In particular, type θ_X signals in the "wrong direction," the direction of the lowest type.

3. Figure SO.17 illustrates the construction.

   As before, the plan is to look at the best (i.e., lowest-effort) equilibrium, both to maximize welfare and to minimize higher-type Bob's incentives to imitate the lowest type. The lowest effort $e_H = e_X = e^*$ that does not make it profitable for type θ_L to deviate from zero effort to e^* solves

$$\theta_L = \theta_H - \frac{e^*}{\theta_L},$$

   whence $e^* = (\theta_H - \theta_L)\,\theta_L$. Type θ_L knows that Alice observes "naughty," and, therefore, when he exerts effort e^*, he will not be mistaken for type θ_X.

   Type θ_X knows that, conditional on message "nice" (which type θ_X knows Alice observes)

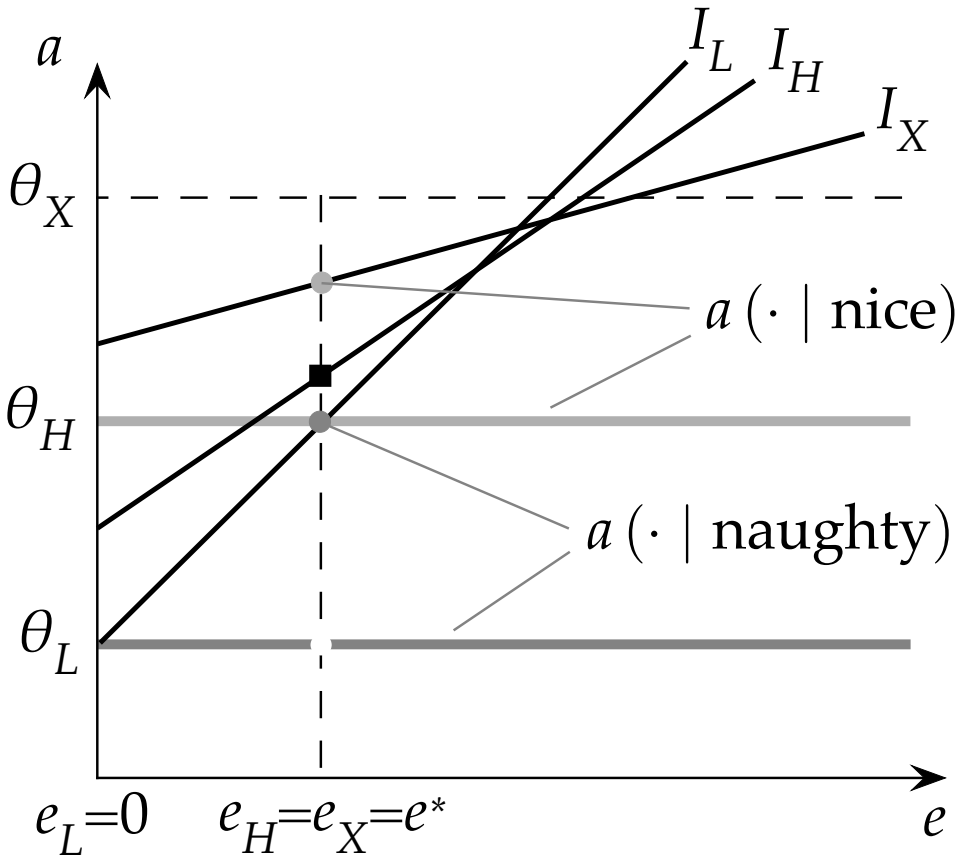


Figure SO.17: (Problem 6.9) A semi-pooling countersignaling equilibrium. Schedule $a(\cdot \mid \text{naughty})$ is Alice's action given the pessimistic belief conditional on message "naughty." Schedule $a(\cdot \mid \text{nice})$ is Alice's action given the pessimistic belief conditional on message "nice." The indifference curves of the three types of Bob are the straight lines labeled I_L, I_H, and I_X. Type θ_L exerts effort $e_L = 0$. Types θ_H and θ_X exert effort $e_H = e_X > 0$. Because type θ_H does not know whether Alice observes "naughty" or "nice," his equilibrium allocation (i.e., effort-payment pair), the solid rectangle, lies between Alice's action schedules conditional on "naughty" and "nice."

and effort e^*, Alice assigns probability

$$\frac{\lambda_X}{\lambda_X + \lambda_H (1-p)}$$

to him being type θ_X. Alice assigns the complementary probability to type θ_H. Hence, type θ_X's expected payment conditional on exerting effort e^* is

$$\frac{\theta_X \lambda_X + \theta_H \lambda_H (1-p)}{\lambda_X + \lambda_H (1-p)}. \tag{SO.4}$$

Type θ_X has no incentive to deviate from effort e^* if and only if he has no incentive to deviate toward zero effort:

$$\theta_H \leq \frac{\theta_X \lambda_X + \theta_H \lambda_H (1-p)}{\lambda_X + \lambda_H (1-p)} - \frac{e^*}{\theta_X} = \frac{\theta_X \lambda_X + \theta_H \lambda_H (1-p)}{\lambda_X + \lambda_H (1-p)} - \frac{(\theta_H - \theta_L)\,\theta_L}{\theta_X}. \tag{SO.5}$$

Type θ_H believes that, with probability p, Alice observes "naughty," and, with probability $1-p$, she observes "nice." If she observes "naughty," effort e^* will reveal Bob's type, and she will pay θ_H. If she observes "nice," effort e^* will not resolve her uncertainty about whether Bob's type is θ_H or θ_X; she will therefore pay him the amount in (SO.4). Thus, type θ_H's expected equilibrium payment is

$$p\theta_H + (1-p)\,\frac{\theta_X \lambda_X + \theta_H \lambda_H (1-p)}{\lambda_X + \lambda_H (1-p)}. \tag{SO.6}$$

Type θ_H has no incentive to deviate from effort e^* if and only if he has no incentive to deviate toward zero effort:

$$p\theta_L + (1-p)\,\theta_H \leq p\theta_H + (1-p)\,\frac{\theta_X \lambda_X + \theta_H \lambda_H (1-p)}{\lambda_X + \lambda_H (1-p)} - \frac{(\theta_H - \theta_L)\,\theta_L}{\theta_H},$$

where the left-hand side reflects type θ_H's uncertainty about the signal Alice has received from Carol.

To summarize, the conjectured semi-pooling equilibrium exists if and only if inequalities (SO.5) and (SO.6) hold. One can verify that the parameter set defined by these two inequalities is nonempty, as figure SO.18 illustrates.

4. The generalization of the intuitive criterion follows closely the spirit of the criterion's definition for two types. A signaling equilibrium fails the intuitive criterion if some type would be able to improve upon his equilibrium utility by exerting some disequilibrium effort e provided that, upon seeing this effort, Alice were to believe that Bob's type were the lowest type who is consistent with Carol's message and who would not be hurt by exerting effort e in return for being paid the amount that equals the highest type consistent with Carol's message. A signaling equilibrium survives the intuitive criterion if it does not fail the intuitive criterion.

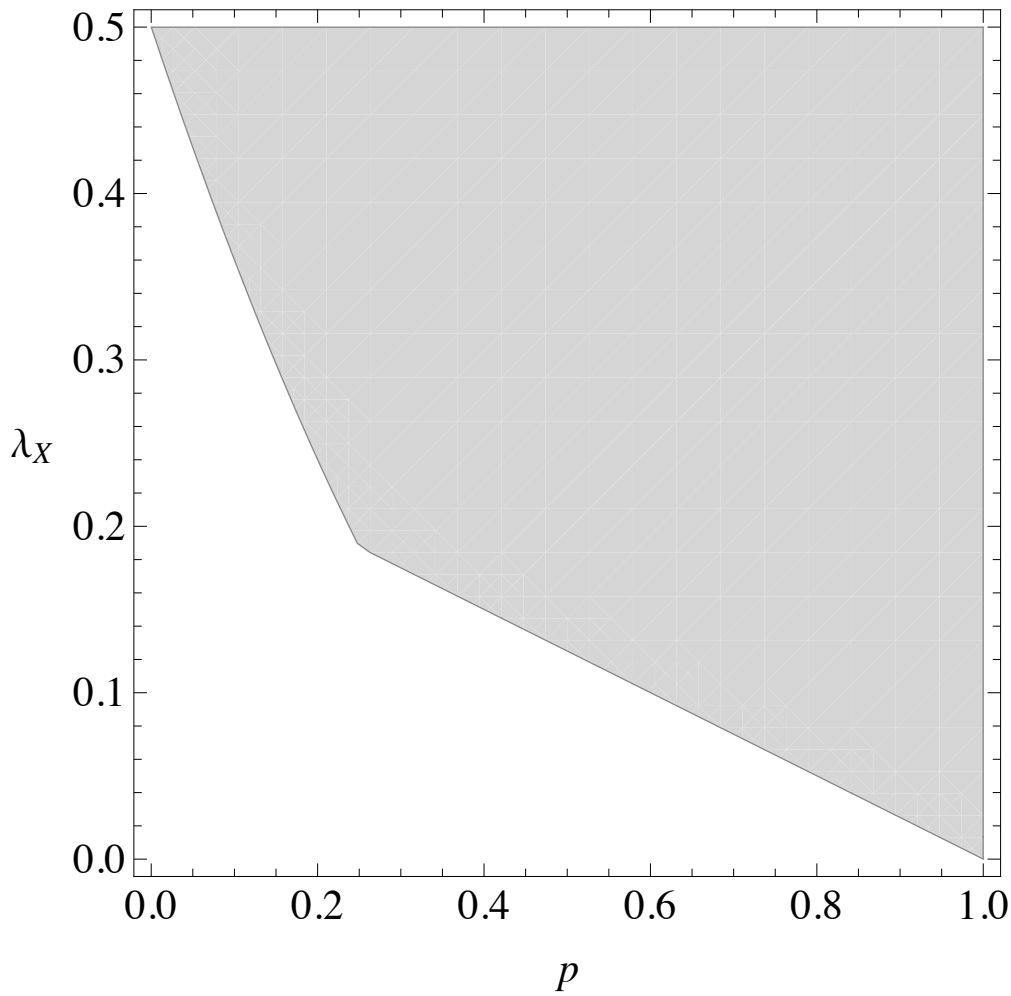


Figure SO.18: (Problem 6.9) Values of parameters p and λ_X for which a semi-pooling countersignaling equilibrium exists. The remaining parameters are taken to be $\theta_L = 1$, $\theta_H = 2$, $\theta_X = 3$, $\lambda_L = 1 - \lambda_H - \lambda_X$, and $\lambda_H = 0.5$.

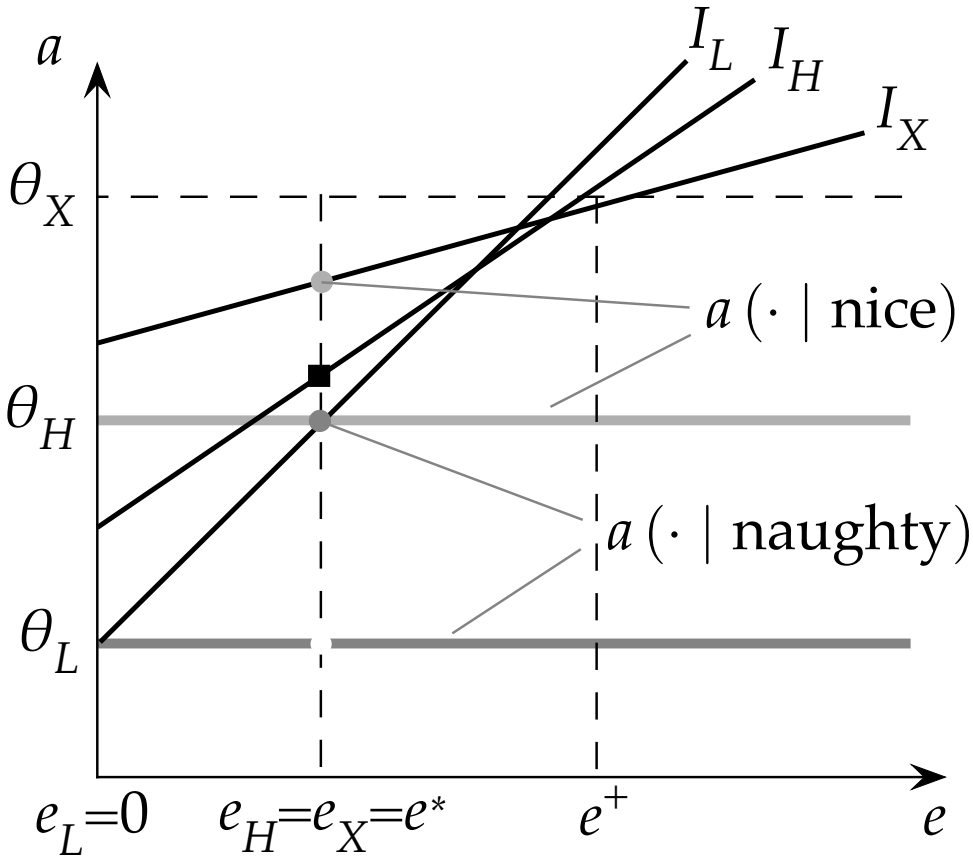


Figure SO.19: (Problem 6.9) The semi-pooling equilibrium fails the intuitive criterion. Type-θ_X Bob exerts the disequilibrium effort e^+ and persuades Alice (who heard "nice") that type θ_H would have never benefited from this deviation, whereas type θ_X could.

The equilibrium in part 2 survives the intuitive criterion.

Equilibrium in part 3 may fail the intuitive criterion. For example, figure SO.19 illustrates how the intuitive criterion destroys the equilibrium in figure SO.17. (Note that it is not generally true that any semi-pooling equilibrium for any parameterization of the model studied in this problem must fail the intuitive criterion.) The idea is the same as the one employed in breaking the pooling equilibrium in the two-type model. Type θ_X deviates to the effort e^+, which would have never benefited θ_H (and θ_L is ruled out by the message "nice"), but can benefit θ_X, for ab appropriate belief of Alice.

5. Suppose that Alice does not discuss Bob with Carol, and Bob knows it. Suppose that the best separating equilibrium prevails. Alice's payoff is the maximal feasible.

   Suppose instead that Alice discusses Bob with Carol, and Bob knows it. Suppose that the best separating countersignaling equilibrium prevails. Alice's payoff is again the maximal feasible. Thus, if one compares the best equilibria, Alice is equally well-off under both scenarios. As far as Bob is concerned, types θ_L and θ_H are indifferent, whereas type θ_X prefers that Alice

talk to Carol, so that he can countersignal in the best separating equilibrium, thereby saving on the cost of effort.

6. Golda Meir's idea may have been that mediocre people signal by not being humble, in order to differentiate themselves from those without any accomplishments, whereas great people signal by being humble, in order to differentiate themselves from the mediocre. To substantiate this idea, one should have to be explicit about what is observed and about the costs of different behavior.

Chapter 7

Problem 7.1

1. Optimality requires $x_1 + x_2 = K$. Substituting $x_2 = K - x_1$ into Bob's utility function gives

$$\ln x_1 + \frac{(K - x_1)^2}{2},$$

whose derivative with respect to x_1,

$$\frac{1}{x_1} + x_1 - K,$$

is nonnegative if $K \leq 2$, implying that $(x_1, x_2) = (K, 0)$ is optimal for $K \leq 2$. When $K > 2$, the derivative in the display above is at first positive, then negative, then positive, implying that an optimal x_1 either sets the derivative to zero, $x_1 = \left(K - \sqrt{K^2 - 4}\right)/2$, or occurs at $x_1 = K$. One can verify numerically that $x_1 = K$ is optimal when $K \leq K^* \approx 2.216$. For all $K > K^*$, $x_1 = \frac{K - \sqrt{K^2 - 4}}{2}$ is optimal and is strictly decreasing in K. Figure SO.20 illustrates the dependence of the optimal bundle (x_1, x_2) on K.

2. Object $(p_1, p_2, r, x_1, x_2, k)$ is a Walrasian equilibrium if the allocation (x_1, x_2, k) solves both the firm's profit maximization problem

$$(x_1, x_2, k) \in \arg\max_{x_1', x_2', k'} \left\{ p_1 x_1' + p_2 x_2' - r k' \mid (x_1', x_2') \in B(k') \right\}$$

and Bob's utility maximization problem

$$(x_1, x_2, k) \in \arg\max_{x_1', x_2', k'} \left\{ u(x_1', x_2') \mid p_1 x_1' + p_2 x_2' \leq r k' + (p_1 x_1 + p_2 x_2 - r k),\ k' \leq K \right\}.$$

The market clearing conditions are trivial and implicit in the notation. In particular, the bundle Bob consumes is the bundle the firm produces. The capital Bob rents out is the capital the firm rents and does not exceed his endowment K.

Guess that Bob's bundle in part 1 is a Walrasian equilibrium bundle, that, at equilibrium, Bob rents out his entire endowment of capital, and that the equilibrium prices are $(p_1, p_2, r) = (1, 1, 1)$. To verify the guess, first note that the firm's profit implied by the conjecture is zero, and so Bob's problem is equivalent to his problem in part 1. Hence, Bob trivially rents out his entire endowment of capital (he has nothing else to do with capital), and then optimally buys the bundle (x_1, x_2) identified in part 1. That very same bundle (x_1, x_2) and Bob's endowment K constitute a profit-maximizing production plan for the firm,

$$(x_1, x_2, K) \in \arg\max_{x_1', x_2', k'} \left\{ x_1' + x_2' - k' \mid (x_1', x_2') \in B(k') \right\},$$

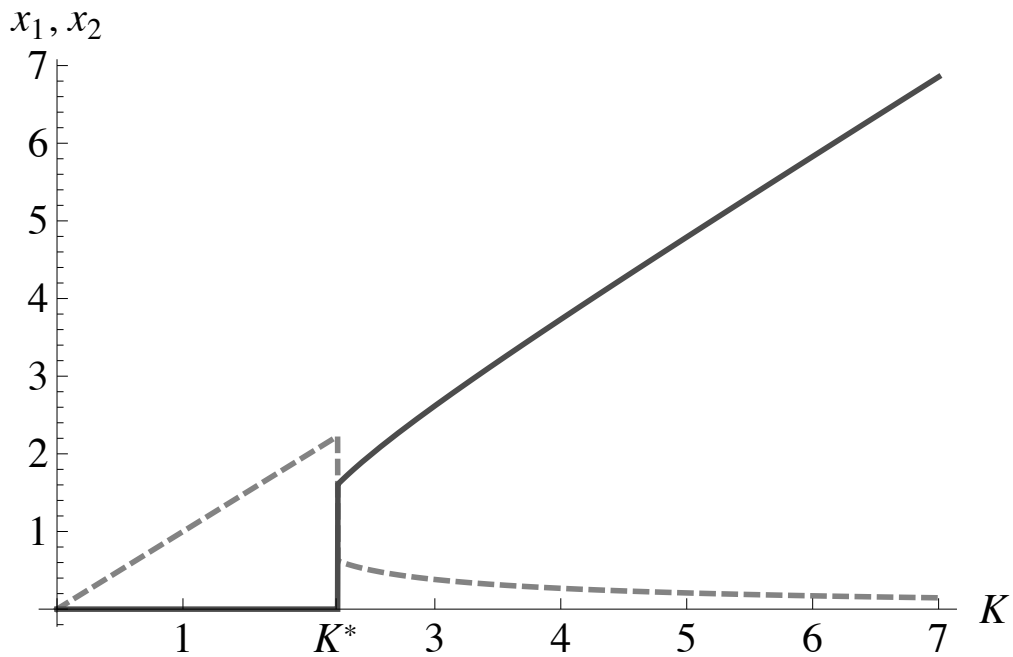


Figure SO.20: (Problem 7.1) Wealth and violence. As K increases, violence (the dashed curve, x_1) first equals K and so increases, then, at $K^* \approx 2.2$, drops abruptly, and finally, falls continuously and converges to zero. Travel (the solid curve, x_2) is first zero, for $K \leq K^*$, then jumps up abruptly at K^*, and continues to increase thereafter.

with the optimal profit being zero, as conjectured. (There are many optimal production plans—just scale (x_1, x_2, K) by an arbitrary positive number.) Thus, the guessed object is indeed a Walrasian equilibrium.

3. The findings are consistent with the historical evolution of violence, for the Western world has witnessed both rising incomes (here, K) and the decline in violence (here, x_1). Steven Pinker in *The Better Angels of Our Nature: Why Violence Has Declined* (Viking, 2011) makes a broad case for the decline in violence. An illustrative example is the decline of violence in children's entertainment. To see the secular reduction in violence, compare *Grimm's Fairy Tales* to their censored instantiations in *Walt Disney* films, *Punch and Judy* puppet shows to *Sesame Street*, and even the *Road Runner*, and *Tom and Jerry* cartoons to the *Curious George*.

   The assumed utility function has the feature that, the more the agent consumes of either good, the more he needs to be compensated with violence for a unit decrease in travel in order to remain indifferent between the old and the new bundles. For low consumption levels, however, the compensation may be quite small, meaning that the agent enjoys the violence so much that he would rather spend an additional dollar on violence than travel.

Problem 7.3

1. Each agent's utility function is concave in his effort, and, hence, the first-order condition for optimality characterizes the agent's optimal effort: $x^i = \theta^i$.

2. Agent i's Kantian-counterfactual problem is

$$\max_{\alpha} \left\{ -\frac{\alpha x^i}{1+\theta^i} + \sum_{j \in \mathcal{I}} \ln\left(1 + \alpha x^j\right) \right\}.$$

The scaling factor $\alpha^i(x)$ that solves this problem satisfies the necessary and sufficient first-order condition for optimality:

$$-\frac{x^i}{1+\theta^i} + \sum_{j \in \mathcal{I}} \frac{x^j}{1+\alpha^i(x)\, x^j} = 0.$$

This condition implies

$$x^i = \left(1+\theta^i\right) \sum_{j \in \mathcal{I}} \frac{1}{\alpha^i(x) + 1/x^j}. \tag{SO.7}$$

From (SO.7),

$$x^i \leq \frac{I\left(1+\bar{\theta}\right)}{\alpha^i(x)} \iff x^i \alpha^i(x) \leq B \equiv I\left(1+\bar{\theta}\right).$$

From (SO.7) again,

$$x^i \geq \frac{1+\underline{\theta}}{\alpha^i(x) + 1/x^i} \iff \alpha^i(x)\, x^i \geq b \equiv \underline{\theta}.$$

Theorem 7.2 then immediately implies the existence of a positive Kantian equilibrium.

3. Kantian equilibrium allocation (x^1, x^2) satisfies

$$1 \in \arg\max_{\alpha > 0} \left\{ -\frac{\alpha x^i}{1+\theta^i} + \sum_{j \in \mathcal{I}} \ln\left(1 + \alpha x^j\right) \right\},$$

or, equivalently, from the first-order condition for optimality of $\alpha = 1$,

$$x^i = \left(1+\theta^i\right) \sum_{j \in \mathcal{I}} \frac{x^j}{1+x^j},$$

wherefrom the sought ratio $x^1/x^2 = \left(1+\theta^1\right) / \left(1+\theta^2\right)$ follows. Substituting the equilibrium effort in the display above into agent i's utility function gives

$$-\sum_{j \in \mathcal{I}} \frac{x^j}{1+x^j} + \sum_{j \in \mathcal{I}} \ln\left(1+x^j\right),$$

which is independent of i.

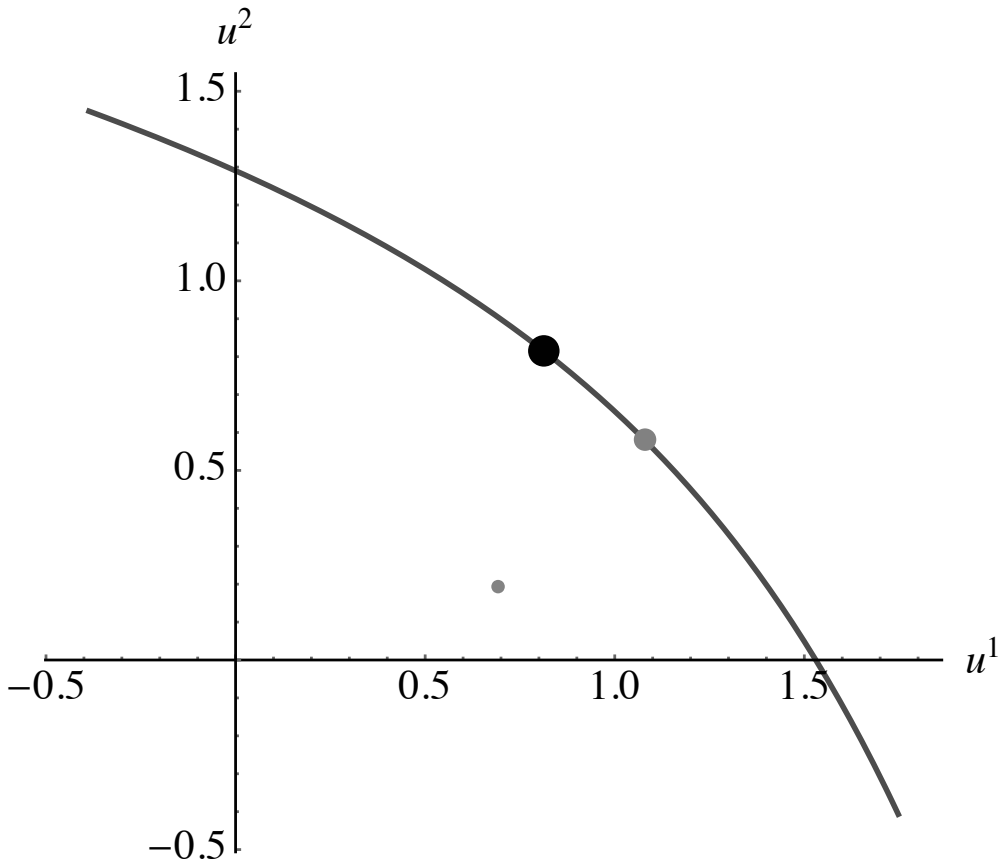


Figure SO.21: (Problem 7.3) The curve is a set of utility pairs induced by Pareto efficient allocations. The utility pairs at the Nash equilibrium, utilitarian optimum, and Kantian equilibrium are, respectively, the small, the intermediate, and the large solid dots.

4. Optimality conditions

$$x^1 = \frac{x^1}{1+x^1} + \frac{x^2}{1+x^2} \qquad \text{and} \qquad x_2 = 2x_1$$

imply the unique positive solution:

$$x^1 = \frac{1+\sqrt{17}}{4} \approx 1.3 \qquad \text{and} \qquad x^2 = \frac{1+\sqrt{17}}{2} \approx 2.6.$$

The Kantian equilibrium effort exceeds the Nash equilibrium effort for each agent. At the Kantian equilibrium, each agent's utility is approximately 0.81.

5. The set of Pareto efficient allocations is

$$\mathcal{P} = \left\{ \left(x^1, x^2\right) = \left(\frac{1+\theta^1}{\omega} - 1, \frac{1+\theta^2}{1-\omega} - 1 \right) \mid \omega \in (0,1) \right\}.$$

6. The set $\mathcal{U}$ is depicted in figure SO.21. The Kantian equilibrium is egalitarian and Pareto

efficient. The Nash equilibrium is Pareto inefficient and favors the agent with the higher disutility of effort. The utilitarian optimum—the outcome of the maximization of the sum of the agents' utilities—is Pareto efficient and favors the agent with the higher disutility of effort. At the utilitarian optimum, the social planner wants the agent who finds effort less costly to exert more effort, for the greater social good; the agent suffers.

Problem 7.5

1. Agent i consumes amount $(1-\lambda)\,\lambda^i$.

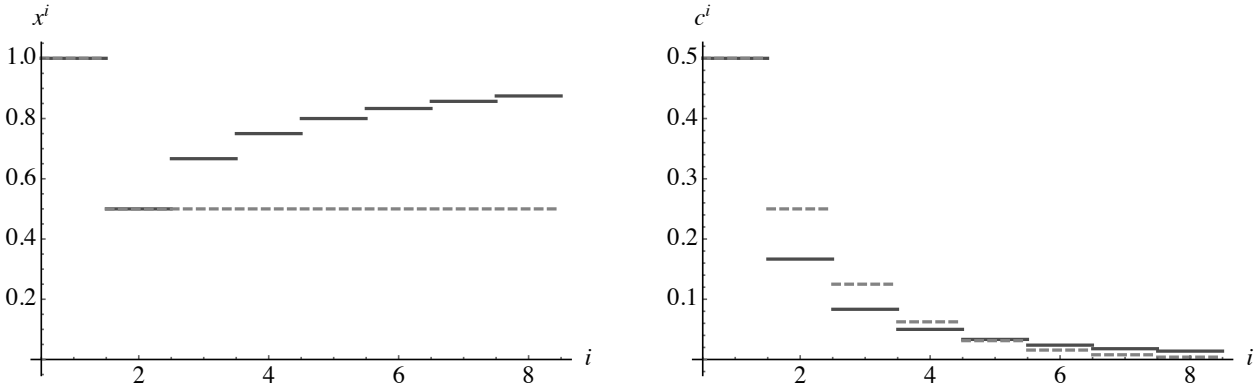
2. The game is peculiar because it has infinitely many agents, who never stop arriving. But let us neglect this peculiarity and assume that it will cause no trouble. The best each agent can do is to expropriate as much as he can, regardless of how much there is to expropriate. So, $x^i = 1$ for each $i \in \mathcal{I}$. Then, at equilibrium, each agent is fully expropriated before he has a chance to consume. Each agent's equilibrium consumption is zero; Nature's endowment keeps changing hands and is never consumed.

3. By contradiction, suppose there exists a Kantian equilibrium in which each agent expropriates fraction $\lambda > 0$. It cannot be that $\lambda = 1$; if $\lambda = 1$, any agent would gain if all agents decreased their expropriation rate a little. Then, because we seek a positive equilibrium, the only candidate for a symmetric equilibrium has $\lambda \in (0,1)$. If each agent i sets $x^i = \lambda$, then agent i's consumption is
$$(\lambda)^i\,(1-\lambda)\,.$$
As a result, at the symmetric Kantian equilibrium, λ must satisfy agent i's first-order condition of optimality:
$$i\,(1-\lambda) - \lambda = 0,$$
which cannot possibly hold for all i. Hence, no symmetric positive Kantian equilibrium exists.

4. An action profile $(x^1, x^2, \ldots, x^i, \ldots)$ is a Kantian equilibrium with the universal counterfactual if, for each $i \in \mathcal{I}$,[SO.5]
$$1 \in \arg\max_{\alpha>0} \left(1-\alpha x^{i+1}\right) \prod_{j=1}^{i} \left(\alpha x^j\right),$$
which implies that x^1 is arbitrary in $(0,1]$, and $x^{i+1} = i/\,(i+1)$ for $i \geq 1$. Agent i's consumes amount $c^i \equiv \left(1-x^{i+1}\right)\prod_{j=1}^{i} x^j = x^1/\,(i\,(i+1))$.

5. An action profile $(x^1, x^2, \ldots, x^i, \ldots)$ is a Kantian equilibrium with the meta-golden-rule counterfactual if, for each $i \in \mathcal{I}$,
$$1 \in \arg\max_{\alpha>0} \left(1-\alpha x^{i+1}\right)\left(\alpha x^i\right) \prod_{j=1}^{i-1} x^j,$$

[SO.5] Implicitly, we cap any agent's expropriation in any Kantian counterfactual at 1.



(a) Universal Kantians (the solid segments) expropriate more aggressively than those who abide by the meta-golden rule (the dashed segments).

(b) Consumption inequality is higher with the meta-golden rule (the dashed segments) than with universal Kantians (the solid segments).

Figure SO.22: (Problem 7.5)

which implies that x^1 is arbitrary in $(0, 1]$, and $x^{i+1} = 1/2$ for $i \geq 1$. Agent i consumes amount $c^i \equiv x^1/2^i$.

6. Universal Kantians expropriate more aggressively than those who abide by the meta-golden rule (figure SO.22a). Both kinds of Kantian reason: "If I expropriate my predecessor more, then my successor will expropriate me more." Only a universal Kantian adds, however: " If I expropriate my predecessor more, then each of my predecessors will also expropriate more, and, hence, there will be more for me to expropriate." Thus, a universal Kantian has a stronger incentive to expropriate.

   Consumption is more unequal with the meta-golden rule than with universal Kantians (figure SO.22b). The economy is endowed with just one unit of the good, at time zero, and aggressive expropriation is needed to spread this unit over time. More aggressive expropriation by universal Kantians ensures that their early generations consume less, and their late generations consume more, than the corresponding Kantians who abide by the meta-golden rule do.

Chapter 8

Problem 8.1

1. Instead of presenting ad hoc arguments for SM2LA, let us borrow the general formula for the taxes from (SO.9), computed in the next step. According to that formula,

$$\tau_S(t_S) = 0 \qquad \text{and} \qquad \tau_F(t_F) = t_F.$$

That is, to reduce congestion on the freeway, its users pay a toll. By contrast, side streets are not liable to congestion, and, hence, their use is not taxed.

2. Again, use (SO.9) to obtain

$$\tau_{10}(t_{10}) = t_{10}, \quad \tau_{134}(t_{134}) = t_{134}, \quad \tau_{110}(t_{110}) = \tau_{405}(t_{405}) = \tau_{101}(t_{101}) = 0.$$

Tolls are only on the congestible segments and discourage the agents from using highway 101—even though it is not liable to congestion and is toll-free.

3. We would like to make sure that each type i chooses route x^i over route y^i if and only if doing so lowers the aggregate costs:

$$\sum_{l\in x^i}(c_l(t_l)+\tau_l(t_l)) < \sum_{l\in y^i}(c_l(t_l)+\tau_l(t_l)) \iff \sum_{l\in x^i}\frac{\mathrm{d}}{\mathrm{d}t_l}(t_l c_l(t_l)) < \sum_{l\in y^i}\frac{\mathrm{d}}{\mathrm{d}t_l}(t_l c_l(t_l)), \tag{SO.8}$$

where the inequality on the right-hand side of the equivalence says that the rate at which the aggregate costs increase if a small measure of agents are routed to x^i is smaller than the rate at which the aggregate costs increase if a small measure of agents are routed to y^i. This inequality can be rewritten as

$$\sum_{l\in x^i}(c_l(t_l)+t_l c_l'(t_l)) < \sum_{l\in y^i}(c_l(t_l)+t_l c_l'(t_l)),$$

where $c_l'(t_l) \equiv \mathrm{d}c_l(t_l)/\mathrm{d}t_l$. Then, the equivalence in (SO.8) holds if

$$\tau_l(t_l) = t_l c_l'(t_l), \qquad l \in \mathcal{L}. \tag{SO.9}$$

With the taxes set as in the display above, utilitarian routing is Nash equilibrium routing; if the aggregate cost function has been minimized, then there is no local improvement and, hence, by (SO.8), no profitable deviation for any driver. Moreover, with the taxes set as in the display above, any Nash equilibrium routing is utilitarian routing if the cost function is concave, meaning that

$$\frac{\mathrm{d}^2}{\mathrm{d}t_l^2}(t_l c_l(t_l)) = 2c_l'(t_l) + t_l c_l''(t_l) \le 0, \qquad l \in \mathcal{L}.$$

What the concavity guarantees is that the absence of a location improvement in costs implies the absence of a global improvement. If the concavity fails, a Nash equilibrium routing can be but a local minimum of the cost function and, hence, not utilitarian.

The taxes in (SO.9) force an agent to recognize that, when he takes a particular route, he makes the route a little slower for measure t_l of other agents at rate $c_l'(t_l)$. That is, each agent is now forced to "internalize the externality" that his route choice impose on others.

Problem 8.3

The table below reports four action profiles and, for each of them, specifies whether it is equilibrium for either friendship graph. Consistent with theorem 8.4, each equilibrium in figure 8.3b remains equilibrium in figure 8.3a.

Action Profile	Utilitarian Welfare	Eqm in Fig. 8.3a?	Eqm in Fig. 8.3b?
$(0,0,0,0,1,0,0,0,1)$	$9-2c$	Yes	No
$(0,0,0,0,1,1,1,1,0)$	$9-4c$	Yes	Yes
$(1,1,1,1,0,0,0,0,1)$	$9-5c$	Yes	Yes
$(1,1,1,1,0,1,1,1,0)$	$9-7c$	Yes	Yes

The example in figure 8.3 does not illustrate the possibility in which an equilibrium with an added link ceases being equilibrium without the link, and a Pareto inferior equilibrium can be found. To illustrate this possibility, simply remove agents 6, 7, and 8 from the graphs in figure 8.3. Then, the action profile in which only agent 5 contributes is equilibrium in figure 8.3b but not in figure 8.3a, in which, at the utilitarian best equilibrium, agent 9 contributes, as well.

Chapter 9

Problem 9.1

1. Alice's objective function is strictly concave and, so, the first-order condition with respect to z implies that Alice equalizes her and each Bob's consumption:

$$e^A - z = e^B + \frac{z}{I-1} \implies z = \left(1 - \frac{1}{I}\right)\left(e^A - e^B\right).$$

   Each agent i (including Alice) consumes

$$x^i = \frac{1}{I}e^A + \left(1 - \frac{1}{I}\right)e^B.$$

2. As $I \to \infty$, $z \to e^A - e^B$ and $x^1 \to e^B$. Alice's consumption is independent of her endowment. If she were to become richer, she would share all the extra endowment with Bobs, and would end up consuming the same as before.

3. Now, when deciding how much to give to charity, Alice compares her utility to the average of Bobs' utilities, instead of their sum, which may diminish her motivation to donate. She now also recognizes that Bobs care about her utility, too, which may further discourage her from donating. Of course, Bobs now also care about other Bobs, which may make donating more attractive to Alice.
   The hint to the problem gives away the answer to rewriting u^is. This answer can be verified by substitution.
   Alice's optimal donation solves

$$\max_{z \in \mathbb{R}_+} \left\{2\ln\left(e^A - z\right) + (I-1)\ln\left(e^B + \frac{z}{I-1}\right)\right\},$$

   which implies

$$z = \frac{I-1}{I+1}\max\left\{0, e^A - 2e^B\right\}.$$

   As $I \to \infty$, $z \to \max\left\{0, e^A - 2e^B\right\}$ and $x^1 = \min\left\{e^A, 2e^B\right\}$.

4. Both formulations have the property that, when $I \to \infty$ (which seems to be the empirically relevant case), Alice's consumption eventually stops increasing in her endowment. In the first formulation, Alice donates everything in excess of Bob's endowment, whereas, in the second formulation, Alice donates everything in excess of twice Bob's endowment. In this, the second formulation appears somewhat more plausible, even though the prediction that Alice would necessarily donate entire increase in her endowment to charity seems counterfactual (with the exception of very few very rich people) and is shared by both formulations.

Problem 9.3

1. Note that any x that solves the problem specified here (denoted by P) also solves (under the section's conditions) the problem in problem 4.14 (denoted by P'), whose solutions are socialist. Indeed, if x solves P, then $u\left(s^1, x^1\right) = u\left(s^2, x^2\right)$. By contradiction, suppose, say, $u\left(s^1, x^1\right) > u\left(s^2, x^2\right)$. Then, one can redistribute stuff from agent 1 to agent 2 so as to raise agent 2's utility while preserving the inequality. Doing so is possible by compensation and u-monotonicity, which imply the continuity of u in the consumption of stuff. This redistribution raises the utility of the worst-off agent, thereby contradicting the optimality of x in P. Thus, x must equalize the utilities in P. Thus, the utility equalization constraint can be added to P without affecting the set of solutions while making the equivalence of P and P' apparent.

2. By contradiction, suppose that x solves P' but not P. Then, one can raise the utility of, say, agent 1 while raising utility of agent 2 even more. But then one can equalize the agents' utilities by redistributing some stuff from agent 2 to agent 1, thereby contradicting the hypothesis that x solves P'.

3. Suppose that an (indivisible) concert ticket must be allocated between Alice and Bob. If the ticket is destroyed, Alice and Bob each get the utility of zero. If Alice gets the ticket, her utility is two, and Bob's is one, for Bob is pleased that the ticket is not wasted. Similarly, if Bob gets the ticket, his utility is two, and Alice's is one. Then, destroying the ticket would equalize the agents' utilities, while the utility of the worst-off individual is maximized by giving the ticket to either Alice or Bob.

Problem 9.5

1. Roughly speaking, according to (9.11), Alice and Bob end up voting for the voting rule that ensures that, in a vote for voting rules conducted according to that voting rule, that very voting rule itself would win. In (9.12), by contrast, the voting rule is not held fixed as Alice and Bob vote.

   Here is a metaphor. Suppose the agents vote to decide how large the threshold in a supermajority vote should be. To adopt a higher threshold, (9.12) requires a greater supermajority, whereas (9.11) uses a fixed supermajority, which is independent of the threshold that is being decided on. In fact, (9.11) requires the very supermajority that ends up being selected. The motivation for (9.11) is a notion of stability. One would not expect that supermajority rule for a constitutional change to prevail which would be immediately overturned in favor of something else as soon as the voters got the chance to vote according to the present constitution (in which the supermajority rule for constitutional change is specified).

2. Each agent prefers more consumption to less and also (rather selflessly) that the social welfare function put a smaller weight on herself or himself.

   In the interior maximization problem, over c, the objective function is concave in c, and so the

first-order condition for optimality characterizes the unique solution: $c = \alpha^*$. The fixed-point problem happens to be independent of the consumption profile just identified and (neglecting the additive terms) is

$$\alpha^* \in \arg\max_{\alpha \in [0,1]} \left\{ -\alpha^* \gamma \ln \alpha + \left(1 - \alpha^*\right) \alpha \right\},$$

whose unique solution is $\alpha^* = 1 - \gamma$. Thus, Alice consumes less and the planner cares about her less if she herself would like the planner to care less about her.

3. Each agent prefers more consumption to less and also that the social welfare function put more weight on herself or himself. By inspection of (9.11), the two extreme solutions are $c = \alpha^* = 1$ and $c = \alpha^* = 0$.