Answers to Selected Exercises

Answer (Exercise 1.1)

(a) $m = 4$ and $n = 4$.

(b) $mn = 16$.

(c) There are zero empty cells.

(d) There are 16 filled cells.

Answer (Exercise 1.3)

(a) Because there is an equal number of rows and columns.

(b) Because the array is symmetric around the diagonal.

(c) Row/column V02 has a sum of 16.

Answer (Exercise 1.5)

(a) Associativity among **A**, **B**, and **C** is illustrated by the expression

$$(\mathbf{A} \oplus \mathbf{B}) \oplus \mathbf{C} \quad = \quad \mathbf{A} \oplus (\mathbf{B} \oplus \mathbf{C})$$

or

$$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} \quad = \quad \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$$

(b) Commutativity among **A**, **B**, and **C** is illustrated by the expression

$$\mathbf{A} \oplus \mathbf{B} \quad = \quad \mathbf{B} \oplus \mathbf{A}$$

or

$$\mathbf{A} \otimes \mathbf{B} \quad = \quad \mathbf{B} \otimes \mathbf{A}$$

(c) Distributivity among **A**, **B**, and **C** is illustrated by the expression

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) \quad = \quad (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C})$$

Answer (Exercise 2.1)

(a) upper left: $m = 4$ and $n = 4$, upper middle: $m = 3$ and $n = 2$, right: $m = 12$ and $n = 1$, and lower middle: $m = 6$ and $n = 3$.

(b) upper left: $mn = 16$, upper middle: $mn = 6$, right: $mn = 12$, and lower middle: $mn = 18$.

(c) There are no empty cells.

(d) upper left: integers, upper middle: words, right: dollars, and lower middle: integers.

Answer (Exercise 2.3)

(a) $m = 6$ and $n = 3 + 6 = 9$.

(b) $mn = 54$.

(c) $6 + 21 = 27$ empty entries.

(d) $12 + 15 = 27$ non-empty entries.

Answer (Exercise 2.5)

(a) The longest side of the red rectangle aligns with the longest diagonal of the blue parallelogram, and the shortest side of the red rectangle aligns with the shortest diagonal of the blue parallelogram.

(b) The sides of the black square align with the sides of the red rectangle.

(c) The longest side of the red rectangle aligns with the semi-major axis of the green dashed ellipse, and the shortest side of the red rectangle aligns with the semi-minor axis of the green dashed ellipse.

(d) Two of the sides of the blue parallelogram are tangent to the green dashed ellipse.

Answer (Exercise 4.1)

(a) $\mathbf{A}$: $m = 24$ and $n = 9$, $\mathbf{A}_1$: $m = 3$ and $n = 2$, and $\mathbf{A}_1$: $m = 3$ and $n = 3$.

(b) $\mathbf{A}$: $mn = 216$, $\mathbf{A}_1$: $mn = 6$, and $\mathbf{A}_2$: $mn = 9$.

(c) $\mathbf{A}$ has 2 empty entries, $\mathbf{A}_1$ has 0 empty entries, and $\mathbf{A}_2$ has 0 empty entries.

(d) $\mathbf{A}$ has $216 - 2 = 214$ non-empty entries, $\mathbf{A}_1$ has 6 non-empty entries, and $\mathbf{A}_2$ has 9 non-empty entries.

(e) $\mathbf{A}$('063012ktnA2 ','Track ') = 'Sugar ', $\mathbf{A}_1(1,1)$ = 'Kitten ', and $\mathbf{A}_2(3,3)$ = 'Pop '.

Answer (Exercise 4.3)

(a) $\mathbf{E}$: $m = 22$ and $n = 31$, $\mathbf{E}_1$: $m = 22$ and $n = 3$, and $\mathbf{E}_1$: $m = 22$ and $n = 5$.

(b) $\mathbf{E}$: $mn = 682$, $\mathbf{E}_1$: $mn = 66$, and $\mathbf{E}_2$: $mn = 110$.

(c) $\mathbf{E}$ has 498 empty entries, $\mathbf{E}_1$ has 38 empty entries, and $\mathbf{E}_2$ has 63 empty entries.

(d) $\mathbf{E}$ has 184 non-empty entries, $\mathbf{E}_1$ has 28 non-empty entries, and $\mathbf{E}_2$ has 47 non-empty entries.

(e) Pop is the most common genre and Kitten is the most common artist.

Answer (Exercise 5.1)

(a) The graph is an unweighted, undirected graph because the edges have no values and no direction. It is not a hyper-graph because there are no edges connecting more than two vertices. It is not a multi-graph because there is no more than one edge between the vertices.

(b) The graph has 8 vertices and 6 edges.

(c) There are $(8-1)8/2 = 28$ possible edges between the 8 vertices.

(d) There are $28-6 = 22$ possible edges that are not in the graph.

Answer (Exercise 5.3)

(a) $\mathbf{A}$: $m = 13$ and $n = 15$, $\mathbf{A}^{\mathsf{T}}$: $m = 15$ and $n = 13$, and $\mathbf{A} \oplus \mathbf{A}^{\mathsf{T}}$: $m = 20$ and $n = 20$.

(b) $\mathbf{A}$: $mn = 195$, $\mathbf{A}^{\mathsf{T}}$: $mn = 195$, and $\mathbf{A} \oplus \mathbf{A}^{\mathsf{T}}$: $mn = 400$.

(c) $\mathbf{A}$ has $195 - 23 = 172$ empty entries, $\mathbf{A}^{\mathsf{T}}$ has $195 - 23 = 172$ empty entries, and $\mathbf{A} \oplus \mathbf{A}^{\mathsf{T}}$ has $400 - 46 = 354$ empty entries.

(d) $\mathbf{A}$ has 23 non-empty entries, $\mathbf{A}^{\mathsf{T}}$ has 23 non-empty entries, and $\mathbf{A} \oplus \mathbf{A}^{\mathsf{T}}$ has 46 non-empty entries. $\mathbf{A} \oplus \mathbf{A}^{\mathsf{T}}$ has double the non-empty entries of $\mathbf{A}$.

Answer (Exercise 5.5)

O5 is the hyper-edge containing the six vertices {V09, V11, V02, V16, V06, V20} that are part of all the other edges in the picture. Thus, because O5 shares edges with every other edge, it has a dense row and column in the array.

Answer (Exercise 6.1)

(a) The graph is an unweighted, directed graph. Is not a hyper-graph. It is not a multi-graph.

(b) The graph is unweighted because there are no values on the edges. The graph is a directed graph because the edges have direction given by arrows. The graph is not a hyper-graph because there are no edges connecting more than two vertices. The graph is not a multi-graph because there is no more than one edge in the same direction between any two vertices.

(c) The adjacency matrix is an unweighted graph because there are no values on the entries. The adjacency matrix is directed because it is not symmetric about the diagonal. The adjacency matrix is not a hyper-graph because an adjacency matrix cannot distinguish an edge with more than two vertices. The adjacency matrix is not a multi-graph because there are no values on the entries that could be used as a count of multiple edges between the same vertices.

Answer (Exercise 6.3)

In all cases, addition of 0 with a nonzero value produces a nonzero value. Likewise, addition of a 0 value with another 0 value results in another 0 value.

Answer (Exercise 7.2)

Formally, an associative array $\mathbf{A}$ is a map from sets of keys $K_1 \times K_2$ to a value set V with a semiring structure

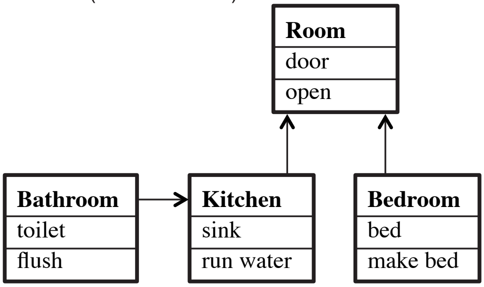
$$\mathbf{A} : K_1 \times K_2 \to V,$$

where $(V, \oplus, \otimes, 0, 1)$ is a semiring with addition operator $\oplus$, multiplication operator $\otimes$, additive-identity/multiplicative-annihilator 0, and multiplicative-identity 1.

Answer (Exercise 7.3)
Adjacency arrays can be used to describe weighted-directed graphs. Incidence arrays can be used to describe weighted-directed-multi-hyper graphs.

Answer (Exercise 7.5)
The result of $\mathbf{VA}$ is the transpose of $\mathbf{AV}$.

Answer (Exercise 8.1)



Answer (Exercise 8.3)

(a) A monoid adds the property of the identity $1 \otimes v = v$ to a semigroup.

(b) A commutative monoid adds the property of commutativity $u \otimes v = v \otimes u$ to a monoid.

(c) A group adds the property of the inverse $v^{-1} \otimes v = 1$ to a monoid.

(d) A commutative group adds the property of commutativity $u \otimes v = v \otimes u$ to a group.

Answer (Exercise 8.5)

(a) A partially ordered set adds the property of $u \leq v$ or $v \leq u$ for some u and v in the set.

(b) A totally ordered set adds the property of $u \leq v$ or $v \leq u$ for all u and v in the set.

(c) A strict partially ordered set adds the property of $u < v$ or $v < u$ for some u and v in the partially ordered set.

(d) A strict totally ordered set adds the property of $u < v$, $u > v$, or $u = v$ for all u and v in the set.

Answer (Exercise 8.7)

(a)

$$\mathbf{AA} = \begin{array}{c} \\ \text{v04} \\ \text{v08} \end{array} \begin{array}{c} \begin{array}{cc} \text{v10} & \text{v12} \end{array} \\ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \end{array}$$

(b)

$$(\mathbf{AA})^{\mathsf{T}} = \begin{array}{c} \\ \text{v10} \\ \text{v12} \end{array} \begin{array}{c} \begin{array}{cc} \text{v04} & \text{v08} \end{array} \\ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \end{array}$$

Answer (Exercise 9.1)

(a)

$$\alpha \otimes \mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \begin{bmatrix} 21 & 6 & 3 \\ 0 & 9 & 9 \end{bmatrix} \end{array}$$

(b)

$$\mathbf{A} \oplus \mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \begin{bmatrix} 11 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} \end{array}$$

(c)

$$\mathbf{A} \otimes \mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \begin{bmatrix} 28 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

(d)

$$\mathbf{A} \oplus.\otimes \mathbf{C} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 21 & 17 & 11 & 7 \\ 0 & 6 & 6 & 12 \end{bmatrix} \end{array}$$

Answer (Exercise 9.3)

(a)

$$\alpha \otimes \mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \left[\begin{array}{ccc} 7 & 3 & 3 \\ \infty & 3 & 3 \end{array} \right] \end{array}$$

(b)

$$\mathbf{A} \oplus \mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \left[\begin{array}{ccc} 4 & 2 & 1 \\ 1 & 3 & 1 \end{array} \right] \end{array}$$

(c)

$$\mathbf{A} \otimes \mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \left[\begin{array}{ccc} 7 & 2 & 5 \\ \infty & \infty & 3 \end{array} \right] \end{array}$$

(d)

$$\mathbf{A} \oplus.\otimes \mathbf{C} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 7 & 1 & 2 & 1 \\ \infty & 3 & 3 & 3 \end{array} \right] \end{array}$$

Answer (Exercise 9.5)

(a)

$$\alpha \otimes \mathbf{A} = \begin{array}{c} \\ \mathsf{a} \\ \mathsf{b} \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} \emptyset & \{0\} \\ \emptyset & \emptyset \end{array} \right] \end{array}$$

(b)

$$\mathbf{A} \oplus \mathbf{B} = \begin{array}{c} \\ \mathsf{a} \\ \mathsf{b} \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} \{0,1\} & \{0,2\} \\ \{1\} & \{0\} \end{array} \right] \end{array}$$

(c)

$$\mathbf{A} \otimes \mathbf{B} = \begin{array}{c} \\ \mathsf{a} \\ \mathsf{a} \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} \emptyset & \emptyset \\ \emptyset & \emptyset \end{array} \right] \end{array}$$

(d)

$$\mathbf{A} \oplus.\otimes \mathbf{C} = \begin{array}{c} \\ \mathsf{a} \\ \mathsf{a} \end{array} \begin{array}{c} \mathsf{a} \\ \begin{bmatrix} \{0\} \\ \emptyset \end{bmatrix} \end{array}$$

Answer (Exercise 9.7) (Proof of Lemma 9.1.)
By definition

$$w \wedge u \leq w$$

and

$$w \wedge u \leq u \leq v$$

so that

$$w \wedge u \leq w \wedge v$$

Likewise, by definition

$$w \leq w \vee v$$

and

$$u \leq v \leq w \vee v$$

so that

$$w \vee u \leq w \vee v$$

Answer (Exercise 10.1)
Yes, it maps into a semiring and has finite support in both the min-max and max-plus algebras. The size in either is given by $(2,3)$.

Answer (Exercise 10.3)
In all three cases, they are associative arrays if the underlying sets of values are semirings. Regardless of the structure chosen, the sizes are the same, given by $(2,3)$, $(2,3)$, and $(3,4)$.

Answer (Exercise 10.5)
The maps are associative arrays if the underlying sets of values are semirings. **G** and **H** both have size $(2,2)$, regardless of the choice of semiring. **I** can have size $(1,1)$ if either $\{0\}$ or $\{0,2\}$ are the additive identity, otherwise it has size $(2,1)$.

Answer (Exercise 10.7)

The relevant computations are as follows.

$$vA = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \end{array} \qquad \mathbf{A} \oplus \mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} \end{array}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{AC} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \end{array}$$

Then the relevant inequalities are satisfied.

$$
\begin{aligned}
\text{size}(v\mathbf{A},1) &= 2 \le 2 = \text{size}(\mathbf{A},1)\\
\text{size}(v\mathbf{A},2) &= 3 \le 3 = \text{size}(\mathbf{A},2)\\
\text{size}(\mathbf{A}\oplus\mathbf{B},1) &= 2 \le 2 = \text{size}(\mathbf{A},1)\oplus\text{size}(\mathbf{B},1)\\
\text{size}(\mathbf{A}\oplus\mathbf{B},2) &= 2 \le 2 = \text{size}(\mathbf{A},2)\oplus\text{size}(\mathbf{B},2)\\
\text{size}(\mathbf{A}\otimes\mathbf{B},1) &= 2 \le 2 = \min(\text{size}(\mathbf{A},1),\text{size}(\mathbf{B},1))\\
\text{size}(\mathbf{A}\otimes\mathbf{B},2) &= 2 \le 2 = \min(\text{size}(\mathbf{A},2),\text{size}(\mathbf{B},2))\\
\text{size}(\mathbf{AC},1) &= 2 \le 2 = \text{size}(\mathbf{A},1)\\
\text{size}(\mathbf{AC},2) &= 4 \le 4 = \text{size}(\mathbf{A},2)
\end{aligned}
$$

Answer (Exercise 10.9)

Row k_1 of $v\mathbf{A}$ is row k_1 of $\mathbf{A}$ multiplied by v. Hence, if row k_1 of $v\mathbf{A}$ is nonzero, then row k_1 of $\mathbf{A}$ must also be nonzero, since any scalar multiple of a row of zeroes is also zero. Thus, $\text{size}(v\mathbf{A},1) \leq \text{size}(\mathbf{A},1)$. Likewise, $\text{size}(v\mathbf{A},2) \leq \text{size}(\mathbf{A},2)$.

Elements of row k_1 of $\mathbf{A}\otimes\mathbf{B}$ are the products of corresponding entries of row k_1 of $\mathbf{A}$ and row k_1 of $\mathbf{B}$. Hence, if row k_1 of $\mathbf{A}\otimes\mathbf{B}$ is nonzero, then row k_1 of both $\mathbf{A}$ and $\mathbf{B}$ must be nonzero. Thus, $\text{size}(\mathbf{A}\otimes\mathbf{B},1) \leq \min(\text{size}(\mathbf{A},1),\text{size}(\mathbf{B},1))$. Likewise, $\text{size}(\mathbf{A}\oplus\mathbf{B},2) \leq \min(\text{size}(\mathbf{A},2),\text{size}(\mathbf{B},2))$.

Elements of row k_1 of $\mathbf{A}\oplus\mathbf{B}$ are the sums of corresponding entries of row k_1 or $\mathbf{A}$ and row k_1 of $\mathbf{B}$. Hence, if row k_1 of $\mathbf{A}\oplus\mathbf{B}$ is nonzero, then at least one of row k_1 of $\mathbf{A}$ or row k_1 of $\mathbf{B}$ must be nonzero. Thus, $\text{size}(\mathbf{A}\oplus\mathbf{B},1) \leq \text{size}(\mathbf{A},1)+\text{size}(\mathbf{B},1)$. Likewise, $\text{size}(\mathbf{A}\oplus\mathbf{B},2) \leq \text{size}(\mathbf{A},2)+\text{size}(\mathbf{B},2)$.

Elements of row k_1 of $\mathbf{AC}$ are the sums of products of entries of row k_1 of $\mathbf{A}$ and the entries in the columns of $\mathbf{C}$. Hence, if row k_1 of $\mathbf{AC}$ is nonzero, then row k_1 of $\mathbf{A}$ must be nonzero. Thus, $\text{size}(\mathbf{AC},1) \leq \text{size}(\mathbf{A},1)$. Likewise, elements of column k_3 of $\mathbf{AC}$ are the sums of products of entries of the rows of $\mathbf{A}$ and the entries of column k_3 of $\mathbf{C}$. Hence, if column k_3 of $\mathbf{AC}$ is nonzero, then column k_3 of $\mathbf{C}$ must be nonzero. Thus, $\text{size}(\mathbf{AC},1) \leq \text{size}(\mathbf{C},2)$.

Answer (Exercise 10.10) (Proof of Theorem 10.2.)

Suppose that some coordinate pair (k_1,k_2) is not in the support of $\mathbf{A}$ or $\mathbf{B}$. Then

$$\mathbf{A}(k_1,k_2) = \mathbf{B}(k_1,k_2) = 0$$

so

$$(\mathbf{A}\oplus\mathbf{B})(k_1,k_2) = \mathbf{A}(k_1,k_2)+\mathbf{B}(k_1,k_2) = 0+0 = 0$$

and so

$$(k_1,k_2) \notin \text{support}(\mathbf{A}\oplus\mathbf{B})$$

Thus, any point not in the support of $\mathbf{A}$ or $\mathbf{B}$ cannot be in the support of $\mathbf{A}\oplus\mathbf{B}$, and so

$$\text{support}(\mathbf{A}\oplus\mathbf{B}) \subset \text{support}(\mathbf{A}) \cup \text{support}(\mathbf{B})$$

If V is zero-sum-free, then

$$(\mathbf{A}\oplus\mathbf{B})(k_1,k_2) = \mathbf{A}(k_1,k_2)+\mathbf{B}(k_1,k_2) = 0$$

if and only if

$$\mathbf{A}(k_1,k_2) = 0 = \mathbf{B}(k_1,k_2)$$

so

$$(k_1,k_2) \in \text{support}(\mathbf{A}\oplus\mathbf{B})$$

implies

$$(k_1,k_2) \in \text{support}(\mathbf{A}) \cup \text{support}(\mathbf{B})$$

Answer (Exercise 10.11) (Proof of Theorem 10.3.)
Firstly, for any sets U and V, if

$$U \subset V$$

then

$$|U| \leq |V|$$

Additionally, for any sets U and V,

$$|U \cup V| = |U| + |V| - |U \cap V| \leq |U| + |V|$$

So by Theorem 10.2

$$\begin{aligned}
\text{nnz}(\mathbf{A} \oplus \mathbf{B}) &= |\text{support}(\mathbf{A} \oplus \mathbf{B})| \\
&\leq |\text{support}(\mathbf{A}) \cup \text{support}(\mathbf{B})| \\
&\leq |\text{support}(\mathbf{A})| + |\text{support}(\mathbf{B})| \\
&= \text{nnz}(\mathbf{A}) + \text{nnz}(\mathbf{B})
\end{aligned}$$

and so

$$\text{nnz}(\mathbf{A} \oplus \mathbf{B}) \leq \text{nnz}(\mathbf{A}) + \text{nnz}(\mathbf{B})$$

On the other hand, $\text{nnz}(\mathbf{A} \oplus \mathbf{B})$ can be bounded from below because every nonzero element of $\mathbf{A}$ or $\mathbf{B}$ contributes a nonzero element of $\mathbf{A} \oplus \mathbf{B}$ if the zero-sum-free criterion is used. Thought of another way,

$$\begin{aligned}
\text{nnz}(\mathbf{A} \oplus \mathbf{B}) &= |\text{support}(\mathbf{A} \oplus \mathbf{B})| \\
&= |\text{support}(\mathbf{A}) \cup \text{support}(\mathbf{B})| \\
&\geq \max(|\text{support}(\mathbf{A})|, |\text{support}(\mathbf{B})|) \\
&= \max(\text{nnz}(\mathbf{A}), \text{nnz}(\mathbf{B}))
\end{aligned}$$

Answer (Exercise 10.12) (Proof of Theorem 10.4.)
Suppose row k_1 of $\mathbf{A}$ and $\mathbf{B}$ are empty (all zeros)

$$\mathbf{A}(k_1,:) = 0$$
$$\mathbf{B}(k_1,:) = 0$$

so that row k_1 does not add to either size($\mathbf{A}$) or size($\mathbf{B}$). Then sum of these rows must also be zero

$$\mathbf{A}(k_1,:) \oplus \mathbf{B}(k_1,:) = 0$$

Thus, the addition of two empty rows cannot add to

$$\text{size}(\mathbf{A}(k_1,:) \oplus \mathbf{B}(k_1,:))$$

The same argument can also be applied to the columns, showing that

$$\text{size}(\mathbf{A} \oplus \mathbf{B}) \leq \text{size}(\mathbf{A}) + \text{size}(\mathbf{B})$$

Additionally, given that the i-th row of either $\mathbf{A}$ or $\mathbf{B}$ is nonzero, by zero-sum-freeness the i-th row of $\mathbf{A} \oplus \mathbf{B}$ is also nonzero. The same is true of the columns. Thus, the number of nonzero rows (respectively, columns) in $\mathbf{A} \oplus \mathbf{B}$ is at least the number of rows (respectively, columns) of either $\mathbf{A}$ or $\mathbf{B}$, so

$$\text{size}(\mathbf{A} \oplus \mathbf{B}) \geq \max(\text{size}(\mathbf{A}), \text{size}(\mathbf{B}))$$

Answer (Exercise 10.13) (Proof of Theorem 10.5.)
Assume V is zero-sum-free. If $\underline{m}_{\mathbf{D}} \times \underline{n}_{\mathbf{D}} = (\underline{m}_{\mathbf{D}}, \underline{n}_{\mathbf{D}})$ denotes size($\mathbf{D}$), then Theorem 10.4 implies

$$\underline{m}_{\mathbf{A}\oplus\mathbf{B}} \geq \max(\underline{m}_{\mathbf{A}}, \underline{m}_{\mathbf{B}})$$
$$\underline{n}_{\mathbf{A}\oplus\mathbf{B}} \geq \max(\underline{n}_{\mathbf{A}}, \underline{n}_{\mathbf{B}})$$

so

$$\underline{m}_{\mathbf{A}\oplus\mathbf{B}}\underline{n}_{\mathbf{A}\oplus\mathbf{B}} \geq \max(\underline{m}_{\mathbf{A}}, \underline{m}_{\mathbf{B}}) \max(\underline{n}_{\mathbf{A}}, \underline{n}_{\mathbf{B}})$$

Now note that in general $\max(a,b) \cdot \max(c,d) \geq \max(a \cdot c, b \cdot d)$ for non-negative integers a,b,c,d since

$$\max(a,b) \cdot c = \max(a \cdot c, b \cdot c) \qquad \max(a,b) \cdot d = \max(a \cdot d, b \cdot d)$$

Then

$$\begin{aligned} \max(a,b) \cdot \max(c,d) &= \max(\max(a \cdot c, b \cdot c), \max(a \cdot d, b \cdot d)) \\ &= \max(a \cdot c, b \cdot c, a \cdot d, b \cdot d) \\ &\geq \max(a \cdot c, b \cdot d) \end{aligned}$$

In particular,

$$\underline{\text{total}}(\mathbf{A} \oplus \mathbf{B}) \geq \max(\underline{\text{total}}(\mathbf{A}), \underline{\text{total}}(\mathbf{B}))$$

Answer (Exercise 10.14) (Proof of Theorem 10.6.)
Suppose

$$\mathbf{w} \in \text{image}(\mathbf{C})$$

so that there is **v** with

$$\mathbf{Cv} = \mathbf{w}$$

But then

$$\mathbf{w} = (\mathbf{A} \oplus \mathbf{B})\mathbf{v} = \mathbf{Av} \oplus \mathbf{Bv}$$

and

$$\mathbf{Av} \in \text{image}(\mathbf{A})$$

and

$$\mathbf{Bv} \in \text{image}(\mathbf{B})$$

The above reasoning shows that

$$\mathbf{w} \in \text{image}(\mathbf{A}) \oplus \text{image}(\mathbf{B})$$

and hence that

$$\text{image}(\mathbf{C}) \subset \text{image}(\mathbf{A}) \oplus \text{image}(\mathbf{B})$$

Answer (Exercise 10.15) (Proof of Theorem 10.7.)

$$\text{image}(\mathbf{A} \oplus \mathbf{B}) \subset \text{image}(\mathbf{A}) \oplus \text{image}(\mathbf{B})$$

By the above condition, it is only required to show that the dimension of the sum of two sets is less than or equal to the sum of the dimensions. If there is a generating set for image(**A**) and a generating set for image(**B**), then these two sets together will generate

$$\text{image}(\mathbf{A}) \oplus \text{image}(\mathbf{B})$$

and hence

$$\text{image}(\mathbf{A} \oplus \mathbf{B})$$

So writing this in terms of ranks gives

$$\text{rank}(\mathbf{A} \oplus \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$$

Answer (Exercise 10.16) (Proof of Theorem 10.8.)
Suppose that some coordinate pair (k_1, k_2) is not in the support of **A**, then $\mathbf{A}(k_1, k_2) = 0$, so

$$(\mathbf{A} \otimes \mathbf{B})(k_1, k_2) = \mathbf{A}(k_1, k_2) \otimes \mathbf{B}(k_1, k_2) = 0 \otimes \mathbf{B}(k_1, k_2) = 0$$

Thus (k_1,k_2) is not in the support of $\mathbf{A}\otimes\mathbf{B}$. Similarly, if (k_1,k_2) is not in the support of $\mathbf{B}$, then $\mathbf{B}(k_1,k_2) = 0$, so

$$(\mathbf{A}\otimes\mathbf{B})(k_1,k_2) = \mathbf{A}(k_1,k_2)\otimes\mathbf{B}(k_1,k_2) = \mathbf{A}(k_1,k_2)\otimes 0 = 0$$

The above argument means that if (k_1,k_2) is not in the support of $\mathbf{A}$ or is not in the support of $\mathbf{B}$, then it is not in the support of $\mathbf{A}\otimes\mathbf{B}$. So if (k_1,k_2) is in the support of $\mathbf{A}\otimes\mathbf{B}$, it must be in the support of $\mathbf{A}$ and in the support of $\mathbf{B}$. Hence

$$\text{support}(\mathbf{A}\otimes\mathbf{B}) \subset \text{support}(\mathbf{A}) \cap \text{support}(\mathbf{B})$$

Now, suppose

$$(k_1,k_2) \in \text{support}(\mathbf{A}) \cap \text{support}(\mathbf{B})$$

The above equation implies that $\mathbf{A}(k_1,k_2) \neq 0$ and $\mathbf{B}(k_1,k_2) \neq 0$ as well. If the entry space of $\mathbf{A}$ and $\mathbf{B}$ has no zero divisors, then for all elements of the entry space, if a and b are nonzero, ab is as well. Since it is known that for this (k_1,k_2)

$$(\mathbf{A}\otimes\mathbf{B})(k_1,k_2) = \mathbf{A}(k_1,k_2)\mathbf{B}(k_1,k_2)$$

is the product of two nonzero elements of the entry space, if the entry space has no zero divisors, then

$$(\mathbf{A}\otimes\mathbf{B})(k_1,k_2) \neq 0$$

Thus in this case, if

$$(k_1,k_2) \in \text{support}(\mathbf{A}) \cap \text{support}(\mathbf{B})$$

then

$$(k_1,k_2) \in \text{support}(\mathbf{A}\otimes\mathbf{B})$$

and hence

$$\text{support}(\mathbf{A}) \cap \text{support}(\mathbf{B}) \subset \text{support}(\mathbf{A}\otimes\mathbf{B})$$

Combining the above equation with

$$\text{support}(\mathbf{A}\otimes\mathbf{B}) \subset \text{support}(\mathbf{A}) \cap \text{support}(\mathbf{B})$$

gives

$$\text{support}(\mathbf{A}\otimes\mathbf{B}) = \text{support}(\mathbf{A}) \cap \text{support}(\mathbf{B})$$

Answer (Exercise 10.17) (Proof of Theorem 10.9.)
Firstly, for any sets U and V, if

$$U \subset V$$

then

$$|U| \leq |V|$$

Secondly, it is the case that

$$U \cap V \subset U$$

and

$$U \cap V \subset V$$

so

$$|U \cap V| \leq |U|$$

and

$$|U \cap V| \leq |V|$$

so

$$|U \cap V| \leq \min(|U|, |V|)$$

So by Theorem 10.8

$$\begin{aligned}
\text{nnz}(\mathbf{A} \otimes \mathbf{B}) &= |\text{support}(\mathbf{A} \otimes \mathbf{B})| \\
&\leq |\text{support}(\mathbf{A}) \cap \text{support}(\mathbf{B})| \\
&\leq \min(|\text{support}(\mathbf{A})|, |\text{support}(\mathbf{B})|) \\
&= \min(\text{nnz}(\mathbf{A}), \text{nnz}(\mathbf{B}))
\end{aligned}$$

Answer (Exercise 10.18) (Proof of Theorem 10.10.)
Suppose that row key k_1 defines a row $\mathbf{A}(k_1,:)$ that is not counted in size($\mathbf{A}$), and that $\mathbf{B}(k_1,:)$ is not counted in size($\mathbf{B}$). If the k_1 row of either $\mathbf{A}$ or $\mathbf{B}$ is zero, then the k_1 row of $\mathbf{A} \otimes \mathbf{B}$ is the element-wise multiplication of these and is also zero. This means that zero rows of $\mathbf{A}$ or $\mathbf{B}$ turn into zero rows of $\mathbf{A} \otimes \mathbf{B}$, so the set of nonzero rows of $\mathbf{A} \otimes \mathbf{B}$ is a subset of both the set of nonzero rows of $\mathbf{A}$ and the set of nonzero rows of $\mathbf{B}$. The same argument can also be made for the columns, so

$$\text{size}(\mathbf{A} \otimes \mathbf{B}) \leq \min(\text{size}(\mathbf{A}), \text{size}(\mathbf{B}))$$

as desired.

Answer (Exercise 10.19) (Proof of Theorem 10.11.)

If $\underline{m}_{\mathbf{D}} \times \underline{n}_{\mathbf{D}} = (\underline{m}_{\mathbf{D}}, \underline{n}_{\mathbf{D}})$ denotes size($\mathbf{D}$), then Theorem 10.10 implies

$$\underline{m}_{\mathbf{A}\otimes\mathbf{B}} \leq \min(\underline{m}_{\mathbf{A}}, \underline{m}_{\mathbf{B}})$$
$$\underline{n}_{\mathbf{A}\otimes\mathbf{B}} \leq \min(\underline{n}_{\mathbf{A}}, \underline{n}_{\mathbf{B}})$$

so

$$\underline{m}_{\mathbf{A}\otimes\mathbf{B}}\underline{n}_{\mathbf{A}\otimes\mathbf{B}} \leq \min(\underline{m}_{\mathbf{A}}, \underline{m}_{\mathbf{B}})\min(\underline{n}_{\mathbf{A}}, \underline{n}_{\mathbf{B}})$$

Now note that in general $\min(a,b) \cdot \min(c,d) \leq \min(a \cdot c, b \cdot d)$ for non-negative integers a,b,c,d since

$$\min(a,b) \cdot c = \min(a \cdot c, b \cdot c) \qquad \min(a,b) \cdot d = \min(a \cdot d, b \cdot d)$$

Then

$$\begin{aligned} \min(a,b) \cdot \min(c,d) &= \min(\min(a \cdot c, b \cdot c), \min(a \cdot d, b \cdot d)) \\ &= \min(a \cdot c, b \cdot c, a \cdot d, b \cdot d) \\ &\leq \min(a \cdot c, b \cdot d) \end{aligned}$$

In particular,

$$\underline{\text{total}}(\mathbf{A} \otimes \mathbf{B}) \leq \min(\underline{\text{total}}(\mathbf{A}), \underline{\text{total}}(\mathbf{B}))$$

Answer (Exercise 10.20) (Proof of Theorem 10.13.)
For

$$\mathbf{C} = \mathbf{AB}$$

it is known that

$$\mathbf{C}(k_1, k_2) = \mathbf{A}(k_1, :)\mathbf{B}(:, k_2)$$

so if

$$\mathbf{A}(k_1, :) = \mathbb{0}$$

then

$$\mathbf{C}(k_1, k_2) = 0$$

for all k_2 and so

$$\mathbf{C}(k_1, :) = \mathbb{0}$$

and if

$$\mathbf{B}(:, k_2) = \mathbb{0}$$

then

$$\mathbf{C}(k_1, k_2) = 0$$

for all k_2 and so

$$\mathbf{C}(:,k_2) = \mathbb{0}$$

The above observations mean that any zero row of $\mathbf{A}$ is also a zero row of $\mathbf{C}$, and any zero column of $\mathbf{B}$ is also a zero column of $\mathbf{C}$. Hence, any nonzero row of $\mathbf{AB}$ must be a nonzero row of $\mathbf{A}$, and any nonzero column must be a nonzero column of $\mathbf{B}$. Thus

$$\text{size}(\mathbf{AB},1) \leq \text{size}(\mathbf{A},1)$$

and

$$\text{size}(\mathbf{AB},2) \leq \text{size}(\mathbf{B},2)$$

as desired.

Answer (Exercise 11.1)
$\mathbf{A}$ is symmetric because it is the adjacency array of an undirected graph.

Answer (Exercise 11.3)

(a) Zero-sum-free is required to prevent an edge weight being summed with its additive inverse, resulting in an elimination of the edge.

(b) No zero divisors are required to prevent a zero divisor edge weight from being multiplied by another edge weight that results in an elimination of the edge.

(c) 0 annihilator is required so that no edges are created in the adjacency array where none exist in the incidence array.

Answer (Exercise 11.4)(Proof of Corollary 11.5.)
Let $\bar{G}$ denote the reverse of G, and let $\bar{\mathbf{E}}_{\text{out}}$ and $\bar{\mathbf{E}}_{\text{in}}$ be out-vertex and in-vertex incidence arrays for $\bar{G}$, respectively. Recall that $\bar{G}$ is defined to have the same edge and vertex sets as G but changes the directions of the edges; in other words, if an edge k leaves a vertex a in G, then it enters a in $\bar{G}$ and vice versa. As such

$$\mathbf{E}_{\text{out}}(k,a) \neq 0 \quad \text{if and only if} \quad \bar{\mathbf{E}}_{\text{in}}(k,a) \neq 0$$

and likewise

$$\mathbf{E}_{\text{in}}(k,a) \neq 0 \quad \text{if and only if} \quad \bar{\mathbf{E}}_{\text{out}}(k,a) \neq 0$$

As such, choosing $\mathbf{E}_{\text{out}} = \bar{\mathbf{E}}_{\text{in}}$ and $\mathbf{E}_{\text{in}} = \bar{\mathbf{E}}_{\text{out}}$ gives valid in-vertex and out-vertex incidence matrices for $\bar{G}$, respectively. Then by Theorem 11.1 it can be shown that

$$\bar{\mathbf{E}}_{\text{out}}^{\mathsf{T}}\bar{\mathbf{E}}_{\text{in}} = \mathbf{E}_{\text{in}}^{\mathsf{T}}\mathbf{E}_{\text{out}}$$

Answer (Exercise 11.5)

(a) $\underline{m} = 21$, $\underline{n} = 3$, nnz = 28

(b) $\underline{m} = 22$, $\underline{n} = 5$, nnz = 47

(c) $\underline{m} = 3$, $\underline{n} = 5$, nnz = 11

(d) Size of $\mathbf{E}_1 \ {+}.{\times}\ \mathbf{E}_1^{\mathsf{T}}$ from Figure 11.2 is square and both the row and columns are equal to the number of columns in $\mathbf{E}_1$ from Figure 11.2 and the number of columns in $\mathbf{E}_1$ from Figure 11.1.

Answer (Exercise 11.10)(Proof of Lemma 11.7.)
The order isomorphism f is defined recursively. Let

$$K = \{k_1, \ldots, k_n\}$$

and let

$$f(1) = \min\{k_1, \ldots, k_n\}$$

Then, given $f(1), \ldots, f(i)$ have been defined for $i < n$, define

$$f(i+1) = \min(K \setminus \{f(1), \ldots, f(i)\})$$

Note that because $<$ is a strict total order on K, it follows that

$$f(i+1) \notin \{f(1), \ldots, f(i)\}$$

since the minimum of any subset of K must lie in that subset. This condition implies that f is a bijection, so it only remains to show that f is monotonic. By induction, if

$$k \in K \setminus \{f(1), \ldots, f(i)\}$$

then

$$f(1), \ldots, f(i) < k$$

For the base case, $f(1)$ is defined to be the smallest element of K, so $f(1) < k$ for any $k \neq f(1)$. If the statement is true for i and

$$k \in K \setminus \{f(1), \ldots, f(i), f(i+1)\}$$

then in particular

$$k \in K \setminus \{f(1), \ldots, f(i)\}$$

so

$$f(1), \ldots, f(i) < k$$

Since $f(i+1)$ is defined to be the smallest element of

$$K \setminus \{f(1), \ldots, f(i)\}$$

then $f(i+1) < k$ and

$$f(1), \ldots, f(i), f(i+1) < k$$

Thus, if

$$1 \leq i < j \leq n$$

then

$$f(j) \notin \{f(1), \ldots, f(i)\}$$

so

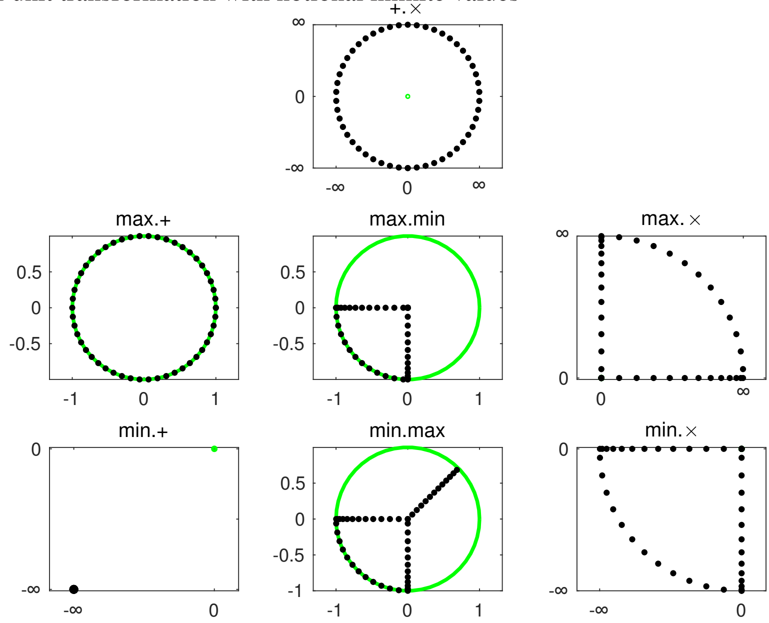
$$f(1), \ldots, f(i) < f(j)$$

In particular, $f(i) < f(j)$, so f is monotonic.

Answer (Exercise 11.11)(Proof of Lemma 11.8.)
Taking the transpose interchanges the roles of rows and columns.

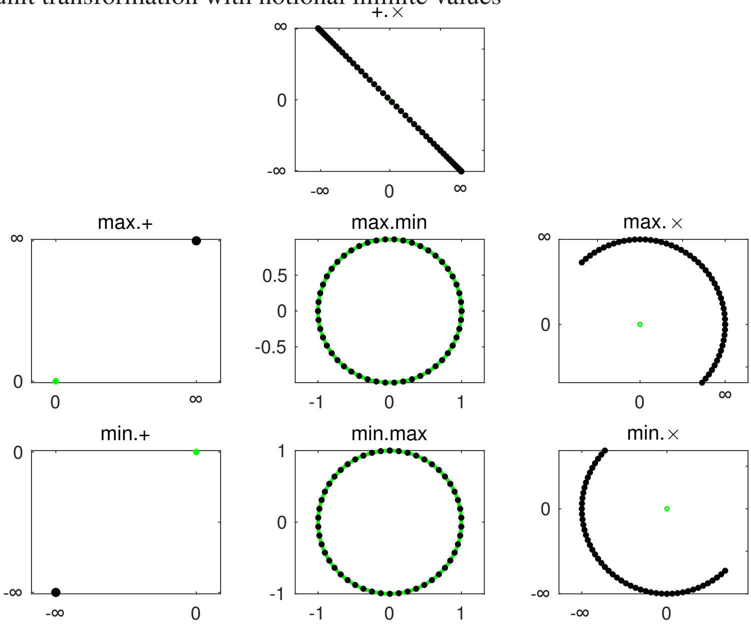
Answer (Exercise 12.1)

Sketch of unit transformation with notional infinite values



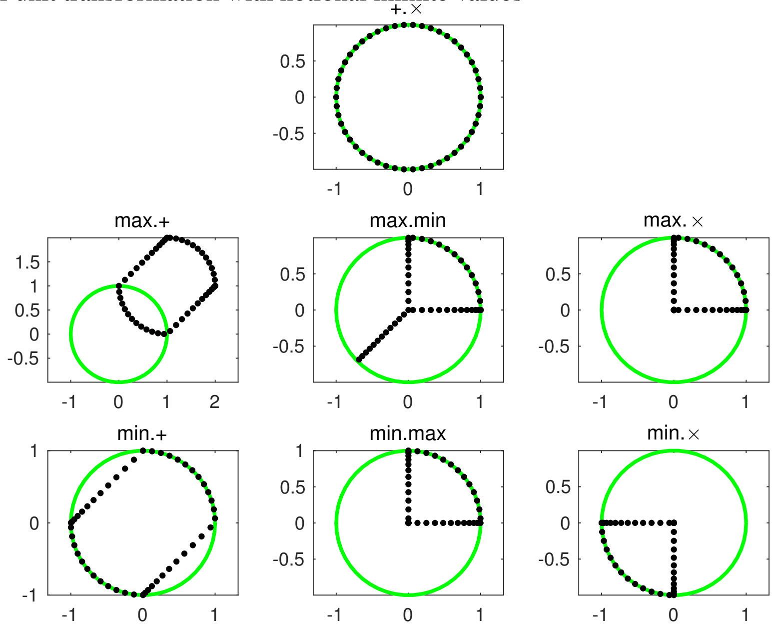
Answer (Exercise 12.3)

Sketch of unit transformation with notional infinite values



Answer (Exercise 12.5)

Sketch of unit transformation with notional infinite values



Answer (Exercise 13.1)
An $n \times m$ matrix with entries in V is the same thing as a function $\{1,\ldots,n\} \times \{1,\ldots,m\} \to V$, so take $K = \{1,\ldots,n\} \times \{1,\ldots,m\}$. Note that since K is finite, there is no issue related to having finite support. Thus, $V^{n\times m} = V^{\boxplus\{1,\ldots,n\}\times\{1,\ldots,m\}}$.

Answer (Exercise 13.3)
Requiring elements to have finite support means that the set $\{\mathbf{e}_k \mid k \in K\}$ actually forms a basis, meaning that $V^{\boxplus K}$ gives a natural semimodule over V with dimension $|K|$. Additionally, having finite support means that when $K = K_1 \times \cdots \times K_d$ for key sets $K_1, \ldots,$ K_d, the elements of $V^{\boxplus K}$ actually are d-dimensional associative arrays, which are objects of in-terest.

Answer (Exercise 13.4) (Proof of Proposition 13.1.)
For uniqueness, given linear functions

$$f', f'' : V^{\boxplus K} \to M$$

that satisfy

$$f'(\mathbf{e}_k) = f''(\mathbf{e}_k) = f(\mathbf{e}_k)$$

For $\mathbf{v} \in V^{\boxplus K}$, there are $k_1, k_2, \ldots, k_n \in K$ such that

$$\mathbf{v} = \mathbf{v}(k_1)\mathbf{e}_{k_1} \oplus \cdots \oplus \mathbf{v}(k_n)\mathbf{e}_{k_n}$$

Then the linearity of f' and f'' implies

$$\begin{aligned}
f'(\mathbf{v}) &= f'(\mathbf{v}(k_1)\mathbf{e}_{k_1} \oplus \cdots \oplus \mathbf{v}(k_n)\mathbf{e}_{k_n}) \\
&= \mathbf{v}(k_1)f'(\mathbf{e}_{k_1}) \oplus \cdots \oplus \mathbf{v}(k_n)f'(\mathbf{e}_{k_n}) \\
&= \mathbf{v}(k_1)f(\mathbf{e}_{k_1}) \oplus \cdots \oplus \mathbf{v}(k_n)f(\mathbf{e}_{k_n}) \\
&= \mathbf{v}(k_1)f''(\mathbf{e}_{k_1}) \oplus \cdots \oplus \mathbf{v}(k_n)f''(\mathbf{e}_{k_n}) \\
&= f''(\mathbf{v}(k_1)\mathbf{e}_{k_1} \oplus \cdots \oplus \mathbf{v}(k_n)\mathbf{e}_{k_1}) \\
&= f''(\mathbf{v})
\end{aligned}$$

The appropriate definition of f' to prove existence is motivated by the uniqueness proof above. Take $\mathbf{v}$ as above and then define

$$f'(\mathbf{v}) = \mathbf{v}(k_1)f(\mathbf{e}_{k_1}) \oplus \cdots \oplus \mathbf{v}(k_n)f(\mathbf{e}_{k_n})$$

This gives a well-defined function $f' : V^{\boxplus K} \to M$ by the fact that if

$$\mathbf{v}(k_1)\mathbf{e}_{k_1} \oplus \cdots \oplus \mathbf{v}(k_n)\mathbf{e}_{k_n} = \mathbf{w}(j_1)\mathbf{e}_{j_1} \oplus \cdots \oplus \mathbf{w}(j_m)\mathbf{e}_{j_m}$$

where each $\mathbf{v}(k_i)$ and $\mathbf{w}(j_\ell)$ are nonzero, then $n = m$, $\{k_1, \ldots, k_n\} = \{j_1, \ldots, j_m\}$, and if $k_i = j_\ell$, then $\mathbf{v}(k_i) = \mathbf{w}(j_\ell)$. Thus, applying f' on both sides gives the same element in M.
Now it just remains to show that f' is linear. Suppose $u \in V$ and $\mathbf{v}, \mathbf{v}' \in V^{\boxplus K}$. Then there are $k_1, \ldots, k_n$ and $j_1, \ldots, j_m$ such that

$$\mathbf{v} = \mathbf{v}(k_1)\mathbf{e}_{k_1} \oplus \cdots \oplus \mathbf{v}(k_n)\mathbf{e}_{k_n}$$

and

$$\mathbf{v}' = \mathbf{v}'(j_1)\mathbf{e}_{j_1} \oplus \cdots \oplus \mathbf{v}'(j_m)\mathbf{e}_{j_m}$$

Then by definition of f', it follows that

$$\begin{aligned} f'(u\mathbf{v}\oplus\mathbf{v}') &= f'(u(\mathbf{v}(k_1)\mathbf{e}_{k_1}\oplus\cdots\oplus\mathbf{v}(k_n)\mathbf{e}_{k_n})\oplus(\mathbf{v}'(j_1)\mathbf{e}_{j_1}\oplus\cdots\oplus\mathbf{v}'(j_m)\mathbf{e}_{j_m})) \\ &= f'(u\mathbf{v}(k_1)\mathbf{e}_{k_1}\oplus\cdots\oplus u\mathbf{v}(k_n)\mathbf{e}_{k_n}\oplus\mathbf{v}'(j_1)\mathbf{e}_{j_1}\oplus\cdots\oplus\mathbf{v}'(j_m)\mathbf{e}_{j_m}) \\ &= u\mathbf{v}(k_1)f(\mathbf{e}_{k_1})\oplus\cdots\oplus u\mathbf{v}(k_n)f(\mathbf{e}_{k_n})\oplus\mathbf{v}'(j_1)f(\mathbf{e}_{j_1})\oplus\cdots\oplus\mathbf{v}'(j_m)f(\mathbf{e}_{j_m}) \\ &= uf'(\mathbf{v})\oplus f'(\mathbf{v}') \end{aligned}$$

Answer (Exercise 13.5)

A general element of $\mathbb{R}^{2\times 1}$ is of the form $a\mathbf{e}_1 + b\mathbf{e}_2 = \begin{matrix} & 1 \\ 1 & \\ 2 & \end{matrix}\begin{bmatrix} a \\ b \end{bmatrix}$, and the function f' is defined by

$$f'(a\mathbf{e}_1 + b\mathbf{e}_2) = af(\mathbf{e}_1) + bf(\mathbf{e}_2) = a - 2b$$

Thus, the matrix representing this linear map is given by

$$\begin{matrix} & 1 \quad 2 \\ 1 & \begin{bmatrix} 1 & -2 \end{bmatrix} \end{matrix} \begin{matrix} & 1 \\ 1 & \\ 2 & \end{matrix}\begin{bmatrix} a \\ b \end{bmatrix} = a - 2b$$

Answer (Exercise 13.6) (Proof of Proposition 13.2.)
First suppose that M is a module.

(i) implies (ii) —If U is linearly dependent, then there are $\mathbf{v}_1,\ldots,\mathbf{v}_n \in U$ and $u_1,\ldots,u_n,w_1,\ldots,w_n \in V$ such that

$$\bigoplus_{k=1}^{n} u_k\mathbf{v}_k = \bigoplus_{k=1}^{n} w_k\mathbf{v}_k$$

and there is k such that $u_k \neq w_k$. Thus

$$\bigoplus_{k=1}^{n} (u_k - w_k)\mathbf{v}_k = \mathbf{0}$$

and at least one of the scalar coefficients $u_k - w_k$ is nonzero by hypothesis.

(ii) implies (i) —Suppose there are $\mathbf{v}_1,\ldots,\mathbf{v}_n \in U$ and $u_1,\ldots,u_n \in V$ not all zero such that

$$\bigoplus_{k=1}^{n} u_k\mathbf{v}_k = \mathbf{0}$$

Then letting $w_1 = \cdots = w_n = 0$ shows that

$$\bigoplus_{k=1}^{n} u_k \mathbf{v}_k = \bigoplus_{k=1}^{n} w_k \mathbf{v}_k$$

where $u_k \neq w_k$ for some k.

Now further suppose M is a vector space.

(ii) implies (iii) —Suppose there are $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $u_1, \ldots, u_n \in V$ not all zero such that

$$\bigoplus_{k=1}^{n} u_k \mathbf{v}_k = \mathbf{0}$$

Suppose without loss of generality that $u_1 \neq 0$. Then

$$\mathbf{v}_1 = \bigoplus_{k=2}^{n} (-u_1^{-1} u_k) \mathbf{v}_k$$

shows that

$$\mathbf{v} \in \mathrm{Span}(U \setminus \{\mathbf{v}\})$$

(iii) implies (ii) —If there is $\mathbf{v} \in U$ such that

$$\mathbf{v} \in \mathrm{Span}(U \setminus \{\mathbf{v}\})$$

then there are $\mathbf{v}_1, \ldots, \mathbf{v}_n \in U \setminus \{\mathbf{v}\}$ and $u_1, \ldots, u_n \in V$ such that

$$\mathbf{v} = \bigoplus_{k=1}^{n} u_k \mathbf{v}_k$$

Writing $-1 = u_0$ and $\mathbf{v} = \mathbf{v}_0$ it follows that

$$-\mathbf{v} + \bigoplus_{k=1}^{n} u_k \mathbf{v}_k = \bigoplus_{k=0}^{n} u_k \mathbf{v}_k = \mathbf{0}$$

Answer (Exercise 13.7)

First suppose U is linearly independent and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset U$ are such that $\bigoplus_{k=1}^{n} u_k \mathbf{v}_k = \mathbf{0}$ for some u_k. Writing $\mathbf{0}$ as $\bigoplus_{k=1}^{n} 0\mathbf{v}_k$ means

$$\bigoplus_{k=1}^{n} u_k \mathbf{v}_k = \bigoplus_{k=1}^{n} 0\mathbf{v}_k$$

so linear independence implies $u_k = 0$ for each k.

Conversely, suppose U satisfies the given statement and $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\} \subset U$ and $u_1,\ldots,u_n,w_1,\ldots,w_n \in V$ are given such that

$$\bigoplus_{k=1}^{n} u_k\mathbf{v}_k = \bigoplus_{k=1}^{n} w_k\mathbf{v}_k$$

Since V is a ring, it makes sense to add $(-w_k)\mathbf{v}_k$ on both sides of the above equation for each k, giving

$$\bigoplus_{k=1}^{n} (u_k - w_k)\mathbf{v}_k = \mathbf{0}$$

But by hypothesis, this implies $u_k - w_k = 0$ for each k, or $u_k = w_k$.

Answer (Exercise 13.9) (Proof of Proposition 13.3.)

Let B be a basis for M. Then consider the V-semimodule $V^{\boxplus B}$. There is a map f sending $\mathbf{e}_\mathbf{v}$ to $\mathbf{v}$, and so by Proposition 13.1 it follows that there is a linear map

$$f' : V^{\boxplus B} \to M$$

It only remains to show that f' is a bijection. By definition of f, if

$$\mathbf{u} = u_1\mathbf{v}_1 \oplus \cdots \oplus u_n\mathbf{v}_n$$

is in M, then

$$f'(u_1\mathbf{e}_{\mathbf{v}_1} \oplus \cdots \oplus u_n\mathbf{e}_{\mathbf{v}_n}) = \mathbf{u}$$

This shows that f' is surjective. To show that it is injective, suppose that $f'(\mathbf{u}) = f'(\mathbf{w})$, where

$$\mathbf{u} = u_1\mathbf{e}_{\mathbf{v}_1} \oplus \cdots \oplus i_n\mathbf{e}_{\mathbf{v}_n}$$

and

$$\mathbf{w} = w_1\mathbf{e}_{\mathbf{v}'_1} \oplus \cdots \oplus w_m\mathbf{e}_{\mathbf{v}'_m}$$

for

$$\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}'_1, \ldots, \mathbf{v}'_m \in B$$

Then

$$u_1\mathbf{v}_1 \oplus \cdots \oplus u_n\mathbf{v}_n = f'(\mathbf{u}) = f'(\mathbf{w}) = w_1\mathbf{v}'_1 \oplus \cdots \oplus w_m\mathbf{v}'_m$$

which implies that $n = m$ and

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_m\}$$

and if $\mathbf{v}_i = \mathbf{v}'_j$, then $u_i = w_j$. In other words, $\mathbf{u} = \mathbf{w}$.
Conversely, suppose that there is a linear isomorphism

$$f' : V^{\boxplus K} \to M$$

Consider the image B of the set $\{\mathbf{e}_k \mid k \in K\}$. Denote the image of $\mathbf{e}_k$ under f' by $\mathbf{v}_k$. Then $\mathrm{Span}(B) = M$ as the image of f' is equal to M by the hypothesis that f' is an isomorphism, so for each $\mathbf{w}$ there is

$$\mathbf{u} = u_1\mathbf{e}_{k_1} \oplus \cdots \oplus u_n\mathbf{e}_{k_n}$$

such that

$$\mathbf{w} = f'(\mathbf{u}) = u_1\mathbf{v}_{k_1} \oplus \cdots \oplus u_n\mathbf{v}_{k_n}$$

B is linearly independent, as given

$$u_1\mathbf{v}_{k_1} \oplus \cdots \oplus u_n\mathbf{v}_{k_n} = w_1\mathbf{v}_{k_1} \oplus \cdots \oplus w_n\mathbf{v}_{k_n}.$$

Then by applying ${f'}^{-1}$:

$$u_1\mathbf{e}_{k_1} \oplus \cdots \oplus u_n\mathbf{e}_{k_n} = w_1\mathbf{e}_{k_1} \oplus \cdots \oplus w_n\mathbf{e}_{k_n}$$

so by the linear independence of $\{\mathbf{e}_k \mid k \in K\}$, it follows that $u_k = w_k$ for each $1 \leq k \leq n$. Thus, B is a basis.

Answer (Exercise 13.10) (Proof of Corollary 13.4.)
Combine Proposition 13.3 and Proposition 13.1.

Answer (Exercise 13.11)
For $\mathbb{Z}/n\mathbb{Z}$, the answer is no because $\mathbb{Z}$ is infinite, and hence every $\mathbb{Z}^{\boxplus K}$ is also infinite (or trivial), whereas $\mathbb{Z}/n\mathbb{Z}$ is neither infinite or trivial, except when $n = 1$.
Proposition 13.5 says that every vector space has a basis. Thus, $\mathbb{R}$ is a free semimodule over $\mathbb{Q}$.

Answer (Exercise 13.12) (Proof of Corollary 13.7.)
Suppose that there is a finite basis of M, and suppose for the sake of a contradiction that there is an infinite basis of M. But by Theorem 13.6, because the size of the finite basis is less than that of the infinite basis, it cannot be that the finite basis is actually a basis at all, contradicting the assumption that there is a finite basis.

Answer (Exercise 14.1)
The resulting system of equations is

$$(a \cap \{1\}) \cup (b \cap \{0\}) = \emptyset$$
$$(a \cap \{0\}) \cup (b \cap \{1\}) = \emptyset$$

and so $a \cap \{1\} = b \cap \{0\} = \emptyset = a \cap \{0\} = b \cap \{1\}$, so $a = b = \emptyset$.

Answer (Exercise 14.3)
Begin by noting that f is injective if and only if it has trivial null space: $f(\mathbf{v}) = f(\mathbf{w})$ if and only if $f(\mathbf{v} - \mathbf{w}) = \mathbf{0}$. Thus, if the null space is trivial, then $\mathbf{v} = \mathbf{w}$, and if $\mathbf{v} \neq \mathbf{w}$, then the null space is nontrivial.
Denote by $[\mathbf{v}]$ the equivalence class containing $\mathbf{v}$.
The map f' is linear, and if $f'([\mathbf{v}]) = f'([\mathbf{w}])$, then by definition $f(\mathbf{v}) = f(\mathbf{w})$ and thus $\mathbf{v} \sim \mathbf{w}$. But this shows that $[\mathbf{v}] = [\mathbf{w}]$, so f is injective. Thus, it is an isomorphism onto its image.

Answer (Exercise 14.4) (Proof of Lemma 14.4.)
Let $U = \{u, v\}$. Then the condition of preserving suprema gives

$$f(u \vee v) = f(u) \vee f(v)$$

Since $u \leq v$ if and only if $u \vee v = v$, it follows that if $u \leq v$, then

$$f(v) = f(u \vee v) = f(u) \vee f(v)$$

so that $f(u) \le f(v)$. An analogous proof follows for the preservation of infima by considering again $U = \{u, v\}$ so that

$$f(u \wedge v) = f(u) \wedge f(v)$$

since $u \le v$ if and only if $u \wedge v = u$, it follows that if $u \le v$, then

$$f(u) = f(u \wedge v) = f(u) \wedge f(v)$$

so that $f(u) \le f(v)$.

Answer (Exercise 14.5) (Proof of Proposition 14.5.)
First it is shown that the product order on V^n is a partial order.

Reflexive — For each i, $\mathbf{v}(i) \le \mathbf{v}(i)$, so $\mathbf{v} \le \mathbf{v}$.

Antisymmetric — Suppose $\mathbf{v} \le \mathbf{w}$ and $\mathbf{w} \le \mathbf{v}$. Then $\mathbf{v}(i) \le \mathbf{w}(i)$ and $\mathbf{w}(i) \le \mathbf{v}(i)$ for each i, so $\mathbf{v}(i) = \mathbf{w}(i)$. Hence, $\mathbf{v} = \mathbf{w}$.

Transitive — Suppose $\mathbf{u} \le \mathbf{v}$ and $\mathbf{v} \le \mathbf{w}$. Then $\mathbf{u}(i) \le \mathbf{v}(i)$ and $\mathbf{v}(i) \le \mathbf{w}(i)$ for each i, so $\mathbf{u}(i) \le \mathbf{w}(i)$ for each i. Hence, $\mathbf{u} \le \mathbf{w}$.

Suppose $\bigvee_{\mathbf{v}\in U} \mathbf{v}(i)$ exists for each i for some subset $U \subset V^n$. $\mathbf{u}(i) \le \bigvee_{\mathbf{v}\in U} \mathbf{v}(i)$ for any $\mathbf{u} \in U$, so

$$\begin{array}{c} \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} 1 \\ \left[\begin{array}{c} \bigvee_{\mathbf{v}\in U} \mathbf{v}(1) \\ \vdots \\ \bigvee_{\mathbf{v}\in U} \mathbf{v}(n) \end{array}\right] \end{array}$$

is an upper bound of U. Now assume that

$$\mathbf{v} \le \mathbf{w}$$

for all $\mathbf{v} \in U$. By definition, $\mathbf{v}(i) \le \mathbf{w}(i)$ for each i, so $\bigvee_{\mathbf{v}\in U} \mathbf{v}(i) \le \mathbf{w}(i)$ for each i. Hence

$$\begin{array}{c} \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} 1 \\ \left[\begin{array}{c} \bigvee_{\mathbf{v}\in U} \mathbf{v}(1) \\ \vdots \\ \bigvee_{\mathbf{v}\in U} \mathbf{v}(n) \end{array}\right] \end{array} \le \mathbf{w}$$

The above expression shows

$$\begin{array}{c} \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} 1 \\ \begin{bmatrix} \bigvee_{\mathbf{v}\in U} \mathbf{v}(1) \\ \vdots \\ \bigvee_{\mathbf{v}\in U} \mathbf{v}(n) \end{bmatrix} \end{array} = \bigvee_{\mathbf{v}\in U} \mathbf{v}$$

is the *least* upper bound of U. Likewise, if $\bigwedge_{\mathbf{v}\in U} \mathbf{v}(i)$ exists for each i, then

$$\begin{array}{c} \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} 1 \\ \begin{bmatrix} \bigwedge_{\mathbf{v}\in U} \mathbf{v}(1) \\ \vdots \\ \bigwedge_{\mathbf{v}\in U} \mathbf{v}(n) \end{bmatrix} \end{array} = \bigwedge_{\mathbf{v}\in U} \mathbf{v}$$

It follows that if V is a lattice, then so is V^n. If V is bounded, meaning that $\bigvee \emptyset \in V$ and $\bigvee V \in V$ exist, then $\bigvee \emptyset \in V^n$ and $\bigvee V^n \in V^n$ exist, so V^n is also bounded. If V is distributive, then since suprema and infima in V^n are calculated entry-wise, it follows that suprema and infima in V^n distribute over one another. Finally, $U \subset V^n$ is non-empty if and only if $\pi_i[U]$ is non-empty for each i, where π_i is projection onto the i-th coordinate. Hence, if V is closed under suprema of non-empty sets, then so is V^n.

Answer (Exercise 14.7) (Proof of Lemma 14.9.)

Let

$$U = \bigcap_{j=1}^{n} \bigcup_{i=1}^{m} U_{i,j}$$

and

$$U' = \bigcup_{1 \le i_1, \ldots, i_j \ldots, i_n \le m} \bigcap_{j=1}^{n} U_{i_j,j}$$

$U = U'$ is shown by proving that $U \subset U'$ and $U \supset U'$. First, it is shown that $U \subset U'$. Suppose that $u \in U$, so that by definition u lies in all of the unions

$$\bigcup_{i=1}^{m} U_{i,j}$$

such that, for every $j \in \{1, \ldots, n\}$, there exists $I_j \in \{1, \ldots, n\}$ such that $u \in U_{I_j,j}$. Thus

$$u \in \bigcap_{j=1}^{n} U_{I_j,j}$$

so that $u \in U'$. Now, it is shown that $U \supset V$. Suppose that $u \in U'$, so that by definition there exists a tuple $I_1, \ldots, I_n$ such that

$$\bigcap_{j=1}^{n} U_{I_j,j}$$

contains u. In particular, u is in every one of the unions

$$\bigcup_{i=1}^{m} U_{i,j}$$

for $j \in \{1, \ldots, n\}$ and thus $u \in U$.

Answer (Exercise 14.8) (Proof of Lemma 14.10.)
Suppose $u \in [v_1, w_1] \cap [v_2, w_2]$, so that $v_1 \le u \le w_1$ and $v_2 \le u \le w_2$. Then $v_1 \vee v_2 \le u \le w_1 \wedge w_2$. Thus,

$$[v_1, w_1] \cap [v_2, w_2] \subset [v_1 \vee v_2, w_1 \wedge w_2]$$

Conversely, suppose $v_1 \vee v_2 \le u \le w_1 \wedge w_2$. Then $v_1, v_2 \le v_1 \vee v_2 \le u \le w_1 \wedge w_2 \le w_1, w_2$ implies $u \in [v_1, w_1] \cap [v_2, w_2]$. Thus,

$$[v_1, w_1] \cap [v_2, w_2] \supset [v_1 \vee v_2, w_1 \wedge w_2]$$

Answer (Exercise 14.9) (Proof of Theorem 14.11.)

$\mathbf{v}$ satisfies $\mathbf{Av} = \mathbf{w}$ if and only if $\mathbf{A}(i,:)\mathbf{v} = \mathbf{w}$ for each i. Hence, by Lemma 14.9 and Lemma 14.10

$$\begin{aligned}
X(\mathbf{A},\mathbf{w}) &= \bigcap_{i=1}^{m} X(\mathbf{A}(i,:),\mathbf{w}(i)) \\
&= \bigcap_{i=1}^{m} \bigcup_{\mathbf{v}\in U_i} [\mathbf{v},\mathbf{x}_i] \\
&= \bigcup_{\mathbf{v}_1\in U_1,\ldots,\mathbf{v}_n\in U_n} \bigcap_{i=1}^{m} [\mathbf{v}_i,\mathbf{x}_i] \\
&= \bigcup_{\mathbf{v}_1\in U_1,\ldots,\mathbf{v}_n\in U_n} \left[\bigvee_{i=1}^{m} \mathbf{v}_i, \bigwedge_{i=1}^{m} \mathbf{x}_i\right]
\end{aligned}$$

Answer (Exercise 14.11) (Proof of Proposition 14.17.)
Suppose $v \leq u$. Then because V is totally-ordered, $\wedge$ is minimum, and it follows that the greatest element w such that

$$v \wedge w \leq u$$

is ∞ because

$$v \wedge \infty = v \leq u$$

and ∞ is the greatest element. On the other hand, if $v > u$, then u is the greatest element w such that

$$v \wedge w \leq u$$

as in fact

$$v \wedge w = v \wedge u = u$$

so that by monotonicity if

$$v \wedge w \leq u$$

then

$$w \leq u$$

Answer (Exercise 14.12) (Proof of Proposition 14.18.)
To show that the above definition makes V into a Heyting algebra, it is first checked that

$$w = \neg v \vee u$$

satisfies

$$v \wedge w \leq u$$

and then that it is the greatest element w satisfying $v \wedge w \leq u$. For the first,

$$v \wedge (\neg v \vee u) = (v \wedge \neg v) \vee (v \wedge u) = 0 \vee (v \wedge u) = v \wedge u \leq u.$$

For the second, suppose w is such that $v \wedge w \leq u$. Then there are the following equivalences:

$$\begin{aligned} v \wedge w \leq u \quad & \text{if and only if} \quad \neg v \vee (v \wedge w) \leq \neg v \vee u \\ & \text{if and only if} \quad (\neg v \vee v) \wedge (\neg v \wedge w) \leq \neg v \vee u \\ & \text{if and only if} \quad w \leq \neg v \wedge w \leq \neg v \vee u \end{aligned}$$

Thus

$$v \Rightarrow u = \neg v \vee u$$

satisfies the required conditions to make V into a Heyting algebra.

Answer (Exercise 14.13) (Proof of Proposition 14.19.)
The supremum in the power set algebra is $\cup$, and negation is the complement in V. Thus

$$\neg U \vee U'$$

becomes

$$U^{\mathsf{c}} \cup U'$$

Then the result follows from Proposition 14.18.

Answer (Exercise 15.1)
In the max-plus algebra, the system of equations becomes

$$\begin{aligned} \max(3+x, y-1) &= \lambda + x \\ \max(2+x, y) &= \lambda + y \end{aligned}$$

Then there are several cases

$2+x \geq y$ — Note that $3+x \geq y-1$ as well. The equations are then $3+x = \lambda + x$ and $2+x = \lambda + y$. The first equation gives either $\lambda = 3$ or $x = \pm\infty$.

In the former case, $2+x = 3+y$, so $x = y+1$, giving eigenvectors

$$\begin{array}{c c} & 1 \\ \begin{array}{c} 1 \\ 2 \end{array} & \left[\begin{array}{c} y+1 \\ y \end{array} \right] \end{array} \quad \text{with } \lambda = 3$$

(note that this does satisfy the condition $2+x \geq y$.)

In the latter case, since $2+x = \lambda + y$, $\lambda + y = \pm\infty$, with the sign the same as that of x. Thus, either $\lambda = \pm\infty$ or $y = \pm\infty$, giving eigenvectors

$$\begin{array}{c c} & 1 \\ \begin{array}{c} 1 \\ 2 \end{array} & \left[\begin{array}{c} \pm\infty \\ \pm\infty \end{array} \right] \end{array} \quad \text{with } \lambda \text{ arbitrary} \quad \text{and} \quad \begin{array}{c c} & 1 \\ 1 & \left[\begin{array}{cc} \pm\infty & y \end{array} \right] \end{array} \quad \text{with } \lambda = \pm\infty$$

$2+x < y, 3+x \geq y-1$ — Note that $2+x < y \leq 4+x$ as well. The equations are then $3+x = \lambda + x$ and $y = \lambda + y$. Then either $\lambda = 3$ or $x = \pm\infty$. However, neither case can occur with the assumption that $2 + x < y \leq 4 + x$: If the former, then $y = \pm\infty$ (to satisfy $y = 3 + y$); to satisfy $2 + x < y$, it must be that $y = \infty$, but then $y \leq 4 + x$ implies $x = \infty$, contradicting $2 + x < y$. The latter case cannot occur, as $2 + x < y$ requires that $x = -\infty$, while then $3 + x = -\infty \geq y - 1$ implies $y = -\infty$, contradicting $2 + x < y$. Thus, there is no eigenvector/eigenvalue pair satisfying this condition.

$2 + x < y, 3 + x < y - 1$ — Note that $3 + x < y - 1$, or $4 + x < y$, as well. The equations are then $y = \lambda + y$ and $y - 1 = \lambda + x$. To satisfy $y = \lambda + y$, either $\lambda = 0$ or $y = \pm\infty$. In the former case, $y = x + 1$, but this contradicts the assumption that $4 + x < y$. In the latter case, $\lambda + x = \pm\infty$, so either $\lambda = \pm\infty$ or $x = \pm\infty$. It cannot be that $x = \pm\infty$ since then $4 + x < y$ does not hold, so $\lambda = \pm\infty$. Likewise, if $y = -\infty$, then $4 + x < y$ cannot be satisfied, so $y = \lambda = \infty$ and $x < \infty$, giving the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{array}{c}1\\2\end{array} & \begin{bmatrix} x \\ \infty \end{bmatrix} \end{array} \quad \text{with } \lambda = \infty \text{ and where } x < \infty$$

In the max-min algebra, the system of equations becomes

$$\max(\min(3, x), \min(-1, y)) = \min(\lambda, x)$$
$$\max(\min(2, x), \min(y, 0)) = \min(\lambda, y)$$

As before, there are several cases, though many more than the max-plus case:

$2 \geq x$, $-1 \geq y$, $y \leq x$ — The equations are then $x = \min(\lambda, x) = \min(\lambda, y)$. Then $\lambda \geq x$ and $x = y$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{array}{c}1\\2\end{array} & \begin{bmatrix} x \\ x \end{bmatrix} \end{array} \quad \text{with } \lambda \geq x \text{ and where } x \leq 2$$

$2 \geq x, -1 \geq y, y > x$ — The equations are then $y = \min(\lambda, x) = \min(\lambda, y)$. Then $\lambda \geq y$ and $x = y$, giving a contradiction.

$2 < x \leq 3, -1 < y \leq 0$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(2, y) = 2$. Then $\lambda \geq x$ and $y = 2$. But this contradicts the fact that $-1 < y \leq 0$.

$3 < x, 0 < y$ — The equations are then $3 = \min(\lambda, x)$ and $2 = \min(\lambda, y)$. That $3 < x$ implies $\lambda = 3$ and so $y = 2$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{array}{c}1\\2\end{array} & \begin{bmatrix} x \\ 2 \end{bmatrix} \end{array} \quad \text{with } \lambda = 3 \text{ and where } x > 3$$

$x \le 2, -1 < y \le 0, y \le x$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(x, y) = x$. Thus, $\lambda \ge x$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{matrix}1\\2\end{matrix} & \begin{bmatrix} x \\ y \end{bmatrix} \end{array} \quad \text{with } \lambda \ge x \text{ and where } x \le 2, -1 < y \le 0, y \le x$$

$-1 \le x \le 2, -1 < y \le 0, x < y$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(x, y) = y$. Then $\lambda \ge y$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{matrix}1\\2\end{matrix} & \begin{bmatrix} x \\ y \end{bmatrix} \end{array} \quad \text{with } \lambda \ge y \text{ and where } -1 \le x \le 2, -1 < y \le 0, x < y$$

$x < -1, -1 < y \le 0, x < y$ — The equations are then $\min(\lambda, x) = \max(x, -1) = -1$ and $\min(\lambda, y) = \max(x, y) = y$. Then $\lambda \ge y$ and $x = -1$, a contradiction.

$3 < x, -1 < y \le 0$ — The equations are then $\min(\lambda, x) = \max(3, -1) = 3$ and $\min(\lambda, y) = \max(2, y) = 2$. Then $\lambda = 3$, so $\min(\lambda, y) = 2$ implies $y = 2$, contradicting $-1 < y \le 0$.

$3 < x, -1 \ge y$ — The equations are then $\min(\lambda, x) = \max(3, y) = 3$ and $\min(\lambda, y) = \max(2, y) = 2$. Thus, $\lambda = 3$ and $y = 2$, giving a contradiction.

$x \le 2, 0 < y, y \le x$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(x, 0) = x$. Thus, $\lambda \ge x$ and $x = y$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{matrix}1\\2\end{matrix} & \begin{bmatrix} x \\ x \end{bmatrix} \end{array} \quad \text{with } \lambda \ge x \text{ and where } 0 < x \le 2$$

$0 \le x \le 2, 0 < y, x < y$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(x, 0) = x$. Thus, $\lambda = x$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{matrix}1\\2\end{matrix} & \begin{bmatrix} x \\ y \end{bmatrix} \end{array} \quad \text{with } \lambda = x \text{ and where } 0 \le x \le 2, 0 < y, x < y$$

$-1 \le x < 0, 0 < y$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(x, 0) = 0$. Thus, $\lambda = 0$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{cc} & 1 \\ \begin{matrix}1\\2\end{matrix} & \begin{bmatrix} x \\ y \end{bmatrix} \end{array} \quad \text{with } \lambda = 0 \text{ and where } -1 \le x < 0, 0 < y$$

$x < -1, 0 < y$ — The equations are then $\min(\lambda, x) = \max(x, -1) = -1$ and $\min(\lambda, y) = \max(x, 0) = 0$. But then $\lambda = 0 = -1$, giving a contradiction.

$2 < x \leq 3, 0 < y, y \leq x$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(2, 0) = 2$. Then $\lambda \geq x$ and $y = 2$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{c c} & 1 \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} x \\ 2 \end{bmatrix} \end{array} \quad \text{with } \lambda \geq x \text{ and where } 2 < x \leq 3$$

$2 < x \leq 3, 0 < y, x < y$ — The equations are then $\min(\lambda, x) = \max(x, -1) = x$ and $\min(\lambda, y) = \max(2, 0) = 2$. Then $\lambda = 2$. This gives the eigenvector/eigenvalue pair

$$\begin{array}{c c} & 1 \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} x \\ y \end{bmatrix} \end{array} \quad \text{with } \lambda = 2 \text{ and where } 2 < x \leq 3, 0 < y, x < y$$

$2 < x \leq 3, y \leq -1$ — The equations are then $\min(\lambda, x) = \max(x, y) = x$ and $\min(\lambda, y) = \max(2, y) = 2$. But then $\lambda \geq x > 2$ so $y = 2$, contradicting the fact that $y \leq -1$.

Answer (Exercise 15.3)

In Example 15.5, an array in the incomplete max-plus algebra is given, namely

$$\mathbf{C} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

Its quasi-inverse is calculated to be

$$\mathbf{C}^* = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix} \end{array}$$

which is not in the incomplete max-plus algebra. This example exhibits that the existence of a quasi-inverse can fail if those order-theoretic completeness properties are not assumed.

However, the same example shows that the array

$$\mathbf{D} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

does have a quasi-inverse in the incomplete max-plus algebra given by

$$\mathbf{D}^* = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Thus, the order-theoretic completeness properties are not necessary, but are sufficient, for the existence of a quasi-inverse.

Answer (Exercise 15.5) (Proof of Lemma 15.3.)
Because $\lambda^2 = \lambda$ by idempotence, it follows that

$$\mathbf{A}(\lambda\mathbf{v}) = \lambda(\mathbf{A}\mathbf{v}) = \lambda\mathbf{v} = \lambda^2\mathbf{v} = \lambda(\lambda\mathbf{v})$$

Answer (Exercise 15.6) (Proof of Corollary 15.4.)
Because

$$\mu \bigoplus_{c \in P_{i,i}} \mathrm{W}(c) \oplus \mu = \mu^2 \oplus \mu = \mu$$

condition (i) of Theorem 15.2 holds and so

$$\mu\mathbf{A}^*(:,i)$$

is an eigenvector of $\mathbf{A}$ with eigenvalue 1. Then by Lemma 15.3, it follows that

$$\lambda\mu\mathbf{A}^*(:,i)$$

is an eigenvector of $\mathbf{A}$ with eigenvalue λ.

Answer (Exercise 15.7)

$$\tilde{\mathbf{A}}(\lambda) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 3 & \text{-}1 & \lambda & \text{-}\infty \\ 2 & 0 & \text{-}\infty & \lambda \\ 0 & \text{-}\infty & 0 & \text{-}\infty \\ \text{-}\infty & 0 & \text{-}\infty & 0 \end{array}\right] \end{array}$$

Then calculating $\det^+(\tilde{A})$ (going through every permutation and moving to the next one as soon as you encounter a -∞) and $\det^-(\tilde{A})$ gives

$$\det{}^+(\tilde{A}) = \max(3, 2\lambda) \quad \det{}^-(\tilde{A}) = \max(1, \lambda + 3)$$

Thus, the characteristic bipolynomial is given by

$$(P^+(\lambda), P^-(\lambda)) = (\max(3, 2\lambda), \max(1, \lambda + 3))$$

Eigenvalues are then solutions to $\max(3, 2\lambda) = \max(1, \lambda + 3)$. First suppose $-2 \leq \lambda < 3/2$, so that this equation becomes $3 = \lambda + 3$, so $\lambda = 0$. Next suppose $\lambda < -2$, so this equation becomes $3 = 1$, which is not true. Finally, suppose $\lambda \geq 3/2$, so the equation becomes $2\lambda = \lambda + 3$, or $\lambda = 3$. This shows that $\lambda = 0, 3$ are the eigenvalues of $\mathbf{A}$ (in the incomplete max-plus algebra).

Answer (Exercise 15.9) (Proof of Lemma 15.11.)
Because $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 1, it follows that

$$\mathbf{A}^k \mathbf{v} = \mathbf{v}$$

for every $k \geq 0$. Thus, by infinite distributivity of $\otimes$ over $\oplus$, this gives

$$\mathbf{A}^* \mathbf{v} = \mathbf{v}$$

Answer (Exercise 15.10) (Proof of Lemma 15.12.)
By Lemma 15.11, it follows that

$$\mathbf{v} \in \mathcal{V}(1)$$

satisfies

$$\mathbf{A}^* \mathbf{v} = \mathbf{v}$$

As such

$$\mathbf{v} = \bigoplus_{i=1}^{n} \mathbf{v}(i) \mathbf{A}^*(:, i)$$

meaning that $\mathbf{v}$ is a linear combination of the columns of $\mathbf{A}^*$.

Answer (Exercise 15.11) (Proof of Lemma 15.16.)
Because $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, it follows that

$$\mathbf{A}^k \mathbf{v} = \lambda^k \mathbf{v}$$

for every $k \geq 0$. Thus, by infinite distributivity of $\otimes$ over $\oplus$, this gives

$$\mathbf{A}^*\mathbf{v} = \mathbf{v} \bigoplus_{k=0}^{\infty} \lambda^k$$

Since 1 is the greatest element of V and $\oplus$ is binary supremum, it follows that

$$\bigoplus_{k=0}^{\infty} \lambda^k = 1 \oplus \bigoplus_{k=1}^{\infty} \lambda^k = 1$$

so

$$\mathbf{A}^*\mathbf{v} = \mathbf{v}$$

Then the proof that

$$\text{span}\{\lambda\mu_i \mathbf{A}^*(:,i) \mid 1 \leq i \leq n\} \subset \mathcal{V}(\lambda) \subset \text{span}\{\mathbf{A}^*(:,i) \mid 1 \leq i \leq n\}$$

follows exactly as in Corollary 15.12 and by Corollary 15.4.

Answer (Exercise 16.1)

In the two-dimensional case, matrices could be considered as linear maps, though this assumption does hold work for $d > 2$ since array multiplication isn't even defined.

Answer (Exercise 16.2)

For determining structure like (weak) diagonality or (weak) upper/lower triangularity, the specific values that the array takes don't matter, so there's no particular reason to record anything more from this perspective.

Adding or multiplying these terms could potentially lead to their becoming zero, and since the presence of those terms at least needs to be recorded, these operations do not necessarily give a useful construction, at least in terms of structure.

Answer (Exercise 16.3)
In the first case, where $\mathbb{R} \cup \{-\infty, \infty\}$ has the max-plus algebra, the two-dimensional projections are given by

$$\mathbf{A}_{1,2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -\infty & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \mathbf{A}_{2,3} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\infty & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}_{1,3} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

In the second case, where $\mathbb{R} \cup \{-\infty, \infty\}$ has the max-min tropical algebra, the two-dimensional projections are the same as in the max-plus case, but with the 0's replaced with ∞'s. This is because the additive identity is the same in each case, $-\infty$, so the only thing that changes is what is written for the multiplicative identity.

Answer (Exercise 16.5)
The key sets become

$$\mathcal{K}_1 = \{\{1\}, \{2, 3\}\}$$
$$\mathcal{K}_2 = \{\{1, 2\}, \{3\}\}$$
$$\mathcal{K}_3 = \{\{1, 2\}, \{3\}\}$$

Then the associated block structure map is given by

$$\mathbf{A}' : \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3 \to \mathbb{A}$$

with slices

$$\mathbf{A}'(:,:,\{1,2\}) = \begin{array}{c c} & \begin{array}{cc} \{1\} & \{2,3\} \end{array} \\ \begin{array}{c} \{1,2\} \\ \{3\} \end{array} & \begin{bmatrix} \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{1\}\times\{1,2\}\times\{1,2\}} & \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{2,3\}\times\{1,2\}\times\{1,2\}} \\ \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{1\}\times\{3\}\times\{1,2\}} & \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{2,3\}\times\{3\}\times\{1,2\}} \end{bmatrix} \end{array}$$

$$\mathbf{A}'(:,:,\{3\}) = \begin{array}{c c} & \begin{array}{cc} \{1\} & \{2,3\} \end{array} \\ \begin{array}{c} \{1,2\} \\ \{3\} \end{array} & \begin{bmatrix} \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{1\}\times\{1,2\}\times\{3\}} & \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{2,3\}\times\{1,2\}\times\{3\}} \\ \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{1\}\times\{3\}\times\{3\}} & \mathrm{pad}_{\{1,2,3\}^3}\mathbf{A}|_{\{2,3\}\times\{3\}\times\{3\}} \end{bmatrix} \end{array}$$

where

$$(\mathbf{A}|_{\{1\}\times\{1,2\}\times\{1,2\}})(:,:,1) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ 1 & \begin{bmatrix} -\infty & 1 \end{bmatrix} \end{array} \quad \text{and} \quad (\mathbf{A}|_{\{1\}\times\{1,2\}\times\{1,2\}})(:,:,2) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ 1 & \begin{bmatrix} -\infty & 0 \end{bmatrix} \end{array}$$

$$(\mathbf{A}|_{\{2,3\}\times\{1,2\}\times\{1,2\}})(:,:,1) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} \infty & -\infty \\ 1 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad (\mathbf{A}|_{\{2,3\}\times\{1,2\}\times\{1,2\}})(:,:,2) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} -\infty & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$(\mathbf{A}|_{\{1\}\times\{3\}\times\{1,2\}})(:,:,1) = \begin{array}{cc} & 3 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \text{and} \quad (\mathbf{A}|_{\{1\}\times\{3\}\times\{1,2\}})(:,:,2) = \begin{array}{cc} & 3 \\ 1 & \begin{bmatrix} \infty \end{bmatrix} \end{array}$$

$$(\mathbf{A}|_{\{2,3\}\times\{3\}\times\{1,2\}})(:,:,1) = \begin{array}{cc} & 3 \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} -1 \\ 3 \end{bmatrix} \end{array} \quad \text{and} \quad (\mathbf{A}|_{\{2,3\}\times\{3\}\times\{1,2\}})(:,:,2) = \begin{array}{cc} & 3 \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

$$(\mathbf{A}|_{\{1\}\times\{1,2\}\times\{3\}})(:,:,3) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ 1 & \begin{bmatrix} -\infty & -\infty \end{bmatrix} \end{array}$$

$$(\mathbf{A}|_{\{2,3\}\times\{1,2\}\times\{3\}})(:,:,3) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 3 \\ -1 & \infty \end{bmatrix} \end{array}$$

$$(\mathbf{A}|_{\{1\}\times\{3\}\times\{3\}})(:,:,3) = \begin{array}{cc} & 3 \\ 1 & \begin{bmatrix} -\infty \end{bmatrix} \end{array}$$

$$(\mathbf{A}|_{\{2,3\}\times\{3\}\times\{3\}})(:,:,3) = \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} 3 \\ \left[\begin{array}{c} -2 \\ 1 \end{array}\right] \end{array}$$