Chapter 0

Preliminaries

Skip or skim this chapter, returning for background explanations as necessary.

§0.1 CONNECTIONS AND COMBINATIONS

0.11 Relational Connections and Posets

Given n non-empty sets $S_1, \ldots, S_n$ and a relation $R \subseteq S_1 \times \cdots \times S_n$, we call the structure $(R, S_1, \ldots, S_n)$ an *n-ary relational connection.* When $s_i \in S_i$ (for $i = 1, \ldots, n$) we often write $Rs_1 \ldots s_n$ for $\langle s_1, \ldots, s_n \rangle \in R$, using the familiar 'infix' notation $s_1 R s_2$ for the case of $n = 2$. We will be especially concerned in what follows with this case – the case of binary relational connections – and will then write "S" (for "source") and "T" (for "target") for S_1 and S_2. A binary relational connection will be said to have the *cross-over property* (or to satisfy the cross-over condition) just in case for all s_1, $s_2 \in S$, and all t_1, $t_2 \in T$:

(*) $\quad (s_1 R t_1 \,\&\, s_2 R t_2) \Rightarrow (s_1 R t_2 \text{ } \mathit{or} \text{ } s_2 R t_1)$

The label "cross-over", for this condition, is explained pictorially. Elements of S appear on the left, and those of T on the right. An arrow going from one of the former to one of the latter indicates that the object represented at the tail of the arrow bears the relation R to that represented at the head of the arrow:

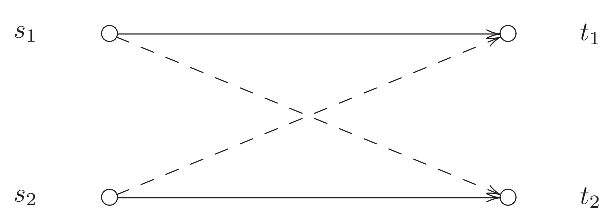

Figure 0.11a: The Cross-Over Condition

Read the diagram as follows: *if* objects are related as by the solid arrows, *then* they must be related as by at least one of the broken arrows. Thus, given the horizontally connected (ordered) pairs as belonging to R, we must have at least one of the crossing-over diagonal pairs also in R. Our main interest in this condition arises through Theorem 0.14.2 below, which will be appealed to more than once in later chapters (beginning with the proof of 1.14.6, p. 69). In the meantime, we include several familiarization exercises.

Exercise 0.11.1 Check that if a binary connection (R, S, T) has the cross-over property then so does the *complementary* connection $(\overline{R}, S, T)$ where $\overline{R} = (S \times T) \smallsetminus R$, and so also does the *converse* connection (R^{-1}, T, S) where $R^{-1} = \{\langle t, s\rangle \mid \langle s, t\rangle \in R\}$.

Exercise 0.11.2 Given $R \subseteq S \times T$, put $R(s) = \{t \in T \mid sRt\}$. Show that (R, S, T) has the cross-over property iff for all s_1, $s_2 \in S$: $R(s_1) \subseteq R(s_2)$ *or* $R(s_2) \subseteq R(s_1)$.

Exercise 0.11.3 (*i*) For S any set containing more than one element, show that the relational connection $(\in, S, \wp(S))$ does not have the cross-over property. (Here $\wp(S)$ is the power set of S, i.e., the set of all subsets of S.)

(*ii*) Where $\mathbb{N}$ is the set of natural numbers and $\leqslant$ is the usual less-than-or-equal-to relation, show that the relational connection $(\leqslant, \mathbb{N}, \mathbb{N})$ has the cross-over property.

Inspired by 0.11.3(i), we say that (R, S, T) is *extensional on the left* if for all s_1, $s_2 \in S$, $R(s_1) = R(s_2)$ implies $s_1 = s_2$, and that it is *extensional on the right* if for all t_1, $t_2 \in T$, $R^{-1}(t_1) = R^{-1}(t_2)$ implies $t_1 = t_2$. The 'Axiom of Extensionality' in set theory says that such connections as that exercise mentions are extensional on the right. (They are also extensional on the left.)

Part (ii) of 0.11.3, on the other hand, serves as a reminder that we do not exclude the possibility, for an n-ary relational connection $(R, S_1, \ldots, S_n)$, that the various S_i are equal, in which case we call the relational connection *homogeneous.* A more common convention is to consider in place of $(R, S_1, \ldots, S_n)$ the structure (S, R), where $S = S_1 = \cdots = S_n$. Such a pair is a special case of the notion of a *relational structure* in which there is only one relation involved. (In general one allows $(S, R_1, \ldots, R_m)$ where the R_i are relations – not necessarily of the same arity – on the set S. Here the *arity* of a relation R_i is that n such that R_i is n-ary. We speak similarly, below, of the arity of a function or operation.) Various conditions on binary relational connections which make sense in the homogeneous case, and so may equivalently be considered as conditions on relational structures, do not make sense in the general case. Three famous conditions falling under this heading, for a set S and a relation $R \subseteq S \times S$ are:

reflexivity: for all $a \in S$, aRa;
transitivity: for all a, b, $c \in S$, aRb & $bRc \Rightarrow aRc$; and
antisymmetry: for all a, $b \in S$, aRb & $bRa \Rightarrow a = b$.

If the first two conditions are satisfied, R is said to be a *pre-ordering* of (or 'pre-order on') S. We will often use the notation "$\precsim$" for R in this case; S together with such a pre-ordering is called a *pre-ordered set.* If all three conditions are satisfied, R is described as a *partial ordering* on S, and the relational structure (S, R) is called a partially ordered set, or *poset* for short. In this case, we will use the notation "$\leqslant$" or else "$\geqslant$" for R, on the understanding that when R is given by either of these symbols, the other stands for the converse of R (i.e., R^{-1}, as in 0.11.1). It is worth remarking that if $(S, \leqslant)$ is a poset, then so is

$(S, \geqslant)$. The latter is called the *dual* of $(S, \leqslant)$, any two statements about posets differing by a systematic interchange of reference to $\leqslant$ and $\geqslant$ (or notation or vocabulary defined in terms of them are) being described as each other's duals. If a statement about lattices is true of all posets, then so is its dual (since if the dual of a statement is false of some dual, the original statement is false of the dual of that poset.) Other standard terminology for properties of binary relations in the same vein as that recalled here (such as for the properties of *irreflexivity*, *symmetry*, and *asymmetry*) will be assumed to be familiar. Recall also that a relation which is reflexive, symmetric, and transitive, is said to be an *equivalence* relation.

Exercise 0.11.4 Given a pre-ordered set $(S, \precsim)$, define, for $s, t \in S$: $s \equiv t$ iff $s \precsim t$ & $t \precsim s$, and put $[s] = \{t \in T \mid s \equiv t\}$. Let $[S]$ be $\{[s] \mid s \in S\}$. Finally, define $[s] \leqslant [t]$ to hold iff $s \precsim t$. Show (1) that this is a good definition (that it introduces no inconsistency in virtue of the possibility that $[s] = [s']$, $[t] = [t']$, even though $s \neq s'$, $t \neq t'$), and (2) that the relational structure $([S], \leqslant)$ is a poset.

Exercise 0.11.5 Show that a relation $R \subseteq S \times S$ is a pre-ordering of S iff for all $s, t \in S$: $sRt \Leftrightarrow R(t) \subseteq R(s)$.

Posets of a special sort (lattices) will occupy us in 0.13. In the meantime, we return to the (generally) non-homogeneous setting of relational connections.

0.12 Galois Connections

Given sets S and T, a pair of functions (f, g) with $f\colon \wp(S) \longrightarrow \wp(T)$ and $g\colon \wp(T) \longrightarrow \wp(S)$ is called a *Galois connection* between S and T if the following four conditions are fulfilled, for all subsets S_0, S_1 of S, and T_0, T_1 of T:

(G1) $S_0 \subseteq g(f(S_0))$

(G2) $T_0 \subseteq f(g(T_0))$

(G3) $S_0 \subseteq S_1 \Rightarrow f(S_1) \subseteq f(S_0)$

(G4) $T_0 \subseteq T_1 \Rightarrow g(T_1) \subseteq g(T_0)$

Note first that the symmetrical treatment of S and f *vis-à-vis* T with g in these conditions has the effect that if (f, g) is a Galois connection between S and T then (g, f) is a Galois connection between T and S, so that we are entitled to the following 'duality' principle: any claim that has been established to hold for an arbitrary Galois connection (f, g) between sets S and T must continue to hold when references to f and g are interchanged in the claim, along with those to S and T. We will call this: *Galois duality*, to contrast it with poset duality (from 0.11 above, or lattice duality, introduced in 0.13 below). (A more explicit notation would have us call (S, T, f, g) a Galois connection between S and T, but we will rely on the context to make clear what S and T are in any given case.)

Given a binary relational connection (R, S, T), if we define, for arbitrary $S_0 \subseteq S$ and $T_0 \subseteq T$:

[Def. f_R] $\quad f_R(S_0) = \{t \in T \mid sRt \textit{ for all } s \in S_0\}$,

[Def. g_R] $\quad g_R(T_0) = \{s \in S \mid sRt \textit{ for all } t \in T_0\}$, then the pair (f_R, g_R) constitutes a Galois connection between S and T: it is not hard to check that (G1)–(G4) are satisfied. To give some feel for why these four conditions are given only as $\subseteq$-requirements, we illustrate with an example that the converse of (G1) need not hold for (f_R, g_R). Let S be a set of people and T a set of cities, and R be the relation of *having visited*. Then $f_R(S_0)$ consists of all the cities (in T) that everyone in S_0—some subset of S—has visited, and $g_R(f_R(S_0))$ is the set of people (drawn from S) who have visited all the cities that everyone in S_0 has visited: obviously this includes everyone in S_0, but there may also be other people who have visited all the cities that everyone in S_0 has visited, in which case $g_R(f_R(S_0))$ will be a proper superset of S_0. From now on, when convenient, we will write "fS" instead of "$f(S)$", etc.

Exercise 0.12.1 Show that (G1)–(G4) above are satisfied if and only if for all $S_0 \subseteq S$, $T_0 \subseteq T$: $T_0 \subseteq fS_0 \Leftrightarrow S_0 \subseteq gT_0$.

Exercise 0.12.2 Prove that if (f, g) is a Galois connection, then $fgfS_0 = fS_0$ and $gfgT_0 = gT_0$, for all $S_0 \subseteq S$, $T_0 \subseteq T$.

Next, we need to consider the way in which the set-theoretic operations of union and intersection are related in $\wp(S)$ and $\wp(T)$ when Galois connections are involved.

Theorem 0.12.3 *If (f, g) is a Galois connection between S and T then for all S_0, $S_1 \subseteq S$, and T_0, $T_1 \subseteq T$: (i) $f(S_0 \cup S_1) = fS_0 \cap fS_1$ and (ii) $g(T_0 \cup T_1) = gT_0 \cap gT_1$.*

Proof. We address part (i); (ii) follows by Galois duality.

First, that $f(S_0 \cup S_1) \subseteq fS_0 \cap fS_1$: $S_0 \subseteq S_0 \cup S_1$, so 'flipping' by (G3): $f(S_0 \cup S_1) \subseteq fS_0$; likewise to show $f(S_0 \cup S_1) \subseteq fS_1$. The desired result follows by combining these two.

It remains to be shown that $fS_0 \cap fS_1 \subseteq f(S_0 \cap S_1)$. First, $fS_0 \cap fS_1 \subseteq fS_0$, so by 0.12.1, $S_0 \subseteq g(fS_0 \cap fS_1)$. Similarly, $S_1 \subseteq g(fS_0 \cap fS_1)$. Putting these together, we get: $S_0 \cup S_1 \subseteq g(fS_0 \cap fS_1)$, from which the desired result follows by another appeal to 0.12.1. □

Theorem 0.12.3 continues to hold if the binary union and intersection are replaced by arbitrary union and intersection (of an infinite family of sets). Notice also that the proof of the first inclusion considered in establishing part (i) here does not involve the mapping g at all. This reflects the fact that if (f, g) is a Galois connection, either of f, g, determines the other *uniquely*:

Exercise 0.12.4 Show that if (f, g) and (f', g) are Galois connections between sets S and T then $f = f'$, and that if (f, g) and (f, g') are Galois connections between S and T, then $g = g'$.

There are two equations missing from Thm. 0.12.3, namely those resulting from (i) and (ii) there on interchanging "$\cap$" and "$\cup$". While these do not hold generally, they can be secured in an important special case. A Galois connection (f, g) between S and T is *perfect on the left* if for all $S_0 \subseteq S$, $gfS_0 = S_0$, and *perfect on the right* if for all $T_0 \subseteq T$, $fgT_0 = T_0$. If it is perfect on the left and perfect on the right, then we call a Galois connection *perfect*.

Observation 0.12.5 *If (f, g) is a perfect Galois connection between S and T, then $f(S_0 \cap S_1) = fS_0 \cup fS_1$ and $g(T_0 \cap T_1) = gT_0 \cup gT_1$, for all S_0, $S_1 \subseteq S$ and T_0, $T_1 \subseteq T$.*

Proof. Assume the antecedent of the Observation; we will show how the first equation follows:

$$S_0 \cap S_1 = gfS_0 \cap gfS_1 = g(fS_0 \cup fS_1),$$

as (f, g) is perfect on the left, by 0.12.3(ii). Applying f to both sides:

$$\begin{aligned} f(S_0 \cap S_1) &= fg(fS_0 \cup fS_1) \\ &= fS_0 \cup fS_1, \end{aligned}$$

as (f, g) is perfect on the right. □

Example 0.12.6 For a simple example of a perfect Galois connection, consider the connection (f, g) between a set S and itself in which for $S_0 \subseteq S$, $f(S_0) = g(S_0) = S \smallsetminus S_0$ (i.e., $\{s \in S \mid s \notin S_0\}$).

Before proceeding, we recall that a function f from S to T is said to map S *onto* T if every $t \in T$ is $f(s)$ for some $s \in S$. When we have a particular source and target in mind, we just say that f is 'onto' or *surjective*, though strictly speaking surjectivity is a property of what we might call the 'functional connection' (f, S, T), by analogy with relational connections above. (Some authors refer to (f, S, T) itself as a function, with f as its *graph*.) Similarly, if $f(s) = f(s')$ only when $s = s'$, the function f is said to be 'one-one' or *injective*. A function $f\colon S \longrightarrow T$ which is both injective and surjective is described as a *bijection* or *one-to-one correspondence between S and T.*

Observation 0.12.7 *Suppose (f, g) is a Galois connection between S and T. Then the following claims are equivalent:*

(i) (f, g) is perfect on the left;
(ii) g is surjective;
(iii) f is injective.

As also are the following three:

(i)′ (f, g) is perfect on the right;
(ii)′ f is surjective;
(iii)′ g is injective.

Proof. We do the proof for the case of (i)–(iii), arguing the cycle of implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: If the given connection is perfect on the left then for any $S_0 \subseteq S$, $gfS_0 = S_0$, so any such S_0 is the value of g applied to some $T_0 \subseteq T$; i.e., g is surjective.

$(ii) \Rightarrow (iii)$: Suppose that g is surjective and $fS_0 = fS_1$. We must show that $S_0 = S_1$. Since g is a surjection, $S_0 = gT_0$ and $S_1 = gT_1$ for some $T_0, T_1 \subseteq T$. Thus, since $fS_0 = fS_1$, $fgT_0 = fgT_1$; so $gfgT_0 = gfgT_1$, whence by 0.12.2, $gT_0 = gT_1$, i.e., $S_0 = S_1$.

$(iii) \Rightarrow (i)$: Suppose that $fS_0 = fS_1$ implies $S_0 = S_1$, for all $S_0, S_1 \subseteq S$. Now for any $S_0 \subseteq S$, we have, by 0.12.2, $fgfS_0 = fS_0$, so we may conclude that $gfS_0 = S_0$.
Since S_0 was arbitrary, the connection is perfect on the left. □

COROLLARY 0.12.8 *If (f, g) is a perfect Galois connection between S and T then f is a bijection from S to T with g as inverse.*

Not only does every binary relational connection give rise to a Galois connection *via* [Def. $f_R g_R$] above, but every Galois connection can be represented as arising in this way. We simply state this as 0.12.9 here; a proof may be found by considering the following way of defining a relation $R_{fg} \subseteq S \times T$ on the basis of a Galois connection (f, g) between S and T:

[Def. R_{fg}] For $s \in S$, $t \in T$: $sR_{fg}t \Leftrightarrow t \in f(\{s\})$.

The rhs of this definition makes no explicit mention of g – *cf.* 0.12.4.

THEOREM 0.12.9 *Every Galois connection between sets S and T is of the form (f_R, g_R) for some relation $R \subseteq S \times T$.*

In fact, starting with a Galois connection, and passing to a relational connection *via* [Def. R_{fg}] and back to a Galois connection *via* [Def. $f_R g_R$], we end up with the same Galois connection we started with. We can also say that starting with a relational connection from which a Galois connection is extracted by [Def. $f_R g_R$], the relational connection delivered from this by [Def. R_{fg}] is the original relational connection itself. There is, then, for any sets S and T a natural one-to-one correspondence between relations $R \subseteq S \times T$ on the one hand, and Galois connections between S and T on the other.

We need the following result for later discussion (in 6.24, p. 840).

EXERCISE 0.12.10 Show that for any Galois connection (f, g) and 'source' subsets S_0, S_1, we have: $fS_0 \cap fS_1 = fg(fS_0 \cap fS_1)$.
(*Hint:* Use 0.12.2, 0.12.3.)

This corresponds – in a way that will become clear in the following subsection – to the fact that the intersection of any two closed sets is closed. The following Remark (and Warning) will not be intelligible until after that is read; the material is included here as it constitutes a commentary on the development above.

REMARK 0.12.11 The definition, by (G1)–(G4), of a Galois connection makes sense for arbitrary posets S and T, when $f\colon S \longrightarrow T$, $g\colon T \longrightarrow S$ and the "$\subseteq$" in (G1), the antecedent of (G3) and the consequent of (G4) is replaced by "$\leqslant_S$" (denoting the partial ordering on S), with the "$\subseteq$" in (G2), the consequent of (G3) and the antecedent of (G4) being similarly replaced by "$\leqslant_T$". Much of the rest of the development – you may care to check *how* much – then goes through if these posets are lattices, in the sense of the following subsection, when unions and intersections are replaced by appropriate joins and meets (explained there). To impose this more general perspective on the formulations above, we should have to describe what we have called Galois connections between sets S and T as being instead between the sets $\wp(S)$ and $\wp(T)$.

Warning: Particular care is needed over adapting 0.12.5 to the more general context introduced in the above Remark. If (f, g) is a perfect Galois connection between posets A_1 and A_2, then (using $\wedge$ and $\vee$ for meets and joins – see 0.13 – in both cases) we can argue, as in the proof of that Observation, for a, $b \in A_1$, that $f(a \wedge b) = f(a) \vee f(b)$. And it may happen that the elements of A_1 and (in particular, for the present illustration) A_2 are themselves sets, with the partial ordering being $\subseteq$; it does not, however, follow that $f(a) \vee f(b)$ is $f(a) \cup f(b)$. The latter union may not belong to A_2 at all.

0.13 Lattices and Closure Operations

If $(S, \leqslant)$ is a poset and $S_0 \subseteq S$ then an element $b \in S$ is called an *upper bound* of S_0 if $c \leqslant b$ for each $c \in S_0$; if in addition it happens that for any $a \in S$ which is an upper bound for S_0, we have $b \leqslant a$, then b is called a *least upper bound* ("l.u.b.", for short) of the set S_0. Note that if b_1 and b_2 are both least upper bounds for the set S_0, then $b_1 = b_2$ (since $b_1 \leqslant b_2$ and $b_2 \leqslant b_1$ and $\leqslant$ is antisymmetric). The concepts of *lower bound* and *greatest lower bound* (g.l.b.) are defined dually. (The duality concerned is poset duality, as in 0.11, not Galois duality, of course.) A poset in which each pair of elements have both a least upper bound and a greatest lower bound is called a *lattice*. By the above observation concerning uniqueness, we can introduce unambiguously the notation $a \vee b$ for the least upper bound of a and b (strictly: of the set $\{a, b\}$) and $a \wedge b$ for their greatest lower bound. Similarly, if there is a least upper bound (greatest lower bound) for the whole lattice, it can unambiguously be denoted by 1 (by 0) and will be called the *unit* (the *zero*) element of the lattice (or just top and bottom elements, respectively). Note that while the existence of such elements follows from, it does not in turn entail, the existence of greatest lower bounds and least upper bounds for arbitrary sets of lattice elements; lattices in which such bounds always exist are called *complete*. In any lattice, however, any finite set of elements $\{a_1, \ldots, a_n\}$ has both a l.u.b. and a g.l.b., namely: $a_1 \vee a_2 \vee \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge \ldots \wedge a_n$, respectively.

Having agreed that the operations $\wedge$ and $\vee$ take elements two at a time, we should strictly insert parentheses into the terms just written, but they are omitted since the different bracketings make no difference to what the terms denote, in view of the third of the conditions listed here, all of which are satisfied by the operations $\wedge$ and $\vee$ in any lattice:

$a \wedge a = a$	$a \vee a = a$	(*Idempotence*)
$a \wedge b = b \wedge a$	$a \vee b = b \vee a$	(*Commutativity*)
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	$a \vee (b \vee c) = (a \vee b) \vee c$	(*Associativity*)
$a \wedge (a \vee b) = a$	$a \vee (a \wedge b) = a$	(*Absorption*)

Such equations are to be understood 'universally', i.e., as claiming that the equalities concerned hold for *all* lattice elements a, b, c. With this universal interpretation, equations are usually called *identities* – thus in this sense the identities of an algebra are the equations holding (universally) in that algebra. Note the need to avoid possible confusion with the use of "identity" to mean *identity element* (or neutral element: see 0.21).

The above equations introduce us to a different perspective on lattices, allowing them to be thought of, not as relational structures, but as *algebras*. An algebra in this sense is a non-empty set together with some operations under which the set is closed. (For more on this concept, see 0.21.) For the moment, we simply note that given $(S, \wedge, \vee)$ satisfying the eight equations above the relation $\leqslant$, defined by: $a \leqslant b \Leftrightarrow a \wedge b = a$, partially orders S, and $\wedge$ and $\vee$ give the g.l.b.s and l.u.b.s of the pairs of elements they operate on. We shall usually think of lattices as algebras in this way, rather than of lattices as (a special kind of) posets. When so thinking, we refer to $a \wedge b$ as the *meet*, and to $a \vee b$ as the *join*, of a, b. The *dual* of $(S, \wedge, \vee)$ is the lattice $(S, \vee, \wedge)$, and the dual of a lattice-theoretic statement is obtained by interchanging "$\vee$" and "$\wedge$" (as well as "$\leqslant$" and "$\geqslant$", if present). The reader is left to verify that this definition of duality for lattices as algebras is consilient with the definition of duality for (lattices as) posets in 0.11.

EXERCISE 0.13.1 Show that for any elements a, b, of a lattice $(S, \wedge, \vee)$: $a \wedge b = a$ iff $a \vee b = b$. Thus the latter equation could equivalently have been used instead of the former as the definition of $a \leqslant b$. (*Hint*: Absorption.)

The two parts of Exercise 0.11.3 suggest some illustrations of these concepts:

EXAMPLES 0.13.2 (*i*) If S is any set, then the poset $(\wp(S), \subseteq)$ is a lattice with $\cap$ (intersection) as meet and $\cup$ (union) as join.

(*ii*) The poset $(\mathbb{N}, \leqslant)$ is also a lattice, the meet of m and n ($m, n \in \mathbb{N}$) being $min(\{m, n\})$ and their join being $max(\{m, n\})$.

Note that the first example here could equally well have been given with any collection A of subsets of S, rather than specifically by insisting on taking all of them, as long as that collection is closed under union and intersection. Considering the lattice we obtain in this way as an algebra $(A, \wedge, \vee)$ we note that it will satisfy an equation which, not following from the eight given above to define the class of lattices, is not satisfied by every lattice, namely the so-called Distributive Law:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (\textit{Distributivity})$$

The second example features a lattice which is not only distributive, but has the stronger property that its associated partial ordering is a *linear* (or 'total') ordering, in the sense that for any elements a, b we have:

$$a \leqslant b \text{ or } b \leqslant a \qquad (\textit{Connectedness})$$

Linearly ordered posets are also called *chains*. Instead of just saying "connected", we shall more often speak of a binary relation R on some set as *strongly* connected when for any elements a, b of that set we have either aRb or bRa; this makes a clear distinction with what is often called *weak* connectedness, for which we require merely that for any *distinct* elements a, b, either aRb or bRa.

EXERCISE 0.13.3 (*i*) Show that all lattices satisfy the condition we get by putting "$\geqslant$" for "$=$" in (Distributivity).

(*ii*) Write down the dual of (Distributivity) and show that this equation is (universally) satisfied in a lattice iff the original equation is.

(*iii*) Show that every chain is, when considered as an algebra with $\wedge$ and $\vee$, a distributive lattice.

There is another way of making lattices out of collections of sets which is commonly encountered and which, by contrast with 0.13.2(i), is not guaranteed to lead to distributive lattices, as we shall see in 0.13.6. The key ingredient is the idea of a *closure operation* (or 'closure operator') on a set S, by which is meant a function C: $\wp(S) \longrightarrow \wp(S)$ satisfying the following three conditions, for all $X, Y \subseteq S$.

(C1) $X \subseteq C(X)$

(C2) $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$

(C3) $C(C(X)) = C(X)$

Observe that in view of (C1), (C3) could be replaced by: $C(C(X)) \subseteq C(X)$, and that the force of all three conditions could be wrapped up succinctly (somewhat in the style of 0.12.1) by the single condition that for all $X, Y \subseteq U$:

$$X \subseteq C(Y) \Leftrightarrow C(X) \subseteq C(Y).$$

It should also be remarked that the conditions could be written with "$\leqslant$" in place of "$\subseteq$", the variables "X", "Y", ranging over elements of any poset; if this poset is a complete lattice, what follows can be taken over to this more general setting, though we shall continue to concentrate on the case in which the partial ordering is the inclusion relation, $\subseteq$, on subsets of some given set. (*Cf.* Remark 0.12.10.) However, it is not necessary to consider *all* subsets of that set, so we shall allow the above definition to stand in even when the source and target of C are not $\wp(S)$ but some proper subset thereof. The origin of the idea of a closure operation is in topology, where the closure of a set of points is that set together with its 'boundary' points; in this case, certain additional features are present – in particular that $C(X \cup Y) = C(X) \cup C(Y)$ and $C(\varnothing) = \varnothing$ – which do not follow from the general definition. These additional features of topological closure operations are not suitable for the main logical application of the idea of closure (to consequence operations: see 1.12) – as is explained in detail in Chapter 1 of Martin and Pollard [1996]. For some practice with the general concept, we include:

Exercise 0.13.4 (*i*) Show that if C is a closure operation on S then for any $X, Y \subseteq S$: $C(X) \cup C(Y) \subseteq C(X \cup Y)$; $C(C(X) \cup C(Y)) = C(X \cup Y)$; and $C(X \cap Y) \subseteq C(X) \cap C(Y)$.

(*ii*) Show that if (f, g) is a Galois connection between S and T then $g \circ f$ is a closure operation on S, where $g \circ f(X) = g(f(X))$, and that $f \circ g$ is a closure operation on T. ($g \circ f$ is called the *composition* of g with f.)

(*iii*) A subset S_0 of a set S is *closed under* an n-ary relation R on S (i.e., $R \subseteq S^n$) if whenever $\{a_1, \ldots, a_{n-1}\} \subseteq S_0$ and $\langle a_1, \ldots, a_n\rangle \in R$, then $a_n \in S_0$. Where $\mathcal{R}$ is a collection of relations of various arities on S, define $C_{\mathcal{R}}(S_0)$ to be the least superset of S_0 to be closed under each $R \in \mathcal{R}$. Verify that $C_{\mathcal{R}}$ is indeed a closure operation.

(*iv*) Show that if $X = C(X)$, $Y = C(Y)$, then $C(X \cap Y) = C(X) \cap C(Y)$.

Parts (ii) and (iii) here have been included since they illustrate the two main ways closure operations enter into discussions of logic; in Chapter 1 it will emerge that the Galois connection route of (ii) is involved in *semantic* specifications of (what we there call) consequence operations, and the relational route of (iii) is involved in specifying them *proof-theoretically.* We further remark, *à propos* of (ii) that when we speak of a set's being closed under an n-ary operation (or function) f we mean that it is closed under the $(n+1)$-ary relation R defined by: $Rx_1 \ldots x_{n+1} \Leftrightarrow f(x_1, \ldots, x_n) = x_{n+1}$.

With a particular closure operation C in mind we call a set X *closed* (more explicitly: C-*closed*) when $C(X) = X$. From information as to which are the closed sets, we can recover C as mapping X to the intersection of all the closed sets $\supseteq X$, exploiting the fact that the intersection of any family of closed sets is itself closed. (Part (iv) of the above Exercise gives this for finite families.)

Exercise 0.13.5 (*i*) Verify that whenever we have a closure operation on some set, the partial ordering $\subseteq$ on the collection $\mathcal{C}$ of closed sets $\mathcal{C}$ gives rise to a lattice (as algebra), $(\mathcal{C}, \wedge, \vee)$ in which $\wedge$ is $\cap$ and $\vee$ is the operation $\dot{\cup}$ defined by: $X \dot{\cup} Y = C(X \cup Y)$. (In fact we get a complete lattice, the join of arbitrarily many elements being the closure of their union.)

(*ii*) Define a set $\mathcal{C} \subseteq \wp(S)$ to be a *closure system* on S if the intersection of arbitrarily many elements of $\mathcal{C}$ is an element of $\mathcal{C}$. As remarked in the text, the set of C-closed subsets of S is a closure system for any closure operation C on S, and C can be recovered from $\mathcal{C}$ by setting $C(X) = \bigcap\{Y \in \mathcal{C} \mid Y \supseteq X\}$. (Here "$X$", "$Y$", range over subsets of S.) Verify that similarly if we start with a closure system $\mathcal{C}$ and use the definition just given to define the closure operation C, then $\mathcal{C}$ can be recovered as the set of C-closed sets (i.e., those $X \subseteq S$ with $C(X) = X$). Thus there is a natural one-to-one correspondence between closure operations and closure systems (on a given set).

Example 0.13.6 Out of the eight subsets of the three-element set $\{1, 2, 3\}$, declare the following to be closed: $\{1, 2, 3\}$, $\{2\}$, $\{1, 3\}$, $\{3\}$, $\varnothing$. (Note that the intersection of any sets listed is also listed, and that to find the closure of any set from the original eight, take its smallest closed superset: e.g., the closure of $\{1\}$ is $\{1, 3\}$.) The closed sets make up a five-element lattice which is not distributive, because, for example, $\{1, 3\} \cap (\{2\} \dot{\cup} \{3\}) = \{1, 3\} \cap \{1, 2, 3\} = \{1, 3\}$, whereas $(\{1, 3\} \cap \{2\}) \dot{\cup} (\{1, 3\} \cap \{3\}) = \varnothing \dot{\cup} \{3\} = \{3\}$.

Next, we record some properties of distributive lattices; (i) gives a kind of generalized transitivity for $\leqslant$ (thought of as defined in terms of $\wedge$ or of $\vee$).

Exercise 0.13.7 (*i*) Show that a lattice $(A, \wedge, \vee)$ is distributive iff for all a, b, $c \in A$ we have: $a \wedge b \leqslant c \;\;\&\;\; a \leqslant b \vee c \Rightarrow a \leqslant c$

(*ii*) Prove that for any a, b, c in a distributive lattice, if $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ then $b = c$.

Note that we have used the letter "A" for the set of elements of our lattice here; the background to this notation will be explained in 0.21. What follow are some exercises on lattices in general.

Exercise 0.13.8 (*i*) Show that in any lattice, if $a \wedge c = a$ and $b \wedge c = b$ then $(a \vee b) \wedge c = a \vee b$. (*Hint:* rephrase everything in terms of joins, using 0.13.1.)

(*ii*) Suppose that $(A_1, \wedge_1, \vee_1)$ and $(A_2, \wedge_2, \vee_2)$ are lattices with $A_2 = A_1$ and $\wedge_2 = \wedge_1$. Show that $\vee_2 = \vee_1$. (*Hint:* Use (i), rewritten with $\wedge_1$ and $\vee_1$ for $\wedge$ and $\vee$, substituting $a \vee_2 b$ for c. This gives:

$$(a \vee_1 b) \wedge_1 (a \vee_2 b) = a \vee_1 b.$$

Similarly, we obtain:

$$(a \vee_2 b) \wedge_2 (a \vee_1 b) = a \vee_2 b.$$

Now use the facts that $\wedge_1 = \wedge_2$ and that this operation is commutative.)

(*iii*) Show that for lattice elements a, b, if $a \wedge b = a \vee b$ then $a = b$.

Exercise 0.13.9 (*i*) Call an element $a \in A$ *join-irreducible* in a lattice $(A, \wedge, \vee)$ if for all $b, c \in A$: $a = b \vee c \Rightarrow a = b$ *or* $a = c$; call $a \in A$ *join-prime* if for all $b, c \in A$: $a \leqslant b \vee c \Rightarrow a \leqslant b$ *or* $a \leqslant c$. Show that in any lattice all join-prime elements are join-irreducible, and that in any distributive lattice, the converse also holds.

(*ii*) Similarly, call $a \in A$ *meet-irreducible* if for all $b, c \in A$, $a = b \wedge c$ implies $a = b$ or $a = c$. Show that in any lattice, an element a is meet-irreducible iff for $b, c \in A$, $a = b \wedge c$ implies $b \leqslant c$ or $c \leqslant b$.

We close with a sort of converse to 0.13.4(ii); the *characteristic function* of a set (a phrase used in the proof) is the function mapping elements of that set to the truth-value True (or "T" as we shall denote this in subsequent chapters) and non-elements of the set to the value False (or "F"; in the work of some authors the numbers 1 and 0 – or even 0 and 1 – are used to play these respective roles):

Observation 0.13.10 *If C is a closure operation on a set S then there is a Galois connection (f, g) between S and some set T such that $C = g \circ f$.*

Proof. Given C and S, let T comprise the characteristic functions of the closed subsets of S, and define f and g via [Def.$f_R\, g_R$] from 0.12, where $R \subseteq S \times T$ is given by: $sRt \Leftrightarrow t(s) =$ True. Since this is automatically a Galois connection, it remains only to check that for all $X \subseteq S$, $C(X) = g(f(X))$. This is left to the reader. □

The characteristic functions employed in the above proof will emerge again in 1.12 under the description "valuations consistent with a consequence operation".

0.14 Modes of Object Combination

We return now to the subject of binary relational connections as in 0.11, to consider some conditions asserting the existence of objects in the source and in the target playing special roles. Since we do not want to disallow the ('homogeneous') possibility that source and target are one and the same set, we will actually speak in terms of *left* and *right* instead. These conditions involve conjunction ("and") and disjunction ("or") in their formulation, so we will use an upward pointing triangle when the characterization is conjunctive (this being suggestive of "$\wedge$") and a downward pointing triangle when it is disjunctive (to recall "$\vee$"). The subscripted "L" and "R" are mnemonic for "left" and "right" (so, in particular, this use of "R" has nothing to do with the relation R). Four conditions are to be introduced, in terms of an arbitrary binary connection (R, S, T); quantifier shorthand "$\forall$", "$\exists$" has been used to make the structure of the conditions more visible:

($\triangle_{\mathrm{L}}$) $\forall s_1, s_2 \in S \exists s_3 \in S$ *such that* $\forall t \in T$: $s_3 Rt \Leftrightarrow s_1 Rt \,\&\, s_2 Rt$.

($\triangledown_{\mathrm{L}}$) $\forall s_1, s_2 \in S \exists s_3 \in S$ *such that* $\forall t \in T$: $s_3 Rt \Leftrightarrow s_1 Rt$ *or* $s_2 Rt$.

($\triangle_{\mathrm{R}}$) $\forall t_1, t_2 \in T \exists t_3 \in T$ *such that* $\forall s \in S$: $sRt_3 \Leftrightarrow sRt_1 \,\&\, sRt_2$.

($\triangledown_{\mathrm{R}}$) $\forall t_1, t_2 \in T \exists t_3 \in T$ *such that* $\forall s \in S$: $sRt_3 \Leftrightarrow sRt_1$ *or* sRt_2.

When the first (second) of these conditions is satisfied by a relational connection, we say that this connection has *conjunctive (disjunctive) combinations on the left*, and call any s_3 with the promised properties a conjunctive (disjunctive) combination of the given s_1 and s_2. Similarly with the remaining two conditions and analogous terminology with *right* replacing *left*.

If either of ($\triangle_{\mathrm{L}}$), ($\triangledown_{\mathrm{L}}$), is satisfied in a relational connection which is extensional on the left, then the element (s_3 above) whose existence is claimed is the only one having the property in question, in which case we can call it *the* conjunctive or disjunctive combination (on the left) of s_1 and s_2, and denote it unambiguously by $s_1 \triangle_{\mathrm{L}} s_2$ or $s_1 \triangledown_{\mathrm{L}} s_2$ respectively. Likewise on the right. Thus, supposing (R, S, T) is an extensional relational connection, we have, using the $R(\cdot)$ notation of 0.11.2 (p. 2):

(*i*) $R(s_1 \triangle_{\mathrm{L}} s_2) = R(s_1) \cap R(s_2)$

(*ii*) $R(s_1 \triangledown_{\mathrm{L}} s_2) = R(s_1) \cup R(s_2)$

(*iii*) $R^{-1}(t_1 \triangle_{\mathrm{R}} t_2) = R^{-1}(t_1) \cap R^{-1}(t_2)$

(*iv*) $R^{-1}(t_1 \triangledown_{\mathrm{R}} t_2) = R^{-1}(t_1) \cup R^{-1}(t_2)$

In fact, even without extensionality, these claims make sense if we think of "$s_1 \triangle_{\mathrm{L}} s_2$" as denoting an arbitrary s_3 satisfying the condition imposed on s_3 for any given s_1, s_2 by ($\triangle_{\mathrm{L}}$), regardless of whether that condition is satisfied for all alternative choices of s_1, s_2, as ($\triangle_{\mathrm{L}}$) itself requires. And similarly in the other cases. We note, without proof, the lattice-theoretic implications of our four existence conditions.

OBSERVATION 0.14.1 *If (R, S, T) is extensional on the left and also has conjunctive and disjunctive combinations on the left, then the structure $(S, \triangle_{\mathrm{L}}, \triangledown_{\mathrm{L}})$ is a distributive lattice; likewise with $(T, \triangle_{\mathrm{R}}, \triangledown_{\mathrm{R}})$ if (R, S, T) is extensional on the right and has conjunctive and disjunctive combinations on the right.*

What follows the "likewise" is not really a separate fact from what precedes it, since we simply apply the preceding assertion to the converse connection (R^{-1}, T, S).

In 0.13 we noted the possibility of generalizing the meet and join operations to form the g.l.b. or l.u.b. of an arbitrary collection of poset elements. Such generalizations of conjunctive and disjunctive combinations also make sense; and in particular we will have need below of the following generalized conjunctive combination. If (R, S, T) is a binary relational connection then a conjunctive combination of any collection $S_0 \subseteq S$, which we may denote by $\triangle(S_0)$, or more explicitly, $\triangle_{\mathrm{L}}(S_0)$, is an element $s \in S$ such that for all $t \in T$, sRt iff for every $s' \in S_0$, $s'Rt$. While the property of having conjunctive combination on the left (to stick with this example) in this more generalized sense is of special interest in requiring the existence of such an s for every *infinite* $S_0 \subseteq S$, it is worth noting the upshot of this definition for the finite case. In particular, if $S_0 = \{s_1, s_2\}$ then any element qualifying as $\triangle(S_0)$ qualifies as $s_1 \vartriangle s_2$, and *vice versa*; if $S_0 = \{s_1\}$ then s_1 qualifies as $\triangle(S_0)$; and finally, that if $S_0 = \varnothing$, then $\triangle(S_0)$ is an element of S which (since the "for every $s' \in S_0$" quantifier is now vacuous) bears R to each $t \in T$. Corresponding sense, *mutatis mutandis*, can of course be made of talk of the *disjunctive* combination of an arbitrary collection of objects from the left or the right of a relational connection. To recall again 0.11.3(i) from p. 2, for any non-empty set S, the relational connection $(\in, S, \wp(S))$ not only has conjunctive and disjunctive combination on the right – with $t_1 \triangledown_{\mathrm{R}} t_2$ being $t_1 \cup t_2$, and $t_1 \vartriangle_{\mathrm{R}} t_2$ being $t_1 \cap t_2$ – it clearly also supports the generalized versions of these modes of combination (arbitrary union and intersection).

Conspicuously absent in the case of $(\in, S, \wp(S))$, as long as S contains more than one element, are conjunctive and disjunctive combinations *on the left.* A disjunctive combination of S-elements s_1 and s_2, for example, would be an object which belonged to precisely the sets at least one of s_1, s_2 belonged to. This object, then, would be a member of $\{s_1\}$, since s_1 is, and of $\{s_2\}$, since s_2 is; this implies $s_1 = s = s_2$, which of course need not be the case if $|S| \geqslant 2$, since we can choose for s_1 and s_2 distinct elements. Now Exercise 0.11.3 asked for a proof that, on the assumption that $|S| \geqslant 2$, the connection $(\in, S, \wp(S))$ does not have the cross-over property. And this is no coincidence. The following result implies that no relational connection with conjunctive and disjunctive combination on the right can have either conjunctive or disjunctive combination on the left without also having the cross-over property.

Theorem 0.14.2 *If a binary connection (R, S, T) has conjunctive combinations on the left and disjunctive combinations on the right, then it is has the cross-over property. The same conclusion follows if such a connection has disjunctive combinations on the left and conjunctive combinations on the right.*

Proof. We prove the first part of the Theorem, since the second will then follow by consideration of the converse connection. Suppose we have (R, S, T) with operations $\vartriangle_{\mathrm{L}}$ and $\triangledown_{\mathrm{R}}$ as in the antecedent of the claim to be proved, and that for s_1, $s_2 \in S$, t_1, $t_2 \in T$, (1) s_1Rt_1 and (2) s_2Rt_2. To demonstrate the cross-over property, we must show that we then have either s_1Rt_2 or else s_2Rt_1. From (1) and (2), we get $s_1R(t_1 \triangledown_{\mathrm{R}} t_2)$ and $s_2R(t_1 \triangledown_{\mathrm{R}} t_2)$, and from these, we infer that $(s_1 \vartriangle_{\mathrm{L}} s_2)R(t_1 \triangledown_{\mathrm{R}} t_2)$. Therefore either $(s_1 \vartriangle_{\mathrm{L}} s_2)Rt_1$ or $(s_1 \vartriangle_{\mathrm{L}} s_2)Rt_2$. From the first, it would follow that s_2Rt_1, and from the second, that s_1Rt_2. □

Three comments on this theorem and its proof are worth making. First, note that neither left nor right extensionality is needed as a hypothesis of the theorem. Secondly, as already remarked, this does not illegitimize the use of the "$s_1 \vartriangle_{\mathrm{L}} s_2$" (etc.) notation, for *some* arbitrarily selected s_3 satisfying for the given s_1, s_2, our condition ($\vartriangle_{\mathrm{L}}$). Finally, observe that the proof fully exploits all these conditions, in that all four of the implications involved in the two biconditionally stated conditions on the combined elements in ($\vartriangle_{\mathrm{L}}$) and ($\triangledown_{\mathrm{R}}$) are used; or, to put it differently, all four of the $\subseteq$-statements implicit in the equations (i) and (iv) above are used. Simple as it is, we shall make considerable use of 0.14.2 in the sequel. (The basic idea in the above proof may be found on pp. 6–7 of Strawson [1974]; the origin of 0.14.3 below may similarly be traced to work of Geach mentioned in Strawson's discussion.)

We turn our attention now to a singulary (or 1-ary) rather than binary mode of object combination, based on negation rather than on conjunction or disjunction. (Many say "unary" for the $n = 1$ case of "n-ary", but this is hard to do for anyone who knows any Latin, and *nobody* says, for example, "duary" for the $n = 2$ case.) Given (R, S, T) and $s_1 \in S$, we call $s_2 \in S$ a *negative object for* s_1 ("on the left") if for all $t \in T$, s_2Rt if and only if it is not the case that s_1Rt. If every $s \in S$ has such a corresponding negative object, we can say that (R, S, T) provides negative objects on the left; as with the conjunctively and disjunctively combined objects, they are uniquely determined if the connection is extensional on the left. Right-sided analogues of these pieces of terminology are to be understood in the obvious way, so that negative object formation on the right provides for $t \in T$, some element of T to which all and only those elements of S bear R that do not bear R to t. Rather than attempting to subscript a symbol reminiscent of complementation or negation (such as $\neg$) with "L" or "R" to indicate "left" or "right", we will use "$neg(\cdot)$" thus subscripted. Given any set S, the set-theoretic connection $(\in, S, \wp(S))$ provides negative objects on the *right*, as in the case of conjunctive and disjunctive combinations, the form of (S-relative) complements. Note that with a binary relational connection (R, S, T) with negative objects on the left and right for $s \in S$, $t \in T$, respectively – in the notation just introduced, $neg_{\mathrm{L}}(s)$ and $neg_{\mathrm{R}}(t)$ – then $sRneg_{\mathrm{R}}(t)$ if and only if $neg_{\mathrm{L}}(s)Rt$. Further (by 'De Morgan's Laws') if a connection has negative objects on the left then it has conjunctive combination on the left iff it has disjunctive combination on the left, and similarly with *right* replacing *left* throughout. Finally, assuming extensionality on the left (right), we have $neg_{\mathrm{L}}(neg_{\mathrm{L}}(s)) = s$ and $neg_{\mathrm{R}}(neg_{\mathrm{R}}(t)) = t$.

OBSERVATION 0.14.3 *(i) If (R, S, T) has negative objects on the right, then the relation $\precsim_{\mathrm{L}}$ is symmetric, where $s_1 \precsim_{\mathrm{L}} s_2 \Leftrightarrow R(s_1) \subseteq R(s_2)$.*

(ii) If (R, S, T) has negative objects on the left, then the relation $\precsim_{\mathrm{R}}$ is symmetric, where $t_1 \precsim_{\mathrm{R}} t_2 \Leftrightarrow R^{-1}(t_1) \subseteq R^{-1}(t_2)$.

Proof. We prove (i). Suppose there are negative objects on the right. We must show that for s_1, $s_2 \in S$, $s_1 \precsim_{\mathrm{L}} s_2 \Rightarrow s_2 \precsim_{\mathrm{L}} s_1$. So suppose further that not $s_2 \precsim_{\mathrm{L}} s_1$, i.e., that for some $t \in T$, s_2Rt but not s_1Rt. So $s_1Rneg_{\mathrm{L}}(t)$. Thus if $s_1 \precsim_{\mathrm{L}} s_2$, we should have $s_2Rneg_{\mathrm{R}}(t)$, which would contradict the fact that s_2Rt. Therefore not $s_1 \precsim_{\mathrm{L}} s_2$. □

Note that extensionality on the left amounts to the relation $\precsim_{\mathrm{L}}$'s being antisymmetric, and that since any relation which is both symmetric and antisymmetric is a subrelation of the identity relation, we have:

COROLLARY 0.14.4 *If (R, S, T) is extensional on the left and has negative objects on the right, then $s_1 \precsim_{\mathrm{L}} s_2 \Rightarrow s_1 = s_2$.*

EXERCISE 0.14.5 Suppose that (R, S, T) is a binary relational connection which has conjunctive combinations on the left and negative objects on the right. Show that for all s, $s' \in S$, $s \precsim_{\mathrm{L}} s'$.