Introduction to Quantitative Finance

A Math Tool Kit

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1 Mathematical Logic

1.1 Introduction

Nearly everyone thinks they know what logic is but will admit the difficulty in formally defining it, or will protest that such a formal definition is not necessary because its meaning is obvious. For example, we all like to stop an adversary in an argument with the statement "that conclusion is illogical," or attempt to secure our own victory by proclaiming "logic demands that my conclusion is correct." But if compelled in either instance, it may be difficult to formalize in what way logic provides the desired conclusion.

A legal trial can be all about attempts at drawing logical conclusions. The prosecution is trying to prove that the accused is guilty based on the so-called facts. The defense team is trying to prove the improbability of guilt, or indeed even innocence, based on the same or another set of facts. In this example, however, there is an asymmetry in the burden of proof. The defense team does not have to prove innocence. Of course, if such a proof can be presented, one expects a not guilty verdict for the accused. The burden of proof instead rests on the prosecution, in that they must prove guilt, at least to some legal standard; if they cannot do so, the accused is deemed not guilty.

Consequently a defense tactic is often focused not on attempting to prove innocence but rather on demonstrating that the prosecution's attempt to prove guilt is faulty. This might be accomplished by demonstrating that some of the claimed facts are in doubt, perhaps due to the existence of additional facts, or by arguing that even given these facts, the conclusion of guilt does not necessarily follow "logically." That is, the conclusion may be consistent with but not compelled by the facts. In such a case the facts, or evidence, is called "circumstantial."

What is clear is that the subject of logic applies to the drawing of conclusions, or to the formulation of inferences. It is, in a sense, the science of good reasoning. At its simplest, logic addresses circumstances under which one can correctly conclude that "B follows from A," or that "A implies B," or again, "If A, then B." Most would informally say that an inference or conclusion is logical if it makes sense relative to experience. More specifically, one might say that a conclusion follows logically from a statement or series of statements if the truth of the conclusion is guaranteed by, or at least compelled by, the truth of the preceding statement or statements.

For example, imagine an accused who is charged with robbing a store in the dark of night. The prosecution presents their facts: prior criminal record; eyewitness account that the perpetrator had the same height, weight, and hair color; roommate testimony that the accused was not home the night of the robbery; and the accused's inability to prove his whereabouts on the evening in question. To be sure, all these facts are consistent with a conclusion of guilt, but they also clearly do not compel such a conclusion. Even a more detailed eyewitness account might be challenged, since this crime occurred at night and visibility was presumably impaired. A fact that would be harder to challenge might be the accused's possession of many expensive items from the store, without possession of sales receipts, although even this would not be an irrefutable fact. "Who keeps receipts?" the defense team asserts!

The world of mathematical theories and proofs shares features with this trial example. For one, a mathematician claiming the validity of a result has the burden of proof to demonstrate this result is true. For example, if I assert the claim,

For any two integers N and M, it is true that $M + N = N + M$,

I have the burden of demonstrating that such a conclusion is compelled by a set of facts. A jury of my mathematical peers will then evaluate the validity of the assumed facts, as well as the quality of the logic or reasoning applied to these facts to reach the claimed conclusion. If this jury determines that my assumed facts or logic is inadequate, they will deem the conclusion "not proved." In the same way that a failed attempt to prove guilt is not a proof of innocence, a failed proof of truth is not a proof of falsehood. Typically there is no single judge who oversees such a mathematical process, but in this case every jury member is a judge.

Imagine if in mathematics the burden of proof was not as described above but instead reversed. Imagine if an acceptable proof of the claim above regarding N and M was: "It must be true because you cannot prove it is false." The consequence of this would be parallel to that of reversing the burden of proof in a trial where the prosecution proclaims: "The accused must be guilty because he cannot prove he is innocent." Namely, in the case of trials, many innocent people would be punished, and perhaps at a later date their innocence demonstrated. In the case of mathematics, many false results would be believed to be true, and almost certainly their falsity would ultimately be demonstrated at a later date. Our jails would be full of the innocent people; our math books, full of questionable and indeed false theory.

In contrast to an assertion of the validity of a result, if I claim that a given statement is false, I simply need to supply a single example, which would be called a "counterexample" to the statement. For example, the claim,

For any integer A, there is an integer B so that $A = 2B$,

can be proved to be false, or disproved, by the simple counterexample: $A = 3$.

What distinguishes these two approaches to proof is not related to the asserted statement being true or false, but to an asymmetry that exists in the approach to the presentation of mathematical theory. Mathematicians are typically interested in whether a general result is always true or not always true. In the first case, a general proof is required, whereas in the second, a single counterexample suffices. On the other hand, if one attempted to prove that a result is always false, or not always false, again in the first case, a general proof would be required, whereas in the second, a single counterexample would suffice. The asymmetry that exists is that one rarely sees propositions in mathematics stated in terms of a result that is always false, or not always false. Mathematicians tend to focus on "positive" results, as well as counterexamples to a positive result, and rarely pursue the opposite perspective. Of course, this is more a matter of semantic preference than theoretical preference. A mathematician has no need to state a proposition in terms of "a given statement is always false" when an equivalent and more positive perspective would be that "the negative of the given statement is always true." Why prove that "$2x = x$ is always false if $x \neq 0$" when you can prove that "for all $x \neq 0$, it is true that $2x \neq x$."

What distinguishes logic in the real world from the logic needed in mathematics is that in the real world the determination that A follows from B often reflects the human experience of the observers, for example, the judge and jury, as well as rules specified in the law. This is reinforced in the case of a criminal trial where the jury is given an explicit qualitative standard such as "beyond a reasonable doubt." In this case the jury does not have to receive evidence of the guilt of the accused that convinces with 100 percent conviction, only that the evidence does so beyond a reasonable doubt based on their human experiences and instincts, as further defined and exemplified by the judge.

In mathematics one wants logical conclusions of truth to be far more secure than simply dependent on the reasonable doubts of the jury of mathematicians. As mathematics is a cumulative science, each work is built on the foundation of prior results. Consequently the discovery of any error, however improbable, would have far-reaching implications that would also be enormously difficult to track down and rectify. So not surprisingly, the goal for mathematical logic is that every conclusion will be immutable, inviolate, and once drawn, never to be overturned or contradicted in the future with the emergence of new information. Mathematics cannot be built as a house of cards that at a later date is discovered to be unstable and prone to collapse.

In contrast, in the natural sciences, the burden of proof allowed is often closer to that discussed above in a legal trial. In natural sciences, the first requirement of a theory is that it be consistent with observations. In mathematics, the first requirement of a theory is that it be consistent, rigorously developed, and permanent. While it is always the case that mathematical theories are expanded upon, and sometimes become more or less in vogue depending on the level of excitement surrounding the development of new insights, it should never be the case that a theory is discarded because it is discovered to be faulty. The natural sciences, which have the added burden of consistency with observations, can be expected to significantly change over time and previously successful theories even abandoned as new observations are made that current theories are unable to adequately explain.

1.2 Axiomatic Theory

From the discussion above it should be no surprise that structure is desired of every mathematical theory:

1. Facts used in a proof are to be explicitly identified, and each is either assumed true or proved true given other assumed or proved facts.

2. The rules of inference, namely the logic applied to these facts in proofs, are to be "correct," and the definition of correct must be objective and immutable.

3. The collection of conclusions provable from the facts in item 1 using the logic in item 2 and known as **theorems**, are to be **consistent**. That is, for no statement P will the collection of theorems include both "statement P is true" and "the negation of statement P is true."

4. The collection of all **theorems** is to be **complete**. That is, for every statement P, either "statement P is a theorem" or "the negation of statement P is a theorem." A related but stronger condition is that the resulting theory is **decidable**, which means that one can develop a procedure so that for any statement P, one can determine if P is true or not true in a finite number of steps.

It may seem surprising that in item 1 the "truth" of the assumed facts was not the first requirement, but that these facts be explicitly identified. It is natural that identification of the assumed facts is important to allow a mathematical jury to do its review, but why not an absolute requirement of "truth"? The short answer is, there are no facts in mathematics that are "true" and yet at the same time dependent on no other statements of fact. One cannot start with an empty set of facts and somehow derive, with logic alone, a collection of conclusions that can be demonstrated to be true.

Consequently some basic collection of facts must be **assumed** to be true, and these will be the **axioms** of the theory. In other words, all mathematical theories are **axiomatic theories**, in that some basic set of facts must be assumed to be true, and based on these, other facts proved. Of course, the axioms of a theory are not arbitrary. Mathematicians will choose the axioms so that in the given context their truth appears undeniable, or at least highly reasonable. This is what ensures that the theorems of the mathematical theory in item 3, that is, the facts and conclusions that follow from these axioms, will be useful in that given context.

Different mathematical theories will require different sets of axioms. What one might assume as axioms to develop a theory of the integers will be different from the axioms needed to develop a theory of plane geometry. Both sets will appear undeniably true in their given context, or at least quite reasonable and consistent with experience. Moreover, even within a given subject matter, such as geometry, there may be more than one context of interest, and hence more than one reasonable choice for the axioms.

For example, the basic axioms assumed for **plane geometry**, or the geometry that applies on a "flat" two-dimensional sheet, will logically be different from the axioms one will need to develop **spherical geometry**, which is the geometry that applies on the surface of a sphere, such as the earth. Which axioms are "true"? The answer is both, since both theories one can develop with these sets of axioms are useful in the given contexts. That is, these sets of axioms can legitimately be claimed to be "true" because they imply theories that include many important and deep insights in the given contexts.

That said, in mathematics one can and does also develop theories from sets of axioms that may seem abstract and not have a readily observable context in the real world. Yet these axioms can produce interesting and beautiful mathematical theories that find real world relevance long after their initial development.

The general requirements on a set of axioms is that they are:

1. Adequate to develop an interesting and/or useful theory.

2. Consistent in that they cannot be used to prove both "statement P is true" and "the negation of statement P is true."

3. Minimal in that for aesthetic reasons, and because these are after all "assumed truths," it is desirable to have the simplest axioms, and the fewest number that accomplish the goal of producing an interesting and/or useful theory.

It is important to understand that the desirability, and indeed necessity, of framing a mathematical theory in the context of an axiomatic theory is by no means a modern invention. The earliest known exposition is in the *Elements* by **Euclid of Alexandria** (ca. 325–265 BC), so Euclid is generally attributed with founding the axiomatic method. The *Elements* introduced an axiomatic approach to two- and three-dimensional geometry (called **Euclidean geometry**) as well as number theory. Like the modern theories this treatise explicitly identifies axioms, which it classifies as "common notions" and "postulates," and then proceeds to carefully deduce its theorems, called "propositions." Even by modern standards the *Elements* is a masterful exposition of the axiomatic method.

If there is one significant difference from modern treatments of geometry and other theories, it is that the *Elements* defines all the basic terms, such as point and line, before stating the axioms and deducing the theorems. Mathematicians today recognize and accept the futility of attempting to define all terms. Every such definition uses words and references that require further expansion, and on and on. Modern developments simply identify and accept certain notions as undefined—the so-called primitive concepts—as the needed assumptions about the properties of these terms are listed within the axioms.

1.3 Inferences

Euclid's logical development in the *Elements* depends on "rules of inference" but does not formally include logic as a theory in and of itself. A formal development of the theory of logic was not pursued for almost two millennia, as mathematicians, following Euclid, felt confident that "logic" as they applied it was irrefutable. For instance, if we are trying to prove that a certain solution to an equation satisfies $x < 100$, and instead our calculation reveals that $x < 50$, without further thought we would proclaim to be done. Logically we have:

"$x < 50$ implies that $x < 100$" is a true statement.

"$x < 50$" is a true statement by the given calculation.

"$x < 100$" is a true statement, by "deduction."

Abstractly: if $P \Rightarrow Q$ and P, then Q. Here we use the well-known symbol $\Rightarrow$ for "implies," and agree that in this notation, all statements displayed are "true." That is, if $P \Rightarrow Q$ and P are true statements, then Q is a true statement. This is an example of the **direct method of proof** applied to the **conditional statement**, $P \Rightarrow Q$, which is also called an **implication**.

In the example above note that even as we were attempting to implement an objective logical argument on the validity of the conclusion that $x < 100$, we would likely have been simultaneously considering, and perhaps even biased by, the intuition we had about the given context of the problem. In logic, one attempts to strip away all context, and thereby strip away all intuition and bias. The logical conclusion we drew about x is true if and only if we are comfortable with the following logical statement in every context, for any meanings we might ever ascribe to the statements P and Q:

If $P \Rightarrow Q$ and P, then Q.

In logic, it must be all or nothing. The rule of inference summarized above is known as **modus ponens**, and it will be discussed in more detail below.

Another logical deduction we might make, and one a bit more subtle, is as follows:

"$x < 50$ implies that $x < 100$" is a true statement.

"$x < 100$" is not a true statement by demonstration.

"$x < 50$" is not a true statement, by deduction.

Again, abstractly: if $P \Rightarrow Q$ and $\sim Q$, then $\sim P$. Here we use the symbol $\sim Q$ to mean "the negation of Q is true," which is "logic-speak" for "Q is false." This is similar to the "direct method of proof," but applied to what will be called the **contrapositive** of the conditional $P \Rightarrow Q$, and consequently it can be considered an **indirect method of proof**. Again, we can apply this logical deduction in the given context if and only if we are comfortable with the following logical statement in every context:

If $P \Rightarrow Q$ and $\sim Q$, then $\sim P$.

The rule of inference summarized above is known as **modus tollens**, and will also be discussed below.

Clearly, the logical structure of an argument can become much more complicated and subtle than is implied by these very simple examples. The theory of mathematical logic creates a formal structure for addressing the validity of such arguments within which general questions about axiomatic theories can be addressed. As it turns out, there are a great many rules of inference that can be developed in mathematical logic, but *modus ponens* plays the central role because other rules can be deduced from it.

1.4 Paradoxes

One may wonder when and why mathematicians decided to become so formal with the development of a mathematical theory of logic, collectively referred to as **mathematical logic**, requiring an axiomatic structure and a formalization of rules of inference. An important motivation for increased formality has been the recognition that even with early efforts to formalize, such as in Euclid's *Elements*, mathematics has not always been formal enough, and the result was the discovery of a host of **paradoxes** throughout its history. A paradox is defined as a statement or collection of statements which appear true but at the same time produce a contradiction or a conflict with one's intuition. Some mathematical paradoxes in history where solved by later developments of additional theory. That is, they were indicative of an incomplete or erroneous understanding of the theory, often as a consequence of erroneous assumptions. Others were more fatal, in that they implied that the theory developed was effectively built as a house of cards and so required a firmer and more formal theoretical foundation.

Of course, paradoxes also exist outside of mathematics. The simplest example is the **liar's paradox**:

This statement is false.

The statement is paradoxical because if it is true, then it must be false, and conversely, if false, it must be true. So the statement is both true and false, or neither true nor false, and hence a paradox.

Returning to mathematics, sometimes an apparent paradox represents nothing more than sleight of hand. Take, for instance, the "proof" that $1 = 0$, developed from the following series of steps:

$$a = 1,$$

$$a^2 = 1,$$

$$a^2 - a = 0,$$

$$a(a-1) = 0,$$

$$a = 0,$$

$$1 = 0.$$

The sleight of hand here is obvious to many. We divided by $a - 1$ before the fifth step, but by the first, $a - 1 = 0$. So the paradoxical conclusion is created by the illegitimate division by 0. Put another way, this derivation can be used to confirm the illegitimacy of division by zero, since to allow this is to allow the conclusion that $1 = 0$.

Sometimes the sleight of hand is more subtle, and strikes at the heart of our lack of understanding and need for more formality. Take, again, the following deduction that $1 = 0$:

$$\begin{aligned} A &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots \\ &= (1 - 1) + (1 - 1) + (1 - 1) + \cdots \\ &= 0. \end{aligned}$$

$$A = 1 - (1 - 1) - (1 - 1) - (1 - 1) - \cdots$$
$$= 1,$$

so once more, $A = 1 = 0$. The problem with this derivation relates to the legitimacy of the grouping operations demonstrated; once grouped, there can be little doubt that the sum of an infinite string of zeros must be zero. Because we know that such groupings are fine if the summation has only finitely many terms, the problem here must be related to this example being an infinite sum. Chapter 6 on numerical series will develop this topic in detail, but it will be seen that this infinite alternating sum cannot be assigned a well-defined value, and that such grouping operations are mathematically legitimate only when such a sum is well-defined.

An example of an early and yet more complex paradox in mathematics is **Zeno's paradox**, arising from a mythical race between Achilles and a tortoise. **Zeno of Elea** (ca. 490–430 BC) noted that if both are moving in the same direction, with Achilles initially behind, Achilles can never pass the tortoise. He reasoned that at any moment that Achilles reaches a point on the road, the tortoise will have already arrived at that point, and hence the tortoise will always remain ahead, no matter how fast Achilles runs. This is a paradox for the obvious reason that we observe faster runners passing slower runners all the time. But how can this argument be resolved?

Although this will be addressed formally in chapter 6, the resolution comes from the demonstration that the infinite collection of observations that Zeno described between Achilles and the tortoise occur in a finite amount of time. Zeno's conclusion of paradox implicitly reflected the assumption that if in each of an infinite number of observations the tortoise is ahead of Achilles, it must be the case that the tortoise is ahead for all time. A formal resolution again requires the development of a theory in which the sum of an infinite collection of numbers can be addressed, where in this case each number represents the length of the time interval between observations.

Another paradox is referred to as the **wheel of Aristotle**. **Aristotle of Stagira** (384–322 BC) imagined a wheel that has inner and outer concentric circles, as in the inner and outer edges of a car tire. He then imagined a fixed line from the wheel's hub extending through these circles as the wheel rotates. Aristotle argued that at every moment, there is a one-to-one correspondence between the points of intersection of the line and the inner wheel, and the line and the outer wheel. Consequently the inner and outer circles must have the same number of points and the same circumference, a paradox. The resolution of this paradox lies in the fact that having a 1:1 correspondence between the points on these two circles does not ensure that they have equal lengths, but to formalize this required the development of the theory of infinite sets many hundreds of years later. At the time of Aristotle it was not understood how two sets could be put in 1 : 1 correspondence and not be "equivalent" in their size or measure, as is apparently the case for two finite sets. Chapter 2 on number systems will develop the topic of infinite sets further.

The final paradox is unlike the others in that it effectively dealt a fatal blow to an existing mathematical theory, and made it clear that the theory needed to be redeveloped more formally from the beginning. It is fair to say that the paradoxes above didn't identify any house of cards but only a situation that could not be appropriately explained within the mathematical theory or understanding of that theory developed to that date. The next paradox has many forms, but a favorite is called the **Barber's paradox**. As the story goes, in a town there is a barber that shaves all the men that do not shave themselves, and only those men. The question is: Does the barber shave himself? Similar to the liar's paradox, we conclude that the barber shaves himself if and only if he does not shave himself. The problem here strikes at the heart of set theory, where it had previously been assumed that a set could be defined as any collection satisfying a given criterion, and once defined, one could determine unambiguously whether or not a given element is a member of the set. Here the set is defined as the collection of individuals satisfying the criterion that they don't shave themselves, and we can get no logical conclusion as to whether or not the barber is a member of this set.

An equivalent form of this paradox, and the form in which it was discovered by **Bertrand Russell** (1872–1970) in 1901 and known as **Russell's paradox**, makes this set theory connection explicit. Let X denote the set of all sets that are not elements of themselves. The paradox is that one concludes X to be an element of itself if and only if it is not an element of itself. This discovery was instrumental in identifying the need for, and motivating the development of, a more careful axiomatic approach to set theory. Of course, the need for the development of a more formal axiomatic theory for all mathematics was equally compelled, since if mathematics went astray by defining an object as simple and intuitive as a set, who could be confident that other potential crises didn't loom elsewhere?

1.5 Propositional Logic

1.5.1 Truth Tables

Much of mathematical logic can be better understood once the concept of **truth table** is introduced and basic relationships developed. The starting point is to define a **statement** in a mathematical theory as any declarative sentence that is either true or false, but not both. For example, "today the sky is blue" and "$5 < 7$" are statements. An expression such as "$x < 7$" is not a statement because we cannot assign T or F to it without knowing what value the variable x assumes. Such an expression will be called a **formula** below. While a formula is not a statement because the variable x is a **free variable**, it can be made into a statement by making x a **bound variable**. The most common ways of accomplishing this is with the **universal quantifier**, $\forall$, and **existential quantifier**, $\exists$, defined as follows:

- $\forall x$ denotes: "for all x."
- $\exists x$ denotes: "there exists an x such that."

For example, $\forall x\ (x < 7)$ and $\exists x\ (x < 7)$ are now statements. The first, "for all x, x is less than 7" is assigned an F; the second, "there exists an x such that x is less than 7" is a T.

A truth table is a mechanical device for deciphering the truth or falsity of a complicated statement based on the truth or falsity of its various substatements. Complicated statements are constructed using **statement connectives** in various combinations. Of course, from the discussion above it should be no surprise that the initial collection of true statements for a given mathematical theory would be the "assumed facts" or axioms of the theory. Truth tables then provide a mechanism for determining the truth or falsity of more complicated statements that can be formulated from these axioms and, as we will see, also provide a framework within which one can evaluate the logical integrity of a given inference one makes in a proof.

If P and Q are statements, we define the following statement connectives and present the associated truth tables. Negation is a **unary** or **singulary connective**, whereas the others are **binary connectives**. In each case the truth table identifies all possible combinations of T or F for the given statements, denoted P or Q, and then assigns a T or F to the defined statements.

1. Negation: $\sim P$ denotes the statement "not P."

P	$\sim P$
T	F
F	T

2. Conjunction: $P \wedge Q$ denotes the statement "P and Q."

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction: $P \vee Q$ denotes the statement "P or Q" but understood as "P and/or Q."

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \vee Q}$
T	T	T
T	F	T
F	T	T
F	F	F

4. Conditional: $P \Rightarrow Q$ denotes the statement "P implies Q."

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$
T	T	T
T	F	F
F	T	T
F	F	T

5. Biconditional: $P \Leftrightarrow Q$ denotes the statement "P if and only if Q."

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Leftrightarrow Q}$
T	T	T
T	F	F
F	T	F
F	F	T

In other words, we have the following truth assignments, which are generally consistent with common usage:

- $\sim P$ has the **opposite** truth value as P.
- $P \wedge Q$ is true only when **both** P and Q are true.
- $P \vee Q$ is true when **at least one of** P and Q are true.
- $P \Rightarrow Q$ is true **unless** P is T, and Q is F.
- $P \Leftrightarrow Q$ is true when P and Q have the **same truth values**.

There may be two surprises here. First off, in mathematical logic the disjunctive "or" means "and/or." In common language, "P or Q" usually means "P or Q but not both." If you are told, "your money or your life," you do not expect an unfavorable outcome after handing over your wallet. Obviously, if the thief is a mathematician, there could be an unpleasant surprise.

An important consequence of this interpretation, which would not be true for the common language notion, is that there is a logical symmetry between conjunction and disjunction when negation is applied:

$$\sim(P \wedge Q) \Leftrightarrow (\sim P) \vee (\sim Q),$$

$$\sim(P \vee Q) \Leftrightarrow (\sim P) \wedge (\sim Q).$$

That is, the statement "$P \wedge Q$" is false if and only if "either P is false or Q is false," and the statement "$P \vee Q$" is false if and only if "both P is false and Q is false."

The equivalence of these statements follows from a truth table analysis that utilizes the basic properties above. For example, the truth table for the first statement is:

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{\sim(P \wedge Q)}$	$\boldsymbol{(\sim P) \vee (\sim Q)}$	$\boldsymbol{\sim(P \wedge Q) \Leftrightarrow (\sim P) \vee (\sim Q)}$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

This demonstrates that the two statements always have the same truth values.

The second surprise relates to the conditional truth values in the last two rows of the table, when P is false. Then, whether Q is true or false, the conditional $P \Rightarrow Q$ is declared true. For example, let

P : There is a mispricing in the market,

Q : I will attempt to arbitrage.

So $P \Rightarrow Q$ is a statement I might make:

"If there is a mispricing in the market, then I will attempt to arbitrage."

The question becomes, How would you evaluate whether or not my statement is true? The truth table declares this statement true when P and Q are both true, and so would you. In other words, if there was a mispricing and I attempted to arbitrage, you would judge my statement true. Similarly, if P was true and I did not make this attempt, you would judge my statement false, consistent with the second line in the truth table.

Now assume that there was not a mispricing in the market today, and yet I was observed to be attempting an arbitrage. Would my statement above be judged false? What if in the same market, I did not attempt to arbitrage, would my statement be deemed false? The truth table for the conditional states that in both cases my original statement would be deemed true, although in the real world the likely conclusion would be "not apparently false." In other words, in these last two cases my actions do not present evidence of the falsity of my statement, and hence the truth table deems my statement "true." Simply said, the truth table holds me truthful unless proved untruthful, or innocent unless proved guilty.

A consequence of this truth table assignment for the conditional is that

$$(P \Rightarrow Q) \Leftrightarrow \sim(P \wedge \sim Q).$$

In other words, $P \Rightarrow Q$ has exactly the same truth values as does $\sim(P \wedge \sim Q)$. The associated truth table is as follows:

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\mathbf{P \Rightarrow Q}$	$\mathbf{\sim(P \wedge \sim Q)}$	$\mathbf{(P \Rightarrow Q) \Leftrightarrow \sim(P \wedge \sim Q)}$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

This truth table analysis and the one above were somewhat tedious, especially when all the missing columns are added in detail, but note that they were entirely mechanical. No intuition was needed; we just apply in a methodical way the logic rules as defined by the truth tables above.

These truth tables have another interpretation, and that is, for any statements P and Q, and any truth values assigned, the statement

$$\sim(P \wedge Q) \Leftrightarrow (\sim P) \vee (\sim Q),$$

is a **tautology**, which is to say that it is always true. The same can be said for the biconditional statements illustrated above. Tautologies will be seen to form the foundation for developing and evaluating rules of inference, and more specifically, the logical integrity of a given proof.

There are many other tautologies possible, in fact infinitely many. One reason for this is that there is redundancy in the list of connectives above:

$$\sim, \wedge, \vee, \Rightarrow, \Leftrightarrow.$$

In a formal treatment of mathematical logic, only $\sim$ and $\Rightarrow$ need be introduced, and the others are then defined by the following statements, all of which are tautologies in the framework above:

$$P \vee Q \Leftrightarrow \sim P \Rightarrow Q,$$

$$P \wedge Q \Leftrightarrow \sim(P \Rightarrow \sim Q),$$

$$(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

Note that the last statement can in turn be expressed in terms of only $\sim$ and $\Rightarrow$ using the second tautology.

There is also redundancy between the universal and existential quantifiers. In formal treatments one introduces the universal quantifier $\forall$ and defines the existential quantifier $\exists$ by

$$\exists x P(x) \Leftrightarrow \sim\forall x(\sim P(x)).$$

In other words, "there exists an x so that statement $P(x)$ is true" is the same as "it is false that for all x the statement $P(x)$ is false."

Admittedly, such definitional connections require one to pause for understanding, and one might wonder why all the terms are simply not defined straightaway instead of in the complicated ways above. The reason was noted earlier in the discussion on axioms. One goal of an axiomatic structure is to be minimal, or at least parsimonious. The cost of this goal is often apparent complexity, as one might spend considerable effort proving a statement that virtually everyone would be more than happy just accepting as another axiom. But the goal of mathematical logic is not the avoidance of complexity by adding more axioms; it is the illumination of the theory and the avoidance of potential paradoxes by minimizing the number of axioms needed. The fewer the axioms, the more transparent the theory becomes, and the less likely the axioms will be in violation of another important goal of an axiomatic structure. And that is consistency.

1.5.2 Framework of a Proof

In later chapters various statements will be made under the heading **proposition**, which is the term used in this book for the more formal sounding **theorem**. These terms are equivalent in mathematics, and the choice reflects style rather than substance. In virtually all cases, a "proof" of the statement will be provided. A **lemma** is yet another name for the same thing, although it is generally accepted that a **lemma** is considered a relatively minor result, whereas a proposition or theorem is a major result. Some authors distinguish between proposition and theorem on the same basis, with theorem used for the most important results.

This terminology is by no means universally accepted. For example, students of finance will undoubtedly encounter *Ito's lemma*, and soon discover that in the theory underlying the pricing of financial derivatives like options, this lemma is perhaps the most important theoretical result in quantitative finance.

Now the typical structure for the statement of a proposition is

If P, then Q.

The statement P is the **hypothesis** of the proposition, and in some cases it will be a complex statement with many substatements and connectives, while the statement Q is the **conclusion**. The goal of this and the next section is to identify logical frameworks for such proofs.

First off, a proof of the statement "If P, then Q" is not equivalent to a proof of the statement '$P \Rightarrow Q$' despite their apparent equivalence in informal language. Specifically,

"If P, then Q" means "if statement P is true, then statement Q is true,"

whereas

'$P \Rightarrow Q$' means "the statement P implies Q is true."

Of course, one is hardly interested in proving statements such as '$P \Rightarrow Q$' unless Q can be asserted to be a true statement. That is the true goal of a proposition, to achieve the conclusion that Q is true. However, the statement $P \Rightarrow Q$ was seen to be true in three of the four cases displayed in the truth table above, and in only one of these three cases is Q seen to be true. Namely the truth of '$P \Rightarrow Q$' assures the truth of Q only when P is true. Consequently, if we want to prove the typical propositional structure above, which is to say that we can infer the truth of statement Q from the truth of statement P, we can prove the following:

If P and $P \Rightarrow Q$, then Q.

If this statement is written in the notation of logic, it is in fact a tautology, and always true. That is, in the truth table of

$$P \wedge (P \Rightarrow Q) \Rightarrow Q, \tag{1.1}$$

we have that for any assignment of the truth values to P and Q, this statement has constant truth value of "true."

This statement is the central **rule of inference** in logic, and it is known as **modus ponens**. It says that:

If statement P is true, and the statement $P \Rightarrow Q$ is demonstrated as true, then Q must be true.

This is the formal basis of many mathematical proofs of "If P, then Q." Of course, the language of the proof usually focuses on the development of the truth of the implication: $P \Rightarrow Q$, while the truth of the statement P, which is the hypothesis of the theorem, is simply implied. Moreover, if P were false, the demonstration of the truth of $P \Rightarrow Q$ would be for naught, since in this case Q could be true or false, as the truth table above attests.

In the next section we investigate proof structures in more detail. The central idea is every logical structure for a valid proof must be representable as a tautology, such as the *modus ponens* structure in (1.1). As we have seen, it is straightforward and mechanical, though perhaps tedious, to verify that a given proof structure, however complicated, is indeed a tautology. Here are a few other possible proof structures that are tautologies intuitively, as well as relatively easy to demonstrate in a truth table. Each is simply related to a single line on one of the basic truth tables given for the connectives:

$$P \wedge (P \wedge Q) \Rightarrow Q,$$

$$(P \vee Q) \wedge {\sim}Q \Rightarrow P,$$

$$(P \Leftrightarrow Q) \wedge {\sim}Q \Rightarrow {\sim}P.$$

For example, on the truth table for $P \wedge Q$, the only row where both P and $P \wedge Q$ are true is the row where Q is also true. In any other row, one or both of P and $P \wedge Q$ are false, and hence the conjunction $P \wedge (P \wedge Q)$ is false, assuring that the conditional $P \wedge (P \wedge Q) \Rightarrow Q$ is true. That is exactly how this statement becomes a tautology, and this logic will be seen to hold in all such cases. Specifically, when the hypothesis of the proposition is a conjunction, as is typically the case, we only really have to evaluate the case where all substatements are true, and assure that the conclusion is then true in this case. In all other cases the conjunction will be false and the conditional automatically true.

1.5.3 Methods of Proof

With *modus ponens* in the background, the essence of virtually any mathematical proof is a demonstration of the truth of the implication $P \Rightarrow Q$. To this end, the first choice one has is to prove the **direct conditional** statement $P \Rightarrow Q$, or its **contrapositive** ${\sim}Q \Rightarrow {\sim}P$. These statements are logically equivalent, which is to say that they have the same truth values in all cases. In other words, the statement

$$(P \Rightarrow Q) \Leftrightarrow ({\sim}Q \Rightarrow {\sim}P) \tag{1.2}$$

is a tautology, in that for any assignment of the truth values to P and Q, this statement has constant truth value of "true."

If *modus ponens* is applied to this contrapositive, we arrive at

$$\sim Q \wedge (\sim Q \Rightarrow \sim P) \Rightarrow \sim P. \tag{1.3}$$

However, because of (1.2), this can also be written as

$$\sim Q \wedge (P \Rightarrow Q) \Rightarrow \sim P, \tag{1.4}$$

which is a rule of inference known as **modus tollens** and exemplified in section 1.2 on axiomatic theory. It is not an independent rule of inference, of course, as it follows from *modus ponens*. In words, (1.4) states that if $P \Rightarrow Q$ is true, and $\sim Q$ is true, meaning Q is false, then $\sim P$ is also true, or P false.

In some proofs, the direct statement lends itself more easily to a proof, in others, the contrapositive works more easily, while in others still, both are easy, and in others still yet, both seem to fail miserably. The only general rule is, if the method you are attempting is failing, try the other. Experience with success and failure improves the odds of identifying the more expedient approach on the first attempt.

For example, assume that we wish to prove $P \Rightarrow Q$, where

$$P : a = b,$$

$$Q : a^2 = b^2.$$

The direct proof might proceed as

$$a = b \Rightarrow [a^2 = ab \text{ and } ab = b^2] \Rightarrow a^2 = b^2.$$

The contrapositive proof proceeds by first identifying the statement negations

$$\sim P : a \neq b,$$

$$\sim Q : a^2 \neq b^2,$$

and constructing the proof as

$$\begin{aligned} \sim Q &\Rightarrow a^2 - b^2 \neq 0 \\ &\Rightarrow (a+b)(a-b) \neq 0 \\ &\Rightarrow [(a+b) \neq 0 \text{ and } (a-b) \neq 0] \\ &\Rightarrow a \neq b. \end{aligned}$$

In the last statement we also can conclude that $a \neq -b$, but this is extra information not needed for the given demonstration.

Once a choice is made between the direct statement and its contrapositive, there are two common methods for proving the truth of the resulting implication. To simplify notation, we denote the implication to be proved as $A \Rightarrow C$, where A denotes either P or $\sim Q$, and C denoted either Q or $\sim P$, respectively.

The Direct Proof

The first approach is what we often think of as the use of "deductive" reasoning, whereby if we cannot prove $A \Rightarrow C$ in one step, we may take two or more steps. For example, proving that for some statement B that $A \Rightarrow B$ and $B \Rightarrow C$, it would seem transparent that $A \Rightarrow C$. One expects that such a partitioning of the demonstration ought to be valid, independent of how many intermediate implications are developed, and indeed this is the case. It is based on a result in logic that is called a **syllogism** and forms the basis of what is known as a **direct proof**. Specifically, we have that

$$(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C) \tag{1.5}$$

is a tautology. That is, for any assignment of the truth values to A, B, and C, this statement has constant truth value of "true."

This direct method is very powerful in that it allows the most complicated implications to be justified through an arbitrary number of smaller, and more easily proved, implications. In the proof above that $P \Rightarrow Q$, this method was in fact used without mention as follows:

$A : a = b,$

$B : a^2 = ab \wedge ab = b^2,$

$C : a^2 = b^2.$

Proof by Contradiction

The second approach to proving an implication is considered an **indirect proof**, and is also known as **reductio ad absurdum**, as well as **proof by contradiction**. In its simplest terms, proof by contradiction proceeds as follows:

To prove P, assume $\sim P$. If $R \wedge \sim R$ is derived for any R, deduce P.

In other words,

If $\sim P \Rightarrow (R \wedge \sim R)$, then P.

If $\sim P \Rightarrow (R \wedge \sim R)$ is true, then since $R \wedge \sim R$ is always false, it must be the case that $\sim P$ is also false, and hence P is true. The logical structure of this is the tautology

$$[\sim P \Rightarrow (R \wedge \sim R)] \Rightarrow P. \tag{1.6}$$

Remark 1.1 *It is often the case that in a given application, what is called a proof by contradiction appears as*

$$\textit{If } \sim P \Rightarrow R, \textit{ and } R \textit{ is known to be false, then } P. \tag{1.7}$$

For example, one might derive that $\sim P \Rightarrow R$, where R is the statement $1 \neq 1$. Implicitly, the truth of the statement $\sim R$, that $1 = 1$, does not need to be explicitly identified, but is understood. Also note that the truth of a statement like $1 = 1$ does not need to "follow" in some sense from the statement $\sim P$. That (1.7) is a valid conclusion can also be formalized by explicitly identifying the truth of $\sim R$ in the tautology

$$[(\sim P \Rightarrow R) \wedge \sim R] \Rightarrow P,$$

which except for notation is equivalent to modus tollens in (1.4). This approach also justifies the terminology of a reductio ad absurdum, namely from the assumed truth of $\sim P$ one deduces an absurd conclusion, R, such as $1 \neq 1$.

The indirect method of proof may appear complex, but with some practice, it is quite simple. The central point is that for any statement R, it is the case that $R \wedge \sim R$ is always false. This is because its negation, $\sim R \vee R$, is always true and

$$\sim (R \wedge \sim R) \Leftrightarrow \sim R \vee R \tag{1.8}$$

is a tautology. That is, for any statement R, either R is true or $\sim R$ is true. This is known as the **law of the excluded middle**.

Before formalizing this further, let's apply this approach to the earlier simple example, taking careful steps:

Step 1 State what we seek to prove: $a = b \Rightarrow a^2 = b^2$.

Step 2 Develop the negation of this implication. Looking at the truth table for the conditional, an implication $A \Rightarrow C$ is false only when A is true, and C is false. So the negation of what we seek to prove is

$$a = b \quad \text{and} \quad a^2 \neq b^2.$$

Step 3 What can we conclude from this assumed statement? This amounts to "playing" with some mathematics and seeing what we get:

$$
\begin{aligned}
a^2 \neq b^2 &\Leftrightarrow a^2 - b^2 \neq 0 \\
&\Leftrightarrow (a+b)(a-b) \neq 0 \\
&\Leftrightarrow a+b \neq 0 \quad \text{and} \quad a-b \neq 0,
\end{aligned}
$$

whereas

$$a = b \Leftrightarrow a - b = 0.$$

Step 4 Identify the contradiction: we have concluded that both $a - b = 0$ and $a - b \neq 0$.

Step 5 Claim victory: $a = b \Rightarrow a^2 = b^2$ is true.

Admittedly, this may look like an ominous process, but with a little practice the logical sequence will become second nature. The payoff to practicing this method is that this provides a powerful and frequently used alternative approach to proving statements in mathematics as will be often seen in later chapters.

Summarizing, we can rewrite (1.6) in the way it is most commonly used in mathematics, and that is when the statement P is in fact an implication $A \Rightarrow C$. To do this, we use the result from step 2 as to the logical negation of an implication. That is,

$$\sim(A \Rightarrow C) \Leftrightarrow A \wedge \sim C.$$

It is also the case that the most common contradiction one arrives at in (1.6) is not a general statement R, but as in the example above, it is a contradiction about A. We express this result first in the common form:

$$\text{If } (A \wedge \sim C) \Rightarrow \sim A, \text{ then } A \Rightarrow C. \tag{1.9}$$

Tautology: $[(A \wedge \sim C) \Rightarrow \sim A] \Rightarrow (A \Rightarrow C)$.

In the more general case,

$$\text{If } (A \wedge \sim C) \Rightarrow R \wedge \sim R, \text{ then } A \Rightarrow C. \tag{1.10}$$

Tautology: $[(A \wedge \sim C) \Rightarrow (R \wedge \sim R)] \Rightarrow (A \Rightarrow C)$.

Remark 1.2 *As in remark 1.1 above, (1.10) can also be applied in the context of $(A \wedge \sim C) \Rightarrow R$, where R is known to be false. The conclusion of the truth of $A \Rightarrow C$ again follows.*

Proof by Induction

A **proof by induction** is an approach frequently used when the statement to be proved encompasses a (countably) infinite number of statements (more on countably infinite sets in chapter 2 on number systems). A somewhat complicated example is the statement in the introduction: For any two integers M and N, we have that $M + N = N + M$. This is complicated because this statement involves two general quantities, and each can assume an infinite number of values. In other words, this statement is an economical way of expressing an infinite number of equalities ($1 + 9 = 9 + 1$, $-4 + 37 = 37 + (-4)$, etc.).

A simpler example involving only one such quantity is as follows:

$$\text{If } N \text{ is a positive integer, then } 1 + 2 + \cdots + N = \frac{N(N+1)}{2}. \tag{1.11}$$

This has the form of an equality, $P = Q$, but neither P nor Q is a simple declarative statement. Instead, both are indexed by the positive integers. That is, we seek to prove

$$\forall N, P(N) = Q(N), \tag{1.12}$$

where we define

$$P(N) = 1 + 2 + \cdots + N,$$

$$Q(N) = \frac{N(N+1)}{2}.$$

Obviously, for any fixed value of N, the proof requires no general theory, and the result can be demonstrated or contradicted by a hand or computer calculation. A proof by induction provides an economical way to demonstrate the validity of (1.12) for all N. The idea can be summarized as follows:

$$\begin{aligned} &\text{If} \quad P(1) = Q(1), \\ &\text{and} \quad [P(N) = Q(N)] \Rightarrow [P(N+1) = Q(N+1)], \\ &\text{then} \quad \forall N, P(N) = Q(N). \end{aligned} \tag{1.13}$$

In other words, proof by induction has two steps:

Step 1 (Initialization Step) Show the statement to be true for the smallest value of N needed, say $N = 1$ (sometimes $N = 0$).

Step 2 (Induction Step) Show that if the result is true for a given N, it must also be true for $N + 1$.

The logic is self-evident. From the initialization step, the induction step assures the truth for $N = 2$, which when applied again assures the truth of $N = 3$, and so forth.

Example 1.3 *To show (1.11), we see that the result is apparently true for $N = 1$. Next, assuming the result is true for N, we get*

$$\begin{aligned}1 + 2 + \cdots + N + (N+1) &= \frac{N(N+1)}{2} + N + 1 \\ &= \frac{N(N+1)}{2} + \frac{2(N+1)}{2} \\ &= \frac{(N+1)(N+2)}{2},\end{aligned}$$

which is the desired result.

*1.6 Mathematical Logic

Mathematical logic is one of the most abstract and symbolic disciplines in mathematics. This is quite deliberate. As exemplified above, the goal of mathematical logic is to define and develop the properties of deductive systems that are context free. We cannot be certain that a given logical development is correct if our assessment of it is encumbered by our intuition in a given application to a field of mathematics. So the goal of mathematical logic is to strip away any hint of a context, eliminate all that is familiar in a given theory, and study the logical structure of a general, and unspecified, mathematical theory.

To do this, mathematical logic must first erase all familiar notations that imply a given context. Also its symbolic structure needs to be very general so that it allows application to a wide variety of mathematical disciplines or contexts. As a result mathematical logic is highly symbolic, highly stylized, leaving the logician with nothing to guide her except the rules allowed by the structure. This way every deduction can be verified mechanically, effectively as an appropriately structured computer program. This program then declares a symbolic statement to be "true" if and only if it is able to construct a symbol sequence, using only the axioms or assumed facts and rules of inference that results in the deductive construction of the statement. No context is assumed, and no intuition is needed or desired.

The preceding section's informal introduction to the mathematical logic of statements, which is referred to as **statement calculus** or **propositional logic**, is a small subset of the discipline of mathematical logic. The axiomatic structure of statement calculus includes:

1. Certain **formal symbols** made up of **logical operators** ($\sim$ and $\Rightarrow$, but excluding $\forall$ and $\exists$), **punctuation marks** (e.g., parentheses), and other **symbols** that are undefined, but in terms of which other needed concepts such as **variable**, **predicate**, **formula**, **operation**, **statement**, and **theorem** are defined.

2. **Axioms** that identify the basic formula structures that will be assumed true.

3. A **rule of inference**: *modus ponens*.

The resulting theory can then be shown to be **complete** because it is **decidable**. The algorithm for determining if a given statement is true or not is the construction of the associated truth table, any one of which requires only a finite number of steps to develop. The key to this result is that a statement is a theorem in statement calculus, meaning it can be deduced from the axioms with *modus ponens* if and only if the statement is a tautology in the sense of the associated truth table.

For many areas of mathematics, however, statement calculus is insufficient in that it excludes statements of the form

$$\forall x P(x) \quad \text{or} \quad \exists x P(x)$$

that are central to the statements in most areas of mathematics. The mathematical theory developed to accommodate these notions is called **first-order predicate calculus**, or simply **first-order logic**.

Landmark results in first-order logic are **Gödel's incompleteness theorems**, published in 1931 by **Kurt Gödel** (1906–1978). Although far beyond the boundaries of this book, the informal essence of Gödel's first theorem is this: In any consistent first-order theory powerful enough to develop the basic theory of numbers, one can construct a true statement that is not provable in this system. In other words, in any such theory one cannot hope to confirm or deny every statement that can be made within the theory, and hence every such theory is "incomplete."

The informal essence of Gödel's second theorem is this: In any consistent first-order theory powerful enough to develop the basic theory of numbers, it is impossible to prove consistency from within the theory. In other words, for any such theory the proof of consistency will of necessity have to be framed outside the theory.

1.7 Applications to Finance

The applications of mathematical logic discussed in this chapter to finance are both specific and general. First off, there are many specific instances in finance when one has to develop a proof of a given result. Typically the framework for this proof is not a formally stated theorem as one sees in a research paper. The proof is more or less an application of, and sometimes the adaptation of, a given theory to a situation not explicitly anticipated by the theory, or entirely outside the framework anticipated.

Alternatively, one might be developing and testing the validity of a variety of hypothetical implications that appear reasonable in the given context. In such specific applications the investigation pursued often requires a very formal process of derivation, logical deduction, and proof, and the tools described in the sections above can be helpful in that they provide a rigorous, or at least semi-rigorous, framework for such investigations.

More specifically, a truth table can often be put to good use to investigate the validity of a subtle logical derivation involving a series of implications and, based on the various identities demonstrated, to provide alternative approaches to the desired result. For example, a proof by contradiction applied to the contrapositive of the desired implication can be subtle in the language provided by the context of the problem. Just as in mathematics, isolating the logical argument from the context provides a better framework for assessing the former without the necessary bias that the latter might convey. In addition, when the investigation ultimately reduces to the proof of a given implication, as often arises in an attempt to evaluate the truth of a reasonable and perhaps even desired implication, the various methods of proof provide a framework for the attack.

There is also a general application of the topics in this chapter to finance, and more broadly, any applied mathematical discipline, and that is as a cautionary tale. All too often the power and rigor of mathematics is interpreted to imply a certain robustness. That is, one assumes that the true results in mathematics are "so true" that they are robust enough to remain true even when one alters the hypotheses a bit, or is careless in their application to a given situation. Actually nothing could be further from the truth.

The most profound thought on this point I recall was made long ago by my thesis advisor and mentor, **Alberto P. Calderón** (1920–1998), during a working visit made to his office. What he said on this point, as perhaps altered by less than perfect recall, was: "*The most interesting and powerful theorems in mathematics are just barely true.*" In other words, the conclusions of the "best theorems" in mathematics are both solid in their foundation and yet fragile; they represent a delicate relationship between the assumed hypothesis and the proved conclusion. In the "best" theorems the hypothesis is in a sense very close to the minimal assumption needed for the conclusion, or said another way, the conclusion is very close to the maximal result possible that follows from the given hypothesis. The "more true" a theorem is, in the sense of excessive hypotheses or suboptimal conclusions, the less interesting and important it is. Such theorems are often revisited in the literature in search of a more refined and economical statement.

The implication of this cautionary tale is that it is insufficient to simply memorize a general version of the many results in mathematics without also paying close attention to the assumptions made to prove these results. A slight alteration of the assumptions, or an attempt to broaden the conclusions, can and will lead to periodic disasters. But more than just the need to carefully utilize known results, it is important to understand the proof of how the given hypotheses provide the given conclusions since, in practice, the researcher is often attempting to alter one or the other, and evaluate what part of the original conclusion may still be valid.

The snippets of mathematical history alluded to in this chapter, and the paradoxes, support this perspective of the fragility of the best results, and the care needed to get them right and in balance. As careful as mathematicians were in the development of their subjects, pitfalls were periodically identified and ultimately had to be overcome. And perhaps it is obvious, but a great many of these mathematicians were intellectual giants, and leaders in their mathematical disciplines. The pitfalls were far less a reflection of their abilities than a testament to the subtlety of their discipline.

As a simple example of this cautionary tale, it is important that in any mathematical pursuit, any quantitative calculation, and any logical deduction, one must keep in mind that the truth of statement Q as promised by *modus ponens*, depends on **both** the truth of the hypothesis P and the truth of the implication $P \Rightarrow Q$. The truth of the latter relies on the careful application of many of the principles discussed above, and it is often the focus of the investigation. But *modus ponens* cautions that equally important is to do what is often the more tedious part of the derivation, and that is to check and recheck the validity of the assumptions, the validity of P.

A simple example is the principle of **arbitrage**, which tends to fascinate new finance students. In an arbitrage, one is able to implement a market trade at no cost, that is risk free over some period of time, and with positive likelihood of producing a profit at the end of the period and no chance of loss. Invariably, students will perform long, detailed, and very creative calculations that identify arbitrages in the financial markets. In other words, they are very detailed and creative in their derivations of the truths of the statements $P \Rightarrow Q$, where in their particular applications, P is the statement "I go long and short various instruments at the market prices I see in the press or online," and Q is the statement "I get embarrassingly rich as the profits come rolling in."

Of course, the poorly trained students make mistakes in this proof of $P \Rightarrow Q$, using the wrong collection of instruments, or not identifying the risks that exist post trade. But the better students produce perfect and sometimes subtle trade analyses. Invariably the finance professor is left the job of bursting bubbles with the question: "How sure are you that the securities are tradable at the prices assumed?" In other words, how sure are you that P is true?

The answer to this question comes from a logical analysis of the following argument using syllogism and *modus tollens*:

If finance students' arbitrages worked,
there would be numerous, embarrassingly rich finance students.
If finance students could trade at the assumed prices,
their arbitrages would work.
There are not numerous embarrassingly rich finance students.

Exercises

Practice Exercises

1. Create truth tables to evaluate if the following statements, $A \Leftrightarrow B$, are tautologies:

(a) $P \vee Q \Leftrightarrow {\sim}P \Rightarrow Q$

(b) $(P \vee Q) \vee (P \Rightarrow Q) \Leftrightarrow P \wedge Q$

(c) $(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

(d) $[P \Rightarrow (Q \vee R)] \wedge [Q \Rightarrow (P \vee R)] \Leftrightarrow R$

2. It was noted that the truth of $P \Rightarrow Q$ does not necessarily imply the truth of Q. Confirm this with a truth table by showing that $(P \Rightarrow Q) \Rightarrow Q$ is not a tautology. Create real world applications by defining statements P and Q illustrating a case where $(P \Rightarrow Q) \Rightarrow Q$ is true, and one where it is false.

3. The contrapositive provides an alternative way to demonstrate the truth of the implication $P \Rightarrow Q$. Confirm that $(P \Rightarrow Q) \Leftrightarrow ({\sim}Q \Rightarrow {\sim}P)$ is a tautology. Give a real world example.

4. Confirm that the structure of the proof by contradiction,

$$[(A \wedge {\sim}C) \Rightarrow {\sim}A] \Rightarrow (A \Rightarrow C),$$

is a tautology.

5. Comedically, the logical deduction

$$[(P \Rightarrow Q) \wedge Q] \Rightarrow P \tag{1.14}$$

is known as **modus moronus**. Show that this statement is not a tautology, and provide a real world example of statements P and Q for which the hypothesis is true and conclusion false.

6. Show by mathematical induction that for any integer $n \geq 0$:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

7. Develop a direct proof of the formula in exercise 6. (*Hint*: Define $S = \sum_{i=0}^{n} 2^i$, consider the formula for $2S$, and then subtract.)

8. Develop a proof by contradiction in the form of (1.6) of the formula in exercise 6. (*Hint*: The formula is apparently true for $n = 0, 1, 2$, and other values of n. Let N be the first integer for which it is false. From the truth for $n = N - 1$, and falsity for $n = N$, conclude that $2^N \neq 2^N$ and recall the remark after (1.6).)

9. It is often assumed that the initialization step in mathematical induction is unnecessary, and that only the induction step need be confirmed. Show that the formula

$$\sum_{i=0}^{n} 2^i = 2^{n+1} + c$$

satisfies the induction step for any c, but that only for $c = -1$ does it satisfy the initialization step.

10. Show by mathematical induction that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

11. A bank has made the promise that for some fixed $i > 0$, an investment with it will grow over every one-year period as $F_{j+1} = F_j(1 + i)$, where F_j denotes the fund at time j in years. Prove by mathematical induction that if an investment of F_0 is made today, then for any $n \geq 1$,

$$F_n = F_0(1 + i)^n.$$

12. Develop a proof using *modus tollens* in the structure of (1.4) that if at some time n years in the future, the bank communicates $F_n \neq F_0(1+i)^n$, then the bank at some point must have broken its promise of one-year fund growth noted in exercise 11. (*Hint*: Define $P : F_{j+1} = F_j(1+i)$ for all j; $Q : F_n = F_0(1+i)^n$ for all $n \geq 1$. What can you conclude from $(P \Rightarrow Q) \wedge \sim Q$?)

Assignment Exercises

13. Create truth tables to evaluate if the following statements, $A \Leftrightarrow B$ or $A \Rightarrow B$, are tautologies:

(a) $P \wedge Q \Leftrightarrow \sim(P \Rightarrow \sim Q)$

(b) $(P \vee Q) \wedge \sim Q \Rightarrow P$

(c) $(P \Rightarrow Q) \wedge (P \wedge R) \Rightarrow Q \wedge R$

(d) $\sim P \vee (Q \wedge R) \Leftrightarrow (\sim R \vee \sim Q) \wedge P$

14. *Modus ponens* identifies the necessary additional fact to convert a proof of the truth of the implication, $P \Rightarrow Q$, into a proof of the conclusion, Q. Confirm that $P \wedge (P \Rightarrow Q) \Rightarrow Q$ is a tautology. Demonstrate by real world examples as in exercise 2 that while $(P \Rightarrow Q) \Rightarrow Q$ can be true or false, $P \wedge (P \Rightarrow Q) \Rightarrow Q$ is always true.

15. Show that *modus ponens* combined with the contrapositive yields $\sim Q \wedge (P \Rightarrow Q) \Rightarrow \sim P$, and show directly that this statement is a tautology. Give a real world example.

16. Identify and label (A, B, etc.) the statements in the argument at the end of this chapter, convert the argument to a logical structure, and demonstrate what conclusion can be derived using syllogism and *modus tollens*.

17. Show by mathematical induction that for $i > 0$ and integer $n \geq 1$,

$$\sum_{j=1}^{n}(1+i)^{-j} = \frac{1-(1+i)^{-n}}{i}.$$

18. Develop a direct proof of the formula in exercise 17. (*Hint*: See exercise 7.)

19. Show by mathematical induction that

$$\sum_{j=1}^{n} j^3 = \left[\sum_{j=1}^{n} j\right]^2.$$

20. A bank has made the promise that for some fixed $i > 0$, an investment with it will grow over every one-year period as $F_{j+1} = F_j(1+i)$, where F_j denotes the fund at time j in years. Develop a proof by contradiction in the form of (1.9) that for any $n \geq 1$,

$$F_n = F_0(1+i)^n.$$

(*Hint*: Define $A : F_{j+1} = F_j(1+i)$ for all $j \geq 0$; $C : F_n = F_0(1+i)^n$ for all $n \geq 1$. If $A \wedge \sim C$ and N is the smallest n that fails in C, what can you conclude about F_N, which provides a contradiction, and about the conclusion $A \Rightarrow C$?)