

in the mathematician's knowledge of them. He even says: "Some such view must be correct." He would hardly have felt in a position to make such a strong claim without argument had there been a respectable rationalist theory of knowledge around. Accordingly, it is not unreasonable to expect that a resuscitated rationalist theory of knowledge can go a long way in defending realism against the charge that it cannot explain mathematical knowledge. In any case, a rationalist epistemology is realism's only hope of explaining how we can have knowledge of objects with which we cannot causally interact.

One of the major themes of this book is the inseparability of realism and rationalism. Realism without rationalism is unbelievable and rationalism without realism is unstable. We have seen how implausible realism can be made to seem when its critics are allowed to assume that an account of mathematical truth has to meet an epistemic requirement set in terms of an empiricist theory of knowledge. We will see in the next chapter how, without rationalism, realism easily slides over into a form of antirealism. The integration of realism and rationalism in a single position provides realism with epistemological credibility and rationalism with ontological stability.

Here is the layout of the book. The core of its argument is contained in chapters 1, 2, 4, and 5. Their agenda is to show that the apparent force of the principal objections to realism rests on the implicit "divide and conquer" strategy which excludes rationalism from the defense of realism and realism from the defense of rationalism. I will argue that once they are integrated into a single position, there are strong replies to these objections. The replies to the epistemological and semantic objections are a matter of providing a comprehensive defense for Gödel's formulation of realism. In the case of the epistemological objection, the defense must supply an appropriate rationalist theory of knowledge. This would block the much too fast dismissal of realism on the grounds that taking numbers to be abstract objects makes them unknowable. What is true is only that they are unknowable on the basis of an empiricist epistemology.

In the case of the semantic objection, the defense must supply an appropriate intensionalist semantics. This would block arguments from the symmetry of intended and deviant interpretations within the mathematical sphere to the indeterminacy of reference to numbers.

In the case of the ontological objection to realism, the defense is a matter of developing a new ontological theory within which the traditional abstract/concrete distinction can be coherently drawn. This new ontological theory turns out to have significant bearings on many of the sciences as well as on a number of philosophical topics.

The first two chapters are a reply to the epistemological challenge to realism. Chapter 1 is concerned with preliminaries and chapter 2 contains the reply to Benacerraf (1983 [1973]) and the philosophers who have tried to turn his argument into a refutation of mathematical realism. It is a testament to the force of his statement of the epistemological criticism that it has also convinced a large number of uncommitted philosophers, many of whom recognize realism's evident philosophical strengths and considerable *prima facie* plausibility in actual mathematics. For example, there is a body of substantial arguments for mathematical realism, such as Frege's (1953, 1964) arguments in the *Grundlagen* and *Grundgesetze* and Benacerraf's (1983 [1973]) own argument in "Mathematical Truth," to the effect that, without mathematical realism, we can't have the same (Tarskian) semantics for mathematical sentences that we have for other sentences and it is then unclear what we are to say about the semantics of mathematical sentences.

Chapter 3 is a digression from the main line of argument, but it complements the reply to the epistemological challenge to realism in chapter 2 by posing an epistemological challenge to antirealists. It argues that antirealists face the epistemological challenge of explaining the special certainty of mathematical and logical knowledge, that Quine's response fails, and, as a consequence, that antirealists stand no chance of meeting this challenge. If both the reply to the epistemological challenge to realism and this argument against antirealism work, we will have met the epistemological challenge to realism in a way that shows that it is the antirealist rather than the realist who faces an apparently insurmountable epistemic challenge. Hence, if the argument up to this point is correct, the proper attitude toward realism and antirealism ought to be the very opposite of what has been the received opinion in contemporary philosophy of mathematics. The doubts about the prospects for an adequate epistemology that have been widely directed toward realism are more appropriately directed toward antirealism.

Chapter 4 replies to the semantic challenge to realism in Benacerraf's (1983 [1965]) "What Numbers Could Not Be." I take a novel approach to Benacerraf's argument. I first develop a strategy for blocking indeterminacy arguments generally and then show that Benacerraf's argument for the indeterminacy of reference to numbers is a special case of such indeterminacy arguments. The strategy blocks not only Benacerraf's symmetry claim about intended and deviant interpretations of arithmetic but also related symmetry arguments such as those in Quine's argument for the indeterminacy of translation, Kripke's rule-following argument, and Putnam's argument for global referential indeterminacy.

Chapter 5 replies to an ontological challenge to realism. This is a challenge to the coherence of realism based on examples that a number of recent philosophers, including some realists, have thought to undermine the traditional abstract/concrete distinction. Some are easily handled on the basis of considerations that have long been part of the realist position but the relevance of which has been overlooked in connection with the putative counterexamples. Others, particularly Frege's (1953, 35) famous equator example, are more difficult and far more interesting, requiring a significant addition to ontological theory. I will argue not only that this addition shows that the distinction is not undermined, but also that it provides a new ontological theory with applications to philosophical and scientific questions.

As we noted above, it would be bad news for philosophy as a whole if nothing works in the philosophy of mathematics. If the line of argument in chapters 1–5 is correct, the news that something works in the philosophy of mathematics ought to be good news for philosophy as a whole.

Chapter 6 presents a rationalist metaphilosophy. It develops it out of the principles underlying the arguments of the previous chapters. The first section of the chapter explains how the rationalist epistemology in chapter 2 for knowledge in the formal sciences can be extended to provide a rationalist epistemology for certain types of philosophical knowledge as well. Our aim is to construct a unified conception of what it is to explain synthetic *a priori* knowledge in the formal sciences and in their philosophical foundations. The second section of the chapter examines some of the philosophical implications of the metaphilosophy. Here I try to set out some new thoughts about the rationalism/empiricism controversy over the existence of synthetic *a priori* knowledge, Carnap's positivist critique and Quine's naturalist critique of metaphysical philosophy, the philosophical distinction between internal and external questions, and the place of skepticism in a world of knowledge.

Chapter 1

Philosophical Preliminaries

1.1 *The Framework*

Poets often comment on the multifariousness of things. Hamlet's rebuke of Horatio is familiar, and Louis MacNeice, in his poem "Snow", tells us that "[t]he world is crazier and more of it than we think, incorrigibly plural." Realists in the philosophy of mathematics make a specific claim about just how much crazier and more populous the world is than the familiar classification "animal, vegetable, or mineral" suggests. Realists think there are things that (necessarily) have neither spatial nor temporal location: abstract objects, such as numbers, sets, propositions, proofs, sentences, and meanings.

Being an object that (necessarily) has no spatial or temporal location is the core of the conception of an abstract object in realist thought from Plato to Gödel. It is also how an abstract object is understood in antirealist criticisms of realism. Thus, Goodman and Quine (1947) write:

We do not believe in abstract objects. *No one supposes that abstract entities—classes, relations, propositions, etc.—exist in space-time; but we mean more than this. We renounce them altogether.* (Italics mine)

Furthermore, taken as the essential definition of "abstract object," this conception is the most compact one that fits the usage of both realists and their critics. Subtracting nonspatiality or atemporality makes the definition inadequate, since the resulting definition no longer captures the notion of abstractness, while adding causal inertness, mind-independence, or some other property that abstract objects are generally taken to have, makes the definition redundant, since the original definition already implies those properties (as will be argued below). Moreover, when we add the definition of "concrete object" as something that can possibly have a spatial or temporal location, we concisely capture the intuitive distinction between the abstract and concrete.

I will call the realist position I am defending in this book “general realism.” General realism is realism in general. It makes the indefinite claim that there are abstract objects. General realism is a view about ontology. In addition to general realism, there are particular realisms, such as mathematical realism, logical realism, and linguistic realism. A particular realism makes a claim that the domain of a formal science contains one or another kind of abstract object, e.g., numbers, propositions, or sentences. A particular realism is a view in the foundations of a formal science concerning what type of objects knowledge in that science is about.¹

What makes someone a realist is his or her acceptance of abstract objects; what makes someone a realist of a particular kind is his or her acceptance of abstract objects of that kind. Kinds here are kinds of structure that abstract objects have, e.g., mathematical, logical, or linguistic structure. Commonly, acceptance of abstract objects of a certain kind is the result of accepting theories about abstract objects of that kind, but being a realist of a certain kind does not depend on having much theoretical knowledge. Plato was a realist about numbers before there was a theory of arithmetic.

The elucidation of the kinds of structure abstract objects have is the task of pure mathematics, pure logic, and pure linguistics. A pure science is, according to our realism, pure because it is about abstract objects pure and simple. Applied sciences are distinguished from pure ones by their concern with concrete objects, but what specific kind of objects those sciences are about is a more complex question, which we shall consider in chapter 5.

If any particular realism is true, general realism is true, but, of course, the truth of general realism does not entail the truth of any particular realism. Moreover, philosophers all of whom would subscribe to general realism might hold different ontological opinions about different particular realisms. Someone might, for example, be a logical and mathematical realist without being a linguistic realist. Frege (1964, 13) took such a position. Some mathematicians seem to be arithmetic realists, but not geometric realists. It might even be possible for someone—Chomsky (1986, 33) seems to be an example—to be a linguistic conceptualist while being a mathematical realist, but there is room for doubt about this because objections to realism in one area seem to apply to realism in other areas. However, some particular realisms are clearly

1. For the time being, I will use “formal science” to denote mathematics, logic, and linguistics, without implying any doctrine about the nature of those sciences. At the end of chapter 3, I will suggest such a doctrine.

interdependent. For example, inscriptionalist nominalists like Goodman and Quine (1947) could not be linguistic realists.

This way of formulating realist and antirealist positions has important consequences for the ontological controversies in the foundations of the various formal sciences. It suggests that the widespread practice of evaluating ontological positions exclusively within the foundations of the directly relevant formal science could be a mistake for one or another of the positions. It can be argued that isolating the controversy between mathematical realists and mathematical antirealists from ontological controversies in the foundations of the other formal sciences has kept realists from making as strong a case for their position as they could.

An example is the following. Inscriptionalist nominalists in mathematics claim that we can do justice to mathematical practice without countenancing abstract objects by taking mathematics to be about mathematical expressions. Such nominalists must mean expression tokens, since expression types—in the standard Peircean sense—are abstract objects. But, since there are not enough actual expression tokens for mathematics to be about them, the nominalist's program in mathematics requires, at the very least, some way of characterizing the class of possible tokens of mathematical expressions on the basis of a sample of actual tokens. Now, this nominalist program in mathematics is a special case of the Bloomfieldian (1936) nominalist program in linguistics. That program takes linguistic reality to be the acoustic phenomena of speech. Given how few sentences of a natural language are exemplified in actual speech, Bloomfieldian linguists had to construct a categorical structure that characterizes the possible sentence tokens of the language on the basis of procedures for segmenting and classifying the items in a corpus of linguistic tokens. Chomsky (1975), however, showed that no such procedures exist, because there is no way to take the inductive step from distributional properties of actual tokens to grammatical categories.

Since the program of the inscriptionalist nominalist in mathematics requires essentially the same bottom-up construction of essentially the same categorical structure, Chomsky's argument applies equally to that program, putting the inscriptionalist nominalist in mathematics in the position of having to do something shown to be a lost cause in linguistics. If these considerations are right, mathematical realists have missed the opportunity to strengthen their case against one form of antirealism.

In this book, I will assume general realism for the sake of argument. The assumption begs no questions, because the main line of argument here is not to establish realism but only to defend it against criticisms

that we cannot have knowledge of abstract objects, that we cannot determinately refer to them, and that we cannot distinguish them from concrete objects. For example, the assumption that there are abstract objects bears no weight in the explanation of how we can have knowledge of abstract objects, since the antirealist who challenges the realist to explain how we can have such knowledge without contact assumes realism for the sake of argument—otherwise the antirealist would have no ontological position to challenge.

In the same spirit, we can also assume mathematical, logical, and linguistic realism. The epistemological, semantic, and ontological challenges to particular realisms arise from their parallel claims that the objects in the domain of their various sciences are abstract, not from any aspect of their mathematical, logical, or linguistic structure. For example, since the question of how we know about objects of one kind with which we can have no contact does not differ from the question of how we can know about objects of other kinds with which we also can have no contact, an epistemology that answers the epistemological challenge for one kind of abstract object answers it for all kinds. This does not mean that the epistemologies for mathematical, logical, and linguistic knowledge, and even for varieties of such knowledge, such as arithmetic and geometric knowledge, will not differ from one another in various ways. But such differences will not affect the general question.

It is not an aim of this book to provide a comprehensive argument for general realism or any particular realism. Explaining how knowledge of abstract objects is possible, how reference to them can be determinate, and how they differ from concrete objects does not establish that there are such objects. Nonetheless, at various points in the course of the book, I will try to strengthen the case for realism by supplying reasons to think that abstract objects exist and by exposing weaknesses of one or another form of antirealism. For example, in section 2.2, I will present two philosophical objections to Hartry Field's (1980) antirealism, and in chapter 2, I will strengthen Benacerraf's (1983 [1973]) argument that realism provides a better account of the semantics of mathematical sentences than antirealism.

Even though it is not our aim to argue for realism, we should explain what form of argument the realist can give for the existence of abstract objects. To establish general realism, it suffices to establish mathematical realism, logical realism, or linguistic realism. The argument for establishing one of them is an argument to show that the particular realism in question is preferable to its rival particular nominalisms and conceptualisms as an account of the objects of knowledge in the relevant formal science. Hence, to explain how a realist can argue

systematically for the existence of abstract objects in the domain of a formal science, we have to look at what is involved in showing that realism provides the best account of the objects of knowledge in that science.

A particular realism is an ontological position in the foundations of a particular formal science. We can distinguish the foundations of mathematics, the foundations of logic, and the foundations of linguistics from mathematics proper, logic proper, and linguistics proper. The former are branches of the philosophy of science, concerned with a philosophical understanding of the results in the particular sciences proper. Mathematicians, logicians, and linguists, like other scientists, typically conduct their professional business with little interest in the philosopher's concern with the nature of numbers, sets, propositions, sentences, and so on. The attempt of philosophers of science to understand the nature of such entities takes the form of a dialectic among nominalists, conceptualists, and realists, where the issue is which position makes the best scientific and philosophical sense of the science proper.

Mathematics, logic, and linguistics tell us that statements like (1)–(4) are true:

- (1) There is a perfect number less than seven.
- (2) There are propositions that imply everything.
- (3) There are English sentences with no phonologically realized subject.
- (4) There are infinitely many numbers (propositions, sentences).

In virtue of our accepting what mathematics, logic, and linguistics tell us, philosophers are committed to the existence of numbers, propositions, and sentences. Philosophers who accept the formal scientist's claims about numbers, propositions, or sentences are required to acknowledge that there are such objects. But that acknowledgment does not require taking a stand on the issue of what kind of things numbers, propositions, and sentences are. Thus, it is mistaken to think that the ontology of mathematics is an issue that can be settled on the basis of a principle of ontological commitment. We may concede for the sake of argument that, as Quine (1961a, 13–14) says, "a theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order for the affirmations made in the theory to be true." But this concession is not enough to establish that "[c]lassical mathematics . . . is up to its neck in commitments to an ontology of abstract objects." The principle of ontological

commitment may show that classical mathematics is up to its neck in numbers, but it can't show that the numbers we are up to our necks in are abstract objects. This requires a philosophical argument to show that numbers are abstract rather than concrete objects.

Realist arguments for the existence of abstract objects are arguments that numbers, sets, propositions, proofs, sentences, meanings, or similar objects are abstract objects. By the same token, antirealist arguments against the existence of abstract objects are arguments against those objects' being abstract objects. Hence, as part of the philosophical dialectic in the foundations of the formal sciences, an argument for realism, as well as one for nominalism or conceptualism, is required to show that the ontology in question best accommodates the full range of facts in the formal science and best satisfies our philosophical intuitions. Thus, an argument that the objects of a formal science are abstract is successful just in case it shows that a realist ontology is best in these respects.²

All sides in this dialectic have a stake in seeing to it that the scientific conclusions from which their philosophical arguments proceed enjoy a large measure of independence from partisan philosophical intrusion. The more such intrusions that philosophers allow themselves, the more they open themselves to the charge that they are arguing from their own theory, and the less convincing their argument for their ontological position becomes. On the other hand, the more they base their ontological position on philosophically unadulterated scientific conclusions, the more their position can claim to have the backing of impartial science, and the more convincing it becomes. The recognition on all sides that they have a stake in keeping such intrusions to a minimum is the gyroscope that restores the balance of the dialectic whenever partisan ontological considerations intrude to deflect it from its proper course.

1.2 *Two Forms of Antirealism*

In this section, I consider briefly two forms of antirealism; in the next section, I want to consider three approaches to the foundations of the formal sciences that, at least as their advocates present them, are forms of realism.

2. The realist's claim that abstract objects exist is sometimes criticized as inchoate because, as the criticism runs, abstract objects are characterized exclusively on the basis of the negative property of not having spatial or temporal location. This is simply false. Their characterization includes various positive properties: being an object, having a formal structure, having the properties and relations in that structure necessarily, existing necessarily (if they exist), being objective, and being knowable (if they are knowable) on the basis of reason alone.

1.2.1 *The Kantian Compromise*

Kant sought a compromise between empiricism and rationalism because he thought neither can explain the full range of our knowledge, especially our mathematical knowledge. Rationalists went too far in allowing reason unrestricted speculative freedom in metaphysics, while empiricists went too far in trying to curb the excesses of reason in metaphysics. As Kant saw it, Hume's view that all genuine knowledge falls into either the category Matters of Fact or the category Relations of Ideas throws out mathematics along with metaphysics.

The arithmetic truth "Seven plus five equals twelve" falls outside the category Relations of Ideas because its predicate "is twelve" is not a component of the concept "seven plus five." Such truths, as Locke had observed, are not trifling, that is, not analytic in Kant's sense of literal concept containment, but express "real knowledge." Yet they also fall outside the category Matters of Fact because they are necessary. Experience, as Kant argued, can teach us that a judgment is true, but not that it couldn't be otherwise. Hence, mathematics as well as metaphysics falls between the two Humean stools. Accordingly, if Hume throws out metaphysics because of the epistemic status of its principles, he has to throw out mathematics too. Mathematical truth is as much a mystery for Humean empiricism as metaphysical truth. Humean empiricism explains why "Squares are rectangles" is true: the definition of "square" categorizes squares as a certain kind of rectangle. But it provides no more of a notion of why "Seven plus five equals twelve" is true than of why a metaphysical principle like "Every event has a cause" is. Kant was awakened from his dogmatic slumbers by the clatter of mathematics going out the window.

Rationalism, as Kant saw it, was principally responsible for the speculative excesses of metaphysics. On the one hand, it draws the sharpest possible distinction between the world and the cognitive faculties of our minds and, on the other hand, it imposes no curb on the use of those faculties. Rationalism thus allows us to use our reason to try to obtain knowledge of objects to which our faculty of sensible intuition can bear no relation. In this, reason's reach exceeds its grasp. This is what Kant saw as the source of the striking lack of progress in metaphysics, and he concluded that the window has to be shut both to keep mathematics in and also to keep metaphysical excesses out.

Kant's Copernican revolution nails the window shut. It makes the existence of objects in the world depend on our cognitive faculties. At the beginning of section 22 of the B-version of the *Transcendental Deduction*—entitled "The Category has no other Application in Knowledge than to Objects of Experience"—Kant ([1787] 1929) states his basic objection to the rationalist claim that we can have knowledge of objects to which our faculty of sensible intuition can bear no relation:

To *think* and to *know* an object are thus by no means the same thing. Knowledge involves two factors: first, the concept, through which an object in general is thought (the category); and secondly, the intuition, through which it is given. For if no intuition could be given corresponding to the concept, the concept would still indeed be a thought, so far as its form is concerned, but would be without any object, and no knowledge of anything would be possible by means of it. So far as I could know, there would be nothing, and could be nothing, to which my thought could be applied.

Kantian philosophy came to be highly influential in the foundations of mathematics. Brouwer ([1913] 1983) developed his “neo-intuitionism”—what is generally called “mathematical intuitionism”—on the basis of, as he put it, (Kant’s) “old intuitionism.” The new intuitionism came from the old by “abandoning Kant’s apriority of space but adhering the more resolutely to the apriority of time.” The mind divides the stream of moments of time into “qualitatively different parts,” thereby “creat[ing] not only the numbers one and two but also all finite ordinal numbers.” Kantian philosophy is also easily recognized in Chomsky’s (1965, 1986) conceptualist view of linguistics. Although Chomsky’s linguistic conceptualism is more sophisticated than Brouwer’s mathematical conceptualism in its account of how the mind creates sentences and is more scientific in its strongly biological flavor, it is the linguistic counterpart of Brouwer’s Kantian conception of mathematics.³

But Kant’s Copernican revolution rests on philosophical doctrines too dubious to ground particular conceptualisms like Brouwer’s and Chomsky’s. Everyone is familiar with the standard problems with transcendental idealism, such as the fact that Euclidean geometry, alleged to be an *a priori* necessary truth, turned out to be an *a posteriori* contingent falsehood. Not only did the fate of Euclidean geometry deal a heavy blow to confidence in the Kantian explanation of synthetic *a priori* knowledge generally, but the role of geometry in relativity theory provided a paradigm for treating other alleged *a priori* necessary truths as *a posteriori* contingent truths—highly theoretical but nonetheless empirical in nature. The proposal to replace classical logic with a more suitable quantum logic can be seen as based on that paradigm.

3. Here is another case where it is a good idea for realists to avoid the parochial practice of defending a particular realism in isolation from the controversies involving other particular realisms. For many of the arguments that I and Postal (1991) brought against Chomsky’s linguistic conceptualism can, with only trivial adaptation, be brought against Brouwer’s mathematical conceptualism.

Of more direct concern in the present context are two other problems. One is that Kant's transcendental idealism does not succeed in solving the problem about necessity that he raised in connection with empiricism. However Kant's transcendental idealism is understood, it locates the grounds of mathematical facts within ourselves in at least the minimal sense that it entails that such facts could not have existed if we (or other intelligent beings) had not existed. But, as Frege (1964, 1–25) pointed out, locating the grounds of necessity within us does not explain the necessity of mathematical truth. It at best explains why we naturally take mathematical truths to be necessary.⁴ In treating necessity as an aspect of the psychology of contingent beings, Kant shifts the basis of the explanation from mathematics to our contingent psychology. The latter provides no grounds to explain why the truths of mathematics couldn't be otherwise.

The other problem is the verificationism in Kant's position. It is clear from the last sentence in the preceding quote that he is saying that the possibility of objects—and hence the possibility of knowledge of them—depends on the possibility of verifying their existence through acquaintance in intuition or inner experience. But why is the fact that “no intuition could be given corresponding to [an] object” a reason to think that the concept couldn't have an object? Frege (1953, 101) famously observed that we can know a great deal about mathematical objects to which our faculty of sensible intuition can bear no relation:

Nought and one are objects which cannot be given to us in sensation. And even those who hold that the smaller numbers are intuitable, must at least concede that they cannot be given in intuition any of the numbers greater than $1000^{1000^{1000}}$, about which nevertheless we have plenty of information.

Kant himself does not provide an external reason to think that the possibility of objects of one or another kind depends on the possibility of our being able to verify their existence in intuition or inner experience. Attempting to motivate such verificationism from within transcendental idealism would beg the question, since the core

4. To see this, consider a contemporary Kantian position like Brouwer's ([1913] 1983). On Brouwer's intuitionist position, the justification for formal beliefs depends on some sort of introspective contact with internal objects. But, as Brouwer ([1913] 1983, 69) readily admits, the objects of mathematical knowledge are created by the mind out of mental stuff. Since the created objects share the contingency of their creator and the mental stuff of which they are created, mathematical conceptualism takes numerical relations to be contingent relations, and truths about such relations to be contingent truths. Psychologist accounts, to echo Frege, at best explain why we think of arithmetic truths as necessary, but not why they are necessary.

verificationist idea that what exists depends on our cognitive makeup is a basic assumption of transcendental idealism. Moreover, there does not seem to be any motivation for verificationism outside transcendental idealism either. The irredeemable failure of the most notable attempt to motivate verificationism independently of transcendental idealism, namely the positivists's linguistic attempt to equate the meaningfulness of a sentence with its verifiability and the meaning of a sentence with its verification conditions, is well known.⁵

Verificationists have failed to motivate an epistemic constraint on what there is, and no account of our cognitive faculties, scientific or otherwise, supports the idea that the limits of those faculties should be the touchstone of existence. Despite the appeal of verificationism as a quick refutation of metaphysics, verificationists have yet to explain why objective reality should have to pass a knowability test framed in terms of human knowledge. There is little behind verificationism but epistemic chutzpah.

1.2.2 Fictionalist Nominalism

In this subsection, I will present arguments against Field's (1980, 1989) fictionalist nominalism. Field's argument against mathematical realism is an attempt to turn Quine's and Putnam's indispensability argument against their conclusion that we are committed to abstract objects because they are indispensable in doing natural science. Field argues that natural science can be done without numbers, and, on the basis of this, that simplicity requires us to limit our ontology to natural objects.

From the standpoint of general realism, it is initially questionable for Field to base his case against realism on the dispensability of numbers for doing natural science. It is, of course, quite legitimate for Quine and

5. The equation is contravened by the simplest facts of meaning in natural language, e.g., the verifiability conditions of "John is five feet tall" and "If John were one foot taller, he would be six feet tall" are the same, but the sentences are not synonymous. Moreover, as a number of philosophers have observed, the equation condemns itself as meaningless because it is not verifiable. Some verificationists have tried to escape this consequence by saying that the equation is a convention, analytic, or meaningful in a noncognitive sense, as ethical injunctions are sometimes thought to be. Taking it as a convention pulls its teeth. Only if one is antecedently in agreement with the positivist is there any point in adopting *that* convention rather than one the metaphysician might propose. Taking the equation as analytic is not helpful either, since the only sense of analyticity on which it is analytic is one based on Carnapian meaning postulates, but again we can adopt anything we like as a meaning postulate. Finally, the positivist cannot get off the hook by taking the equation to have the force of an ethical injunction. A sentence expressing the equation still has to be meaningful in the cognitive sense. How can someone use the sentence to recommend that meaningfulness be taken as verifiability if the sentence has no cognitive content?

Putnam to use an indispensability argument restricted to natural science to argue for realism, but Field's use of dispensability to argue against realism begs the question. Unless Field assumes epistemological naturalism, the dispensability of reference to numbers in natural science does nothing to show that a commitment to abstract objects can be avoided, since they might be indispensable in formal science. Even if, as Field claims, we can do natural science without a commitment to abstract objects, this provides no reason to think that formal science can be done without such a commitment.

Field does not consider the prospect of establishing realism on the basis of an argument for the indispensability of abstract objects in pure mathematics, logic, or linguistics. What he (1980, viii, 6) says is that Quine's and Putnam's indispensability argument is the only serious argument for realism that he knows, and that other arguments for realism are "unpersuasive." But he doesn't say what other arguments he has considered or why he finds them "unpersuasive." There are serious arguments for mathematical, logical, and linguistic realism, from Frege's arguments to current arguments in the foundations of linguistics. No doubt Field finds them "unpersuasive," but, having provided no critical examination of them, the real work of refuting realism is yet to be done.

It may be here that a tacit assumption of epistemological naturalism enters Field's argument. But this assumption will again beg the question against the realist if there is no argument to support it. I can find no direct argument for the assumption in Field's work, but it is plausible to think that he would want to argue that there is simply no plausible alternative to epistemological naturalism. Field (1989, 59) thinks that realists have "to postulate some *aphysical connection*, some *mysterious grasping*," and hence he presumably would want to say that this rules out non-naturalist epistemologies. If this is the argument for epistemological naturalism, the next chapter will show that it is not a good one.

Field's positive view is that the truths in mathematics proper are truths in a fiction. The problem with this view has been missed because philosophers have been too parochial, failing to consider issues in the foundations of mathematics in relation to issues in the foundations of the other formal sciences. From an unparochial standpoint, a test of the adequacy of a view taken in the foundations of one formal science is whether the view is adequate in the foundations of other formal sciences. Hence, we ask: Is the sort of view that Field has about the truths of mathematics adequate as a view about the truths of linguistics? What happens when we try to carry it over to the foundations of linguistics?

The counterpart in the foundations of linguistics of Field's claim that there are no numbers is the claim that there are no sentences. The counterpart to Field's claim that mathematical truths like "Two plus two is four" are not about numbers is the claim that linguistic truths like "'Visiting relatives can be annoying' is ambiguous" are not about sentence types. Types are abstract objects. Further, the sentences that are intended to express Field's fictionalist nominalism can't be treated as tokens of grammatical types, since if there are no sentence types, there are no tokens of sentence types either. This raises the problem of whether sense can be made of the discourse in which Field expresses his mathematical fictionalism, say, *Science without Numbers*, Field's positive view. To avoid the consequence that his view is self-defeating because it makes its own linguistic expression impossible, Field has to interpret his discourse as consisting of concrete objects, deposits of ink on paper. But so construing the discourse drives Field's fictionalist nominalism right back to the dubious inscriptionalist nominalism that he originally—and for good reasons—was at such pains to avoid. Recall Field's (1980, 6) statement that his nominalism

. . . is in sharp contrast to many other nominalistic doctrines, e.g., doctrines which reinterpret mathematical statements as statements about linguistic entities or about mental constructions.

Field was quite right to try to avoid inscriptionalist and conceptualist attempts to understand the vastness of mathematical reality in terms of a paltry finite collection of deposits of ink, graphite, and chalk, or of mental events. My point is that, in the end, Field can't avoid this. He cannot restrict his view to the foundations of mathematics, since the success of linguistic realism entails the success of realism in general. Since Field has to defend inscriptionalist nominalism in linguistics, not only is the plan for a clean surgical strike against realism that Field announces in *Science Without Numbers* unworkable, but the problems with inscriptionalist nominalism, which doomed the Goodman and Quine (1947) enterprise, undercut Field's nominalist program. (See Katz 1996c for a discussion of these issues.)

There is a further criticism concerning Field's (1980, 1989) claim that reference to numbers is reference to fictional entities and mathematical truth is truth in the fiction of mathematics. Field (1989, 2–3) says: "The sense in which ' $2 + 2 = 4$ ' is true is pretty much the same as the sense in which 'Oliver Twist lived in London' is true." For Field, the former statement is true according to the well-known arithmetic story, while the latter statement is true according to the well-known Dickens story. Some such view is needed in Field's position, since

otherwise it does not distinguish truths like ' $2 + 2 = 4$ ' from falsehoods like ' $2 + 2 = 17$.'

There is, however, an essential difference between mathematics and fiction that shows that truth in mathematics cannot be taken to be truth in fiction. It is that consistency is a necessary condition for truth in mathematics but not for truth in fiction. A fictional character's having incompatible properties (in the fictional corpus) doesn't rule out the existence of the fictional character, but a mathematical object's having incompatible properties does rule out the existence of the mathematical object. If I recall correctly, Dr. Watson is attributed incompatible properties. At one place in the Sherlock Holmes corpus, Watson's Jezail bullet wound is in his shoulder, while at another place in the corpus, the wound is in his leg. However, the discovery of this inconsistency does not show that Dr. Watson does not have the fictional existence we took him to have. The inconsistency doesn't show that he has the status of Hamlet's children. Nor does it show that the adventures of Sherlock Holmes are committed to implying everything, e.g., that Holmes is Inspector Lestrade or that Watson is Professor Moriarty. Contrast this with the discovery of an inconsistency in mathematics, which automatically establishes nonexistence. (This criticism does not depend on the example. It is pointless to quibble about examples, e.g., by arguing that the locations are mentioned in different Arthur Conan Doyle stories [so one of them must be false in the Sherlock Holmes corpus]. There are other actual cases in literature, but hypothetical cases will do just as well.)⁶

Further, it does not help to try to distinguish mathematical fiction from, as it were, fictional fiction. It doesn't help to say that it is part of our logical pretense about mathematics (but not about fiction) that nothing can have incompatible properties. If arithmetic and literature are both fiction, why should there be this logical pretense in the one case but not in the other? If mathematics is simply a story that mathematicians tell, in which part of the story is that consistency is a condition for existence, then, on the one hand, their story *could* have been like fictional fiction in tolerating inconsistency, and, on the other hand, their story *could* change so that future mathematical fiction is like fictional fiction in tolerating inconsistency. But it is clear that neither of these scenarios is possible. There can't be inconsistency in mathematics—that's a logical impossibility; there can be inconsistency in fiction—

6. Kaufmann (1961, 375–377) discusses the *prima facie* contradiction in *Macbeth* and two clearcut contradictions in Goethe's *Faust*. Goethe is quoted as saying, "The more incommensurable and incomprehensible . . . a poetic production is, the better."

that's both a logical possibility and an actuality. There is a basic difference between mathematics and literature: consistency is an absolute constraint in mathematics but not in fiction. The explanation is obviously that fiction is fiction and mathematics is fact.

1.3 *Wrong Turns that Point in the Right Direction*

To prepare for the response to the epistemological challenge to realism in the next chapter, I want to examine a number of unsuccessful epistemological approaches to knowledge in the formal sciences that philosophers, including realists, have taken. My interest in these approaches is with what can be learned from them. I will argue that the nature of their failure suggests the direction that we should take in looking for a successful account of formal knowledge.

We will look at three epistemological approaches: the classical Platonist's, the contemporary Aristotelian's, and the naturalized realist's. I will argue that all of them pursue a reconstruction of formal knowledge as knowledge that is *au fond* a matter of acquaintance and that this underlying empiricism is what has been responsible for the failure of realists who take one of these approaches. Neither the notion of acquaintance with abstract objects nor the notion of acquaintance with concrete objects is a viable option for realists. The only way for realists to have any chance of meeting the epistemological challenge is for them to eschew an epistemology based on acquaintance. As will be seen in the next chapter, this means that experience cannot be allowed to enter in any way, shape, or form—that nothing outside reason can provide grounds for mathematical, logical, or linguistic knowledge.

I want to show that, even though classical Platonism, contemporary Aristotelianism, and naturalized realism claim full realist credentials, and even though they share important features of traditional realism and rationalism, their epistemology is empiricist at the core, and hence their failure has no implications for a genuinely rationalist approach. If this can be shown, it will make it clear that their failure does not reflect adversely on the realist's prospects for meeting the epistemological challenge.

1.3.1 *Classical Platonism*

Plato's doctrine of *anamnesis* was the first epistemology proposed for realism. It is condemned by antirealists as myth, but this is too strong, since myths can be compelling ways of expressing ideas that would have been hard or impossible to express literally at the time. Plato's myth of the metals in the *Republic* can be seen in retrospect as a figurative expression of the idea of genetic determinism. But Plato's

myths of the cave and of recollection, even demythologized in a way that provides us with a literally expressed epistemology, ought to be rejected, especially by realists, as articulating an incoherent epistemology for abstract objects.

The strategy behind the doctrine is to extend the range of the perceivable to include abstract objects. But the strategy makes knowledge of abstract objects depend on acquaintance with them. The idea is that the source of the knowledge we obtain through recollection is perceptual contact with the objects known. What is bad about this idea is that it buys into the core empiricist notion that all our knowledge ultimately derives from acquaintance. That notion is not transplantable from a naturalistic ontology to a realist ontology. Since abstract objects are outside the nexus of causes and effects and thus perceptually inaccessible, they cannot be known through their causal effects on us. In buying into the empiricist idea that knowledge is based on acquaintance, classical Platonists render their overall position incoherent. It is as senseless to suppose that we can be acquainted with atemporal abstract objects prior to our entrance into the spatiotemporal world as it is to suppose that we can be acquainted with them afterwards. Acquaintance requires a point of contact, some temporal position that both we and the object occupy, but there cannot be such a point in the case of objects that have no temporal location whether during the soul's existence in this world or prior to its incarnation.⁷

It has seemed to many—myself (1981, 200–202) included—that Gödel was a classical Platonist of some kind. In contemporary philosophy of mathematics, Gödel's ([1947] 1983, 483–84) remarks about perceiving abstract objects are widely interpreted as expressing the view that we have perceptual acquaintance with abstract objects. But, despite the fact that this interpretation seems to fit those remarks themselves, one cannot help having qualms about attributing so obvious an inconsistency to so subtle and powerful a thinker.

Such qualms have induced no less an authority on Gödel than Hao Wang to suggest that those problematic references should be taken as a metaphor for some noncausal form of apprehension. As Yourgrau (1989, 399) reports:

. . . we are able to “see” [mathematical objects] only because there “is” [an objective] mathematical world. How can we, however, apart from using our five senses, see anything that is not in our

7. Even supposing there are beings in another world that can make some sort of contact with abstract objects, they would have to be atemporal and then the same problem would arise when we try to imagine that we could be *those* beings or be continuous with *them*.

minds? The “interaction” must be something different from that between us and the physical world.

But, desirable as it is to try to find a more charitable interpretation, this suggestion does not help. The metaphorical construal removes the incoherence by denying that the interaction is anything like causal interaction with natural objects. But that is all it does. As Benacerraf ([1973] 1983, 415–16) pointed out, in telling us only what contact is not, the construal does not tell us what Gödel’s special sort of contact is. It affords us no idea of what the metaphorical use of “perception,” “seeing,” or “interaction” might amount to. We can set up the Aristotelian scheme: grasping is to abstract objects as perceiving is to concrete ones. But once the notion of sensory contact is factored out, as it must be in order to obtain a charitable interpretation of Gödel’s references to perception, all that is left of the interaction analogy is the unhelpful claim that mathematical intuition and sense perception are similar in that both are ways of coming to know.

It is clear that the strategy of trying to extend the range of perceivable objects to include abstract objects presents the classical Platonist with the dilemma of choosing between literalness and incoherence on the one hand or nonliteralness and inanity on the other. But Gödel ought not to be interpreted as a classical Platonist. True enough, his statement that we have “something like a perception . . . of the objects of set theory” seems to encourage such an interpretation, but, in the broader context, his use of “perception” does not seem to have been intended in this way. The passage that immediately follows the one containing this statement suggests that Gödel is anything but a classical Platonist. Gödel (1947 [1983], 484) says:

It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that . . . we *form* our ideas also of those objects on the basis of something else which is immediately given. Only this something else here is *not*, or not primarily, the sensations. . . . It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things on our sense organs, are something purely subjective, as Kant asserted. Rather, they, too, may represent an aspect of objective reality, but, as opposed to sensations, their presence in us may be due to another kind of relationship between ourselves and reality.

Gödel is clearly saying that the relationship between ourselves and mathematical reality is *not* the kind of relationship that we have to

physical reality in the case of sensations. Mathematical knowledge, he says explicitly, is not the causal effect of “actions of certain things on our sense organs.”

Gödel is not much more informative about what he does mean when he talks about this relationship than Frege is when he talks about grasping—though Gödel is more explicit than Frege about what he does not mean. Gödel’s reluctance to say what this relationship is can be easily explained. Gödel was reluctant to try to characterize the relationship because he had no epistemology to put in place of classical Platonism. This explanation not only fits what he says, but it fits the fact that Gödel spent a great deal of time studying Husserl’s phenomenology in the hope of obtaining insight into the given in mathematics (Wang, personal communication).

On this more charitable interpretation, Gödel should be credited with taking the significant step of breaking with classical Platonism. Realists who recognize the dilemma in which classical Platonism puts them ought to acknowledge the importance of Gödel’s break with classical Platonism and its strategy of trying to extend the range of perceivable objects to include abstract objects. They should follow Gödel’s lead in looking for a different strategy.

1.3.2 Contemporary Aristotelianism

Like classical Platonists, the philosophers I am calling “contemporary Aristotelians,” who sometimes call themselves “Platonists,” also pursue the strategy of seeking to extend the range of the perceivable to include the objects of mathematics, but, unlike classical Platonists, they locate the objects of mathematical knowledge in the natural world. Maddy (1980, 1989, 1990) is a prominent example of such a philosopher. Her aim is to avoid the problem of access to abstract objects, thought of as denizens of a Platonic realm, on the basis of the idea that mathematical knowledge is knowledge by acquaintance. According to her (1980, 179), if there are three eggs to be seen in a carton, then the set of the eggs is to be seen there too. On this approach, we have *a posteriori* perceptual knowledge of sets just as we have of eggs.

No doubt Maddy’s position avoids the dilemma facing classical Platonism, since, on her position, acquaintance with mathematical objects is acquaintance with natural objects. But it looks as if her attempt to naturalize abstract objects is a classic case of out of the frying pan into the fire. Since natural objects are not nonspatial, atemporal, or causally inert entities, her set located in that egg carton must be just another concrete object like the eggs. But if sets are sets and her position is Platonism, the set can’t, logically speaking, be a concrete object like the egg carton. So we have a new dilemma: either the terms in question

have their standard meanings and her Platonism is as incoherent as classical Platonism, or the terms do not have their standard meanings and it is misleading of her to refer to her position as "Platonism."

Maddy has faced up to this dilemma. Confronted with the fact that, on the standard sense of the term "abstract," her notion of a mathematical object is not that of an abstract mathematical object, she (1990, 59) responds: "So be it; I attach no importance to the term." For her, then, mathematical objects are simply natural objects, as spatially and temporally located, as causally active, and presumably as contingent, as spotted owls. Nonetheless, she claims to be a Platonist, *albeit* one with an eccentric ontological terminology. No doubt she would dismiss the complaint that she is presenting a very un-Platonistic claim about mathematical objects under the term "Platonism" as a mere quibble about who gets to use the term "Platonism." The term "Platonism" thus goes the way of the term "abstract." Initial appearances to the contrary, it turns out that there is no contribution to meeting Benacerraf's challenge to realism to be found in Maddy's position.

Maddy's real position is Aristotelian. This becomes clear when the term "abstract" in her various claims about mathematical objects is replaced with the term "concrete": mathematical objects are concrete objects within the natural universe of space and time, mathematical knowledge is natural knowledge, and mathematical epistemology is empiricist in perhaps some broadly Quinean sense. Such an Aristotelian view certainly escapes the epistemological questions about realism, but it faces more difficult questions. How can numbers and pure sets be naturalized? How can it even be meaningful to ascribe physical location to a number or a set? How can there be enough natural objects for all the numbers and sets? Where is the null set? Is it in more than one place? What explanation can be given for the special certainty, if not the necessity, of mathematical and logical truths?

In the case of the last question, the obvious move is to go Quinean, deny necessity, and explain mathematical and logical certainty in terms of centrality. Such a move will enable Maddy to use Quine's doctrine of science as first philosophy to explain the commitment to numbers and sets. But how does her Aristotelian view that numbers and sets are concrete objects square with Quine's Platonist view of numbers and sets as abstract objects? That Platonist view is supposed to follow from mathematics and the Quinean doctrine of science as first philosophy. Furthermore, even though Quine's empiricism offers a better account of the certainty of mathematical and logical truths than Mill's empiricism, it is incoherent, as I shall argue in chapter 3. If both that argument and the argument in chapter 2 that realism can meet Benacerraf's challenge succeed, Maddy has traded an epistemological challenge

that can be met for one that can't. (See also the criticism of Maddy in Chihara 1992.)

1.3.3 *Naturalized Realism*

To escape the epistemological challenge to realism, some philosophers have tried to frame a position between the extremes of classical Platonism and contemporary Aristotelianism. Instead of trying to extend the range of the perceivable to include mathematical objects or trying to make such objects perceivable by making them natural, their strategy is to combine a realist ontology with an empiricist epistemology. Scientific knowledge of the abstract objects in a formal domain is to be explained on the basis of an empirical investigation of the mental/neural structures that constitute our knowledge of mathematical, logical, and linguistic reality.

The proponents of such a "naturalized realism" hope to obtain the virtues of both an empiricist epistemology and a realist ontology without incurring the vices of either. Since psychological theories are about natural objects, naturalized realism will be free of the epistemological difficulties about causal contact that are supposed to plague realism. Since naturalized realism accepts the existence of abstract objects, it can hold that formal truths are about such objects, and hence avoid the difficulties of trying to account for mathematics, logic, and linguistics on an exclusively naturalist ontology.

Such a position sounds too good to be true—and it is. Its problems arise from the very distinction on which naturalized realism is based. This is the separation of the (abstract) objects that formal knowledge is about from the (natural) objects with which, according to the position, our epistemic faculties interact in the acquisition of such knowledge. Given this separation, the abstract objects that the claims of formal scientists are about are not the objects that those scientists study to determine the truth or falsehood of those claims. Rather, they examine our inner mental states or their neurophysiological underpinnings. Empirical facts about those states or their underlying wetware are the basis for claims about numbers, propositions, and sentences in mathematics, logic, and linguistics.

What rationale is there for denying the highly intuitive and widely accepted principle that the nature of the objects that constitute the subject matter of a discipline determines the nature of the discipline? The proposition that six is a perfect number asserts that the number six is equal to the sum of all its divisors except for itself, but, according to naturalized realism, it is not the number six that we focus on to determine if it is a perfect number. Physics and biology are parts of natural science because they study natural objects, and the issue of

whether mathematics is also part of natural science is the issue of whether numbers are psychological objects, as Kant and Brouwer thought, or abstract objects, as Frege and Gödel thought.

George (1989) offers a rationale for distinguishing the ontological nature of the objects with which a science is concerned from the ontological nature of the science. He agrees with the linguistic realist that linguistics is concerned with grammars in the abstract sense, but holds nonetheless that linguistics is a psychological science. George (1989, 106–7) claims that linguistic realists are “confused” because they

slide from the view that linguistics is not about [internal mental states] to the view that linguistics is not psychological. [Katz] seems to assume that the nature of the objects one is investigating determines the nature of one’s investigation.

George (1989, 98) rejects this assumption on the grounds that “Entities can be referred to in many different ways,” arguing that

Just as an inquiry into the identity of Z’s favorite planet is not plausibly considered part of planetary astronomy, so an inquiry into the identity of Z’s grammar is not plausibly considered part of mathematics. . . . identification of that grammar, an abstract object, is a fully empirical inquiry.

Since the case “Z’s favorite number” is completely parallel to the case of “Z’s favorite grammar,” it follows, by parity of reasoning, that arithmetic is “a fully empirical inquiry.” Since so momentous a substantive conclusion could hardly be gotten with such paltry linguistic means, the reasoning must be fallacious.

The fallacy results from an ambiguity in phrases like “inquiries about Z’s favorite planet.” Such phrases have both a referential sense, on which the inquirer can be an astronomer investigating a certain planet—which just happens to be Z’s favorite planet—and a nonreferential sense, on which the inquirer can be a psychologist investigating Z’s taste in planets (Nishiyama in conversation). The sentence “Linguistics is an inquiry into the grammar that a speaker knows” is ambiguous in the same way. On its referential sense, it expresses the claim that linguistics is an inquiry into an abstract object—which happens to be referred to here in a scientifically quaint way. On its nonreferential sense, it expresses the claim that linguistics is a psychological inquiry into the speakers’ epistemic states, namely, an inquiry to discover which grammar they know. Conflating these two senses, George (1989, 89) infers that “identifying a speaker’s grammar, an abstract object, is already part of the psychological enterprise.” When the two senses are kept apart, it is clear that George’s conclusion about

the nature of linguistics does not follow. On the referential sense, there is no identification that is a matter of psychology; on the nonreferential sense, there is such an identification, but the only conclusion that can be drawn is one about the nature of the speaker's psychology.

Finally, as suggested by the parallel of George's argument for arithmetic, his view entails a collapse of the formal sciences into psychology, with the consequence that there is no room left for the study of abstract mathematical, logical, and linguistic objects themselves. The shift from the mathematical domain to the domain of psychology replaces discoveries about the structure of numbers, propositions, and sentences with discoveries about the cognitive states of human beings. The mathematical, logical, and linguistic investigations into the structure of numbers, propositions, and sentences has been lost.

McGinn (1993) proposes an answer to the objection that naturalized realism provides no place for the study of the structure of the abstract objects in the domain of a formal science. His answer is that the mathematical and logical competences that are investigated in psychology mirror the structure of the abstract objects that mathematical and logical truths are about. Given this mirroring relation, those competences can serve as the source of knowledge of the abstract objects of which they are knowledge. Hence, mathematical and logical investigations into the structure of numbers and propositions have not been lost. They are alive and well in the naturalistic areas which study the human mind/brain.

But what entitles McGinn to assume that our mathematical and logical competences mirror the abstract mathematical and logical reality of which they are knowledge? The assumption is not a necessary truth, because whether the mirroring relation holds depends on contingent spatiotemporal creatures. It is thus possible that our mathematical, logical, and linguistic competences do not mirror how things actually stand with numbers, propositions, and sentences. Given this possibility, unless there is an argument for the assumption, it begs the question against realists (e.g., Frege 1964, 12–15) who claim that there is a difference between how we take or represent abstract reality and how it actually is. But since such an argument would have to be based on an independent way of finding out how abstract reality actually is, if there were such a way, then naturalized realism would be otiose.

The doubt that our mathematical, logical, or linguistic competence mirrors how things actually stand with numbers, propositions, and sentences is not a skeptical doubt about whether we have knowledge of them. We can have knowledge of them in the usual sense, but that knowledge can fall short of mirroring their structure. The possibility arises from the fact that the knowledge of relation is loose enough to