Chapter 1

BASIC FORMULAS FOR CLASSICAL RADIATION PROCESSES

In many ways the classical description of a given radiation process is the easiest one to visualize. Usefulness rather than rigor is the goal of this book, so the classical or semi-classical descriptions will be used whenever possible.

In general, classical physics applies when the de Broglie wavelength of the radiating particle is small compared to the typical dimensions of the problem, i.e., when

$$h/p \ll r$$

here h is Planck's constant, p is the momentum of the radiating particle and r is a typical dimension of the system. Stated another way, the uncertainty in the position of the particle must be much less than the characteristic dimensions of the problem. The dimension r may refer to the effective radius of the interaction, or to the wavelength of the radiation. The classical domain can also be defined in terms of the energy W of the radiating particle and the frequency ν of the emitted radiation. Since $W \simeq pv$ and $\nu \simeq v/r$, where v is the velocity of the particle, we have

$$h\nu \ll W$$

This condition states that a classical particle cannot convert a significant amount of its energy into one photon, or alternatively, that the classical approximation holds only for transitions in which the relative change in the principle quantum number is small.

In this chapter the basic formulas needed to calculate classical radiation processes are summarized.

1.1 The Electromagnetic Field Equations

The classical theory of radiation is based on Maxwell's theory of the electromagnetic field. For a given distribution of charge density ρ and current density $\vec{j}$ the field is determined by Maxwell's equations:

$$\text{curl}\ \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{1-1}$$

$$\text{curl}\ \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{1-2}$$

$$\text{div}\ \vec{E} = 4\pi\rho \tag{1-3}$$

$$\text{div}\ \vec{B} = 0 \tag{1-4}$$

(CGS units will be used unless specifically stated otherwise.) Here $\vec{E}$ and $\vec{B}$ denote the electric and magnetic fields and c denotes the speed of light in vacuum. From these equations it follows that the electric charge is conserved, i.e., that it satisfies a continuity equation:

$$\text{div}\ \vec{j} + \frac{\partial\rho}{\partial t} = 0 \tag{1-5}$$

The motion of the particles is described by Newton's second law

$$d\vec{p}/dt = \vec{F} \tag{1-6}$$

where $\vec{p}$ is the momentum of the particle and $\vec{F}$ the Lorentz force, which for particles of charge e and velocity $\vec{v}$ is given by

$$\vec{F} = e(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}) \qquad (1\text{-}7)$$

The fields which are to be used in the Lorentz equation are the external fields as well as the fields produced by the charge itself. This self-produced field will also affect the motion of the particle. The reaction of the field in general has only a small influence on the short-term motion of the particle, so that to a first approximation, the external fields only may be used in (1-6).

The rate of change of the kinetic energy W_k of a charge in an electromagnetic field is given by

$$dW_k/dt = \vec{v} \cdot d\vec{p}/dt = e\, \vec{v} \cdot \vec{E} \qquad (1\text{-}8)$$

The magnetic field does not enter into this equation since the force which the magnetic field exerts on the charge is always perpendicular to its velocity, and hence does no work on it. The rate of increase of energy of all the particles in a unit volume is found by summing (1-8) over all particles in that volume. The result is

$$dU_p/dt = \vec{j} \cdot \vec{E} \qquad (1\text{-}9)$$

where U_p is the kinetic energy density of the particles. In a given volume it changes with time because of changes in the energy of the particles, and because of the flow of particles out of the volume. Thus

$$dU_p/dt = \partial U_p/\partial t + \text{div}\,\vec{Q} = \vec{j}\cdot\vec{E} \qquad (1\text{-}10)$$

where $\vec{Q}$ is the energy flux density vector. For a particle distribution function $f(\vec{r},\vec{p},t)$, $\vec{Q}$ is given by

$$\vec{Q} = \int W\,\vec{v}\, f(\vec{r},\vec{p},t)d^3\ V \qquad (1\text{-}11)$$

Through the use **of Maxwell's equations and** some vector algebra, **the scalar product $\vec{j}\cdot\vec{E}$** can be written in the **form**

$$j\cdot E = -\frac{\partial}{\partial t}\frac{(E^2 + B^2)}{8\pi} - \text{div}\,\frac{(c\vec{E} \times \vec{B})}{4\pi} \qquad (1\text{-}12)$$

Equation (1-10 **can now be rewritten as**

$$\frac{\partial}{\partial t}\{U_p + \frac{(E^2 + B^2)}{8\pi}\} + \text{div}\{\vec{Q} + \frac{c\vec{E} \times \vec{B}}{4\pi}\} = 0$$

$$(1\text{-}13)$$

If we intergrate (1-13) over a volume V, and use Gauss' theorem to express the divergence term as a surface integral, then we obtain

$$\frac{\partial}{\partial t}\int \{U_p + \frac{(E^2 + B^2)}{8\pi}\}\ dV = -\int(\vec{Q} + \vec{S})\circ d\vec{A}$$

$$(1\text{-}14)$$

where

$$\vec{S} = (c\vec{E} \times \vec{B})/4\pi \qquad (1\text{-}15)$$

is called the Poynting vector.
For a closed system in which there are no fields at the boundary and no heat transfer

across the boundary, the surface integral vanishes, and the quantity on the left hand side of equation (1-14) is conserved. The first term in the brackets represents the kinetic energy density, so the second term must be the energy density of the electromagnetic field:

$$U_{em} = (E^2 + B^2)/8\pi \qquad (1\text{-}16)$$

In general the surface integral will not vanish; the first term gives the rate at which heat flows into or out of the volume. The second term therefore gives the flux of electromagnetic energy across the boundary, and the Poynting vector $\vec{S}$ is the amount of electromagnetic energy passing through a unit surface area in a unit time.

The equation for the rate of change of linear momentum can be expressed in a form which is similar to equation (1-14). From equations (1-6) and (1-7) it follows that the rate of change of the momentum density of the particles is (assuming that the particle pressure is negligible):

$$d\vec{\Pi}_p/dt = \rho\vec{E} + (\vec{J} \times \vec{B})/c \qquad (1\text{-}17)$$

Using Maxwell's equations to eliminate ρ and $\vec{J}$, we may write equation (1-17) in the form (see Jackson, 1962)

$$\frac{\partial}{\partial t} \int \{\vec{\Pi}_p + \frac{\vec{E} \times \vec{B}}{4\pi c}\} \, dV = \int \overleftrightarrow{T}\cdot d\vec{A} \qquad (1\text{-}18)$$

Here the tensor $\overleftrightarrow{T}$, called Maxwell's stress tensor, is in dyadic notation

$$\overleftrightarrow{T} = \{\vec{E}\,\vec{E} + \vec{B}\,\vec{B} - \frac{1}{2}\,\overleftrightarrow{I}\,(E^2 + B^2)\}/4\pi \qquad (1\text{-}19)$$

In tensor notation, the elements are given by

$$T_{ij} = \{E_i\, E_j + B_i\, B_j - \frac{1}{2}\,\delta_{ij}(E^2 + B^2)\}/4\pi \qquad (1\text{-}20)$$

For a closed system the integral on the right hand side of (1-17) vanishes, so the second term on the left is to be identified with the electromagnetic momentum. The electromagnetic momentum density, $\vec{\Pi}_{em}$, is therefore

$$\vec{\Pi}_{em} = (\vec{E} \times \vec{B})/4\pi c \qquad (1\text{-}21)$$

1.2 Constant Electromagnetic Fields

In the special case when $\vec{B} = 0$ and all time derivatives vanish, Maxwell's equations become

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad (1\text{-}22)$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad (1\text{-}23)$$

Equation (1-23) shows that the electric field E can be expressed in terms of a scalar potential

$$\vec{E} = -\,\vec{\nabla}\varphi \qquad (1\text{-}24)$$

Equation (1-22) then becomes

$$\nabla^2 \varphi = -\,4\pi\rho \qquad (1\text{-}25)$$

Solving this equation yields the potential

$\varphi(\vec{R}_o)$ at $\vec{R}_o$ due to a charge distribution ρ:

$$\varphi(\vec{R}_o) = \int \frac{\rho(\vec{R}')\, dV'}{|\vec{R}_o - \vec{R}'|} \qquad (1\text{-}26)$$

For a system of **point** charges e_i located at $\vec{R}_i$ the potential at $\vec{R}_o$ is

$$\varphi(\vec{R}_o) = \Sigma_i\, e_i \,/\, |\vec{R}_o - \vec{R}_i| \qquad (1\text{-}27)$$

For the case when $\vec{E} = 0$ and all time derivatives vanish, the magnetic field must satisfy the equations

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (1\text{-}28)$$

$$\vec{\nabla} \times \vec{B} = 4\pi\vec{j}/c \qquad (1\text{-}29)$$

If we introduce the <u>vector potential</u> $\vec{A}$ which satisfies the conditions

$$\vec{\nabla} \times \vec{A} = \vec{B} \qquad (1\text{-}30)$$

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad (1\text{-}31)$$

then the equation (1-29) becomes

$$\nabla^2 \vec{A} = -4\pi\vec{j}/c \qquad (1\text{-}32)$$

In analogy with (1-26) the desired solution for $\vec{A}$ is

$$\vec{A}(\vec{R}_o) = \frac{1}{c}\int \frac{\vec{j}(\vec{R}')\, dV'}{|\vec{R}_o - \vec{R}'|} \qquad (1\text{-}33)$$

For a system of point charges e_i with velocities $\vec{v}_i$

$$\vec{A}\,(\vec{R}_o) = \frac{1}{c}\,\Sigma_i\; e\vec{v}_i/(\vec{R}_o - \vec{R}_i) \qquad (1\text{-}34)$$

1.3 The Dipole and Higher Moments

Consider the field produced by a system of charges at distances large compared with the dimensions of the system.

For $R_o >> R_i$ equation (1-27) becomes

$$\varphi = \frac{\Sigma\, e_i}{R_o} - \vec{d} \cdot \vec{\nabla}\, \frac{1}{R_o} + \ldots \qquad (1\text{-}35)$$

where the neglected terms are of the second order or greater in the small quantity $(R_i/R_o)^2$. The sum

$$\vec{d} = \Sigma\, e_i\, \vec{R}_i \qquad (1\text{-}36)$$

is called the <u>electric dipole moment</u> of the system of charges. Note that if $\sum_i e_i = 0$, the dipole moment is independent of the choice of the origin of the coordinates. In this case the potential of the field at large distances is

$$\varphi = -\,\vec{d} \cdot \vec{\nabla}\, \frac{1}{R_o} \qquad (1\text{-}37)$$

Thus the potential of the field at large distances produced by a system of charges with total charge equal to zero is inversely proportional to the square of the distance and the field intensity αR_o^{-3}. This field has axial symmetry around the direction of $\vec{d}$.

The third term in the expansion of the potential in powers of $1/R_o$ is

$$\varphi^{(2)} = \frac{D_{\alpha\beta}\, n_\alpha\, n_\beta}{2\, R_o^3} \qquad (1\text{-}38)$$

where

$$D_{\alpha\beta} = \sum_i (3\, x_{\alpha i}\, x_{\beta i} - r_i^2\, \delta_{\alpha\beta}) e_i \qquad (1\text{-}39)$$

is the <u>electric quadrupole moment</u> of the system and n_α are the components of a unit vector along $\vec{R}_o$.

In a similar fashion we could write the succeeding terms of the expansion of φ, using the theory of spherical harmonics. For the

At a distance which is large compared with the dimensions of the system, the vector potential of the fields produced by all the charges at the point having the radius vector $\vec{R}_o$ is

$$\vec{A} = (\vec{M} \times \vec{R}_o) \,/\, R_o^3 \qquad (1\text{-}40)$$

where

$$\vec{M} = \frac{1}{2c} \Sigma_i\, e_i\, \vec{R}_i \times \vec{v}_i \qquad (1\text{-}41)$$

is the <u>magnetic dipole moment</u> of the system.

1.4 The Field of a Uniformly Moving Charge

Consider a charge e moving uniformly with velocity $\vec{v}$ along the x-axis in the laboratory frame of reference K. The charge is at rest in the frame K' which is moving with a velocity $\vec{v}$ along the x-axis of K. The axes y and z are parallel to y' and z'. At time t = 0 the origins of the two systems coincide and the charge is at its closest distance to the observer, who is located at the point P which has the coordinates (0,b,0) in the K frame. In the frame K' the observer's point P has the coordinates (- vt, b',0) and is a distance $R' = (b^2 + (vt')^2)^{1/2}$ away from the charge. In the rest frame K' the electric and magnetic fields are

$$\vec{E}' = e\vec{R}'/\vec{R}'^3 \qquad \vec{B}' = 0 \tag{1-42}$$

The coordinates in the two reference frames are related by the Lorentz transformation

$$x' = \gamma(x-vt);\ y' = y;\ z' = z;\ t' = \gamma\left(t- \frac{\beta x}{c}\right) \tag{1-43}$$

where

$$\beta = v/c\ ; \qquad \gamma = (1 - \beta^2)^{-1/2} \tag{1-44}$$

In terms of the coordinates of K the components of the electric field in K' is given by

$$E_x' = \frac{- e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}\ ,\quad E_y' \frac{eb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{1-45}$$

The components of the electric and magnetic field parallel and perpendicular to the direction of motion of K' relative to K are given by the Lorentz transformation for the fields:

$$\vec{E}_{\parallel} = \vec{E}'_{\parallel} \qquad \vec{B}_{\parallel} = \vec{B}'_{\parallel}$$

$$\vec{E}_{\perp} = \gamma(\vec{E}'_{\perp} - \vec{\beta} \times \vec{B}') \qquad \vec{B}_{\perp} = \gamma(\vec{B}'_{\perp} + \vec{\beta} \times \vec{E}') \tag{1-46}$$

In this case we have $B' = 0$ so

$$E_x = E_x' = \frac{-e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad B_x = 0$$

$$E_y = \gamma E_y' - \frac{\gamma eb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad B_y = 0 \tag{1-47}$$

$$E_z = 0 \qquad B_z = \gamma\beta E_y' = \beta E_y$$

Introducing the angle θ between the direction of motion and the radius vector $\vec{R}$ from the charge e to the field point (x,y,z), we can write the expression for $\vec{E}$ in another form

$$\vec{E} = \frac{e\vec{R}}{R^3} \frac{(1 - \beta^2)}{(1 - \beta^2 \sin^2\theta)^{3/2}} \tag{1-48}$$

Along the direction of motion ($\theta = 0, \pi$) the field has the smallest value, equal to

$$\vec{E}_{\parallel} = e(1 - \beta^2) / R^2 \tag{1-49}$$

The largest field is for $\theta = \pi/2$:

$$\vec{E}_{\perp} = \gamma\ e/R^2 \tag{1-50}$$

Note that as the velocity increases, the field $\vec{E}_{\parallel}$ decreases, while $\vec{E}_{\perp}$ increases. For velocities close to the velocity of light, the denominator in (1-48) is close to zero in a narrow interval of values θ around $\theta = \pi/2$, with a width of the order

$$\Delta\theta \sim \gamma^{-1} \tag{1-51}$$

so that the electric field of a relativistic charge is large only in a narrow range of angles in the neighborhood of the equatorial plane. Thus as γ increases the peak fields increase $\alpha\gamma$, but the duration of the peak field at the field point decreases $\alpha\gamma^{-1}$. For large γ the observer sees nearly equal transverse and mutually perpendicular electric and magnetic fields, which are indistinguishable from a pulse of plane polarized radiation propagating in the x direction.

1.5 The Wave Equation

In a vacuum $\rho = 0$, and $\vec{j} = 0$ so Maxwell's equations become

$$\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{1-52}$$

$$\text{curl } \vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{1-53}$$

$$\text{div } \vec{E} = 0 \tag{1-54}$$

$$\text{div } \vec{B} = 0 \tag{1-55}$$

These equations possess non-zero solutions, so an electromagnetic field can exist even in the absence of any charges. Such fields are called electromagnetic waves, since they must necessarily by time-varying. Otherwise the solution, given by (1-26) and (1-33) with $\rho = 0$, and $\vec{j} = 0$ is $\varphi = 0$, $\vec{A} = 0$.

In general the vector potential $\vec{A}$ and the scalar potential φ are defined by the equations (cf. equations (1-24) and (1-30) for constant fields)

$$\vec{B} = \text{curl } \vec{A} \quad , \qquad \vec{E} = - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi \qquad (1\text{-}56)$$

Equations (1-53) and (1-54) then become

$$\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \circ \vec{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t}) = 0 \qquad (1\text{-}57)$$

$$\nabla^2 \varphi + \frac{1}{c} \vec{\nabla} \cdot \frac{\partial \vec{A}}{\partial t} = 0 \qquad (1\text{-}58)$$

It is desirable to decouple these equations. One way to do this is to choose $\vec{A}$, φ, such that they satisfy the Lorentz condition

$$\text{div } \vec{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0 \qquad (1\text{-}59)$$

In this case the equations for the potentials become

$$[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}] \vec{A} = - 4\pi \vec{j}/c = 0 \qquad (1\text{-}60)$$

$$[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}] \varphi = - 4\pi\rho = 0 \qquad (1\text{-}61)$$

It is always possible to find potentials to satisfy the Lorentz condition, because the vector potential is arbitrary to the extent that the gradient of some scalar function X can be added. Thus $\vec{B}$ is left unchanged by the transformation

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla X \tag{1-62}$$

since curl (grad X) = 0. In order that the electric field be left unchanged by this transformation we must simultaneously have

$$\varphi \rightarrow \varphi' = \varphi - \frac{1}{c}\frac{\partial X}{\partial t} \tag{1-63}$$

The demand that $\vec{A}$ and φ satisfy the Lorentz condition can now be satisfied by choosing X appropriately. For example, if div A + $(1/c)\ \partial\varphi/\partial t = Y$ for one choice of $\vec{A}$ and φ, the transformations (1-62) and (1-63) can be used to obtain new potentials satisfying the Lorentz condition, provided

$$(\nabla^2 - \frac{\partial^2}{\partial t^2})\ X = -\ Y \tag{1-64}$$

The different possible choices one can make for $\vec{A}$ and φ, leaving $\vec{B}$ and $\vec{E}$ unchanged are called gauges, and the transformations (1-62) and (1-63) are called gauge transformations. The invariance of $\vec{E}$ and $\vec{B}$ under these transformations is called gauge invariance. The class of gauges satisfying (1-59) is called the Lorentz gauge. Another important gauge is the Coulomb or transverse gauge. Here one chooses X such that

$$\text{div}\ \vec{A} = 0 \tag{1-65}$$

$$\varphi = 0 \qquad (1\text{-}66)$$

for free space. The name, Coulomb gauge, is due to the fact that the scalar potential φ is given by the instantaneous Coulomb potential due to the charge density ρ in the general case. It is also called the transverse gauge because the condition (1-65) ensures that the fields are always perpendicular to the direction of propagation.

For plane waves propagating along the + $\underline{x}$ axis the fields are functions only of $t - x/c$. Therefore if the plane wave is monochromatic, its fields are simply periodic functions of $t - x/c$:

$$f = C \cos \omega(t - \frac{x}{c}) + D \sin \omega(t - \frac{x}{c}) \qquad (1\text{-}67)$$

here ω is the frequency in radians/sec and f denotes the scalar potential φ or one of the components of the vector potential $\vec{A}$, or one of the components of the electric or magnetic fields. It is usually more convenient to write the fields as the real parts of complex expressions:

$$f = \mathrm{Re}\ \{\ f_o \exp(-\ i\omega(t - \frac{x}{c}))\ \} \qquad (1\text{-}68)$$

where f_o is a constant complex number.

The period of variation of the field with the coordinate x at a fixed time t is called the wavelength and is here denoted by λ:

$$\lambda = 2\pi c/\omega \qquad (1\text{-}69)$$

The quantity

$$k = \omega/c \tag{1-70}$$

is called the <u>wave number</u>. For the general case of propagation in an arbitrary direction x is replaced by the radius vector $\vec{R}$ and the wave number is replaced by the <u>wave vector</u>:

$$\vec{k} = \omega\hat{n}/c \tag{1-71}$$

where $\hat{n}$ is the unit vector along the direction of propagation of the wave:

$$\hat{n} = \vec{R}/R \tag{1-72}$$

Rewriting (1-68) in terms of the wave vector, we have

$$f = \text{Re}\ \{\ f_o\ \exp(i\vec{k}\cdot\vec{R} - i\omega t) \tag{1-73}$$

The quantity $\vec{k}\cdot\vec{R}$ is called the <u>phase</u> of the wave. As long as we perform only linear operations, we can omit the sign Re for taking the real part and operate with complex quantities. Thus the expression for the vector potential of a plane, monochromatic wave can be written simply as

$$\vec{A} = \vec{A}_o \exp\ \{\ i\ (\vec{k}\cdot\vec{R} - \omega t)\ \} \tag{1-74}$$

Substituting into equation (1-56) we find

$$\vec{E} = i\ k\vec{A}\quad ; \qquad \vec{B} = i\vec{k} \times \vec{A} \tag{1-75}$$

i.e., the electric and magnetic fields in a monochromatic plane wave are perpendicular to each other and to the direction of propagation of the wave.

1.6 Doppler Effect

Introduce the four dimensional wave vector with components

$$k_\mu = (\vec{k},\ i\omega/c) \tag{1-76}$$

The phase of a monochromatic wave is just the scalar product of k_μ with the four-vector $x_\mu = (\vec{R},\ ict)$,

$$k_\mu\, x_\mu = \vec{k} \cdot \vec{R} - \omega t \tag{1-77}$$

According to special relativity such scalar products are invariant under Lorentz transformations. Therefore for two frames in relative motion along the x-axis with velocity v the phase of a wave is the same:

$$\vec{k}' \cdot \vec{R}' - \omega' t' = \vec{k} \circ \vec{R} - \omega t \tag{1-78}$$

Using a Lorentz transformation to express $\vec{R}'$ and t' in terms of $\vec{R}$ and t and equating coefficients of the components of $\vec{R}$ and t on both sides of the equation we find

$$k_y{}' = k_y \qquad k_z{}' = k_z$$

$$k_x{}' = \gamma\left(k_x - \frac{v}{c^2}\,\omega\right) \tag{1-79}$$

$$\omega' = \gamma(\omega - v\, k_x)$$

For light waves $|k| = \omega/c$, $|k'| = \omega'/c$, so

$$\omega' = \gamma\omega(1 - \beta \cos\theta) \tag{1-80}$$

where θ is the angle between the direction of $\vec{k}$ and $\vec{v}$. It is related to the angle θ'

between $\vec{k}'$ and $\vec{v}'$ by

$$\tan\theta' = \sin\theta/\gamma(\cos\theta - \beta) \qquad (1\text{-}81)$$

1.7 Polarization

For a monochromatic plane wave the electric field $\vec{E}$ is given by

$$\vec{E} = \vec{E}_o\, e^{i(\vec{k}\cdot\vec{R} - \omega t)} \qquad (1\text{-}82)$$

where $\vec{E}_o$ is a complex vector. Suppose

$$\vec{E}_o = \vec{E}_1 + i\vec{E}_2 = (E_{1y} + iE_{2y})\hat{y} + (E_{1z} + iE_{2z})\hat{z} \qquad (1\text{-}83)$$

Then (if we suppress the spatial dependence for the moment)

$$\begin{aligned} E_y &= E_{1y}\cos\omega t + E_{2y}\sin\omega t \\ E_z &= E_{1z}\cos\omega t + E_{2z}\sin\omega t \end{aligned} \qquad (1\text{-}84)$$

Define α_1, α_2 by means of

$$\sin\alpha_1 = \frac{E_{1y}}{\sqrt{E_{1y}^2 + E_{2y}^2}} \qquad \sin\alpha_2 = \frac{E_{1z}}{\sqrt{E_{1z}^2 + E_{2z}^2}} \qquad (1\text{-}85)$$

The equations (1-84) become

$$\begin{aligned} E_y &= E_{oy}\sin(\omega t + \alpha_1) \\ E_z &= E_{oz}\sin(\omega t + \alpha_2) \end{aligned} \qquad (1\text{-}86)$$

where

$$E_{oy} = \sqrt{E_{1y}^2 + E_{2y}^2} \qquad E_{oz} = \sqrt{E_{1z}^2 + E_{2z}^2} \tag{1-87}$$

When the phase difference $\alpha_1 - \alpha_2 = \pi$, or some integral multiple thereof, the components E_y and E_z vary in phase, and the vector $\vec{E}$ traces out a straight line in the $E_y - E_z$ plane as t varies. The wave is said to be <u>linearly polarized</u> in this case. When the phase difference $= \pi/2$, or some odd integral multiple thereof, and $E_{oy} = E_{oz}$, $\vec{E}$ traces out a circle as t varies. The wave is said to be <u>circularly polarized</u>. If $E_{oy} \neq E_{oz}$ the wave is <u>elliptically polarized</u>. The wave is said to have <u>right hand polarization</u> if $\vec{E}$ rotates clockwise as seen by the observer and vice versa for left hand polarization.

If the major axis of the ellipse described by E_y and E_z makes an angle χ with the E_y axis, then

$$E_{\chi} = E_o \cos\xi \sin \omega t$$

$$E_{\chi+\pi/2} = E_o \sin\xi \cos \omega t \tag{1-88}$$

where

$$E_o^2 = E_{oy}^2 + E_{oz}^2 \tag{1-89}$$

and $\tan\xi$ is the ratio of the axes of the ellipse. The angles χ and ξ are related to α_1 and α_2 as follows:

$$\tan\alpha_1 = -\tan\xi \tan\chi$$

$$\tan\alpha_2 = \tan\xi \cot\chi \tag{1-90}$$

The case $\xi = 0$ corresponds to linear polarization, the case $\xi = \pi/4$ to circular polarization. Still another group of parameters is useful in practice for analyzing the polarization of a wave. They are the Stokes parameters, defined by (see Chandrasekhar, 1960):

$$I = E_{oy}^2 + E_{oz}^2 = E_o^2$$

$$Q = E_{oy}^2 - E_{oz}^2 = E_o^2 \cos 2\xi \cos 2\chi$$

$$U = 2E_{oy}E_{oz}\cos(\alpha_2 - \alpha_1) = E_o^2 \cos 2\xi \sin 2\chi$$

$$V = 2E_{oy}E_{oz} \sin(\alpha_2 - \alpha_1) = E_o^2 \sin 2\xi \tag{1-91}$$

Linear polarization implies $U = V = 0$, whereas for circular polarization $Q = U = 0$.

In practice the amplitudes and phases are not constants; however, due to the high frequency of vibration, we may assume that the amplitudes and phases are constant for many vibrations and yet change irregularly many times during the period of observation. The Stokes parameters may then be defined as a time average over many vibrations:

$$I = \overline{E_{oy}^2 + E_{oz}^2} \quad ; \text{ etc.} \tag{1-92}$$

This has the consequence that, for a number of independent waves, the Stokes parameters for the mixture is the sum of the respective

Stokes parameters of the separate streams:

$$I = \sum_i I_i \text{ ; etc.} \tag{1-93}$$

For an arbitrarily polarized beam, there always exists among the quantities I, Q, U and V the inequality

$$I^2 \geq Q^2 + U^2 + V^2 \tag{1-94}$$

The equality holds for the case when the ratio of the amplitudes and the difference in phase remain constant through all fluctuations. These are the same as the conditions for the radiation to be elliptically polarized.

The degree of elliptical polarization Π is defined as the ratio

$$\Pi = (Q^2 + U^2 + V^2)^{\frac{1}{2}}/ I \tag{1-95}$$

For circular polarization, $\Pi = V/I$, whereas for linear polarization $\Pi = Q/I$.

1.8 The Lienard-Wiechert Potentials

In Section 1.5 we saw that in the general case of non-zero charge and current density the vector and scalar potentials satisfy the equations

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = - 4\pi \vec{j}/c \tag{1-96}$$

$$\nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = - 4\pi\rho \tag{1-97}$$

The solution of the inhomogeneous equations (1-96) and (1-97) is

$$f = f_c + f_p \tag{1-98}$$

where f represents any one of the components of $\vec{A}$ or φ, f_c is the solution of the equation with the right-hand side equal to zero (complementary solution) and f_p is a particular integral of the equation. To find the particular integral we introduce the Green's function, G, defined by

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\vec{R}_o - \vec{R}')\delta(t - t') \tag{1-99}$$

Note: The δ-function $\delta(x)$ is defined so that $\delta(x) = 0$ for all $x \neq 0$, and $\delta(x) \to \infty$ as $x \to 0$, so that the integral over a finite interval including $x = 0$ is equal to unity:

$$\int_{-b}^{b} \delta(x)\, dx = 1$$

where b is any non-zero number.
Therefore for any continuous function f(x),

$$\int_{-b}^{b} f(x)\delta(x)\, dx = f(0)$$

and

$$\int g(x)\, \delta\{f(x)\}\, dx = \{g(x)/f'(x)\}_{f(x) = 0}$$

where $f'(x) = df(x)/dx$.
Another useful equality is

$$\delta(\omega) = (1/2\pi)\int_{-\infty}^{\infty} e^{i\omega t}\, dt$$

The three-dimensional δ-function $\delta(\vec{R})$ is defined as

$$\delta(\vec{R}) = \delta(x)\delta(y)\delta(z)$$

Physically $\underset{\sim}{G}(\vec{R}_o, t, \vec{R}', t')$ represents the disturbance at $\vec{R}_o$ caused by a point source at $\vec{R}'$ turned on for only an infinitesimal interval at $t' = t$. Because of the linearity of the field equations, the actual field will be the sum of the fields produced by all such point sources (see Figure 1.1).

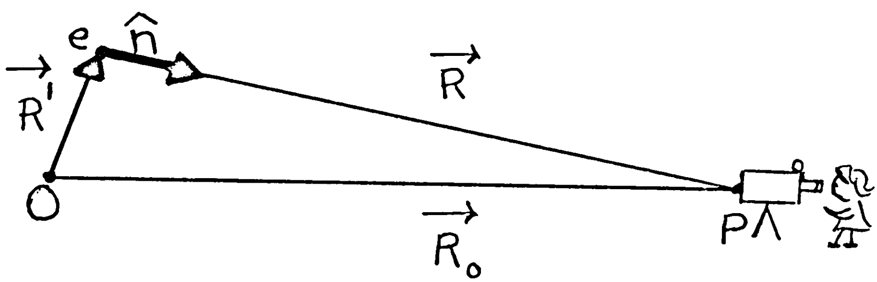

Figure 1.1.

The particular solution for φ, for example, is then given by

$$\varphi_\rho = \int d^3R' \int dt\ G(\vec{R}_o, t, \vec{R}', t')\rho(\vec{R}', t) \qquad (1\text{-}100)$$

Everywhere except at $\vec{R}_o = \vec{R}'$, $\delta(\vec{R}_o - \vec{R}') = 0$ so we have

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = 0 \qquad (1\text{-}101)$$

Since in this case G should be spherically symmetric about $\vec{R}'$, we can assume that it is a function only of $\vec{R} = |\vec{R}_o - \vec{R}'|$. Upon making the substitution $G = g/R$ we find

$$\frac{\partial^2 g}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = 0 \qquad (1\text{-}102)$$

The solution to this equation has the form

$$g = f_1\left(t - \frac{R}{c} - t'\right) + f_2\left(t + \frac{R}{c} - t'\right) \qquad (1\text{-}103)$$

Since we only want a particular solution we need use only one of f_1 and f_2. In this situation we are clearly dealing with outgoing waves so we take $f_2 = 0$. Then everywhere except at $\vec{R}' = \vec{R}_o$, G has the form

$$G = g\left(t - \frac{R}{c} - t'\right) / R\ . \qquad (1\text{-}104)$$

We must now choose g such that G has the correct value at R = 0. The potential of a point charge becomes infinite at the locus of the charge and so do its spatial derivatives, so the time derivatives in G can be neglected at $\vec{R}' = \vec{R}_o$, and (1-99) becomes

$$\nabla^2 G = -4\pi\delta(R)\delta(t - t') \qquad (1\text{-}105)$$

This is just the Poisson equation (1-25) with φ replaced by G and ρ replaced by $\delta(R)\delta(t-t')$. The solution at the origin is therefore (see equation (1-27)) $G = \delta(t - t')/R$ and

$g = \delta(t - \frac{R}{c} - t')$. The Green's function for this problem is

$$G = \delta(t - \frac{R}{c} - t')/R \tag{1-106}$$

The general solutions of equations (1-96) and (1-97) are

$$\varphi = \int \frac{\delta(t - \frac{R}{c} - t')\rho(\vec{R}', t')}{R} dt' \, dV' + \varphi_o \tag{1-107}$$

$$\vec{A} = \int \frac{\delta(t - \frac{R}{c} - t')\vec{j}(\vec{R}', t')}{R} dt' \, dV' + \vec{A}_o \tag{1-108}$$

where φ_o and $\vec{A}_o$ are solutions of the homogeneous wave equation. These solutions are determined by the initial or boundary conditions. $R = |\vec{R}_o - \vec{R}'|$ is the distance between the source coordinate $\vec{R}'$ and the point $\vec{R}_o$ at which the field is observed. The potentials are sometimes called the retarded potentials because they exhibit the causal behavior associated with a wave disturbance. The effect observed at the point $\vec{R}_o$ at time $\underline{t}$ is due to the behavior of the current or charge density at an earlier or retarded time $t' = t - R/c$ at the point $\vec{R}'$.

For a point charge with a position vector $\vec{r}(t')$ and velocity vector $\vec{v}(t')$ the charge and current densities are given by

$$\rho(\vec{R}', t') = e\,\delta\{\vec{R}' - \vec{r}(t')\} \tag{1-109}$$

$$\vec{j}(\vec{R}',t') = e\,\vec{v}\,\delta\{\vec{R}' - \vec{r}(t')\} \tag{1-110}$$

Upon substituting these expressions into equations (1-107) and (1-108) and performing the integrations, using the properties of the δ-function given in the note following equation (1-99) the potentials are found to be (see Jackson, 1962):

$$\varphi = \{e/\kappa R\}_{ret} \tag{1-111}$$

$$\vec{A} = \{e\vec{v}/\kappa R\}_{ret} \tag{1-112}$$

where $\{\ \}_{ret}$ means that the quantity inside the brackets is to be evaluated at the retarded time $t' = t - R/c$, and

$$\kappa = 1 - \hat{n} \cdot \vec{\beta} \tag{1-113}$$

$$\hat{n} = \vec{R}/R \tag{1-114}$$

For non-relativistic motion $\kappa \to 1$. For relativistic motion κ becomes small for some angles, which implies large potentials.

By expanding the charge and current density into monochromatic waves, the potentials can be expressed in terms of Fourier components:

$$\varphi(\omega) = \frac{e}{2\pi}\int_{-\infty}^{\infty} \frac{e}{R}^{\,i\omega(t+R/c)}\, dt \tag{1-115}$$

and

$$\vec{A}(\omega) = \frac{e}{2\pi}\int_{-\infty}^{\infty} \frac{\vec{\beta}e}{R}^{\,i\omega(t+R/c)}\, dt \tag{1-116}$$

where

$$\varphi(t) = \int_{-\infty}^{\infty} \varphi(\omega)e^{-i\omega t}\, d\omega, \text{ etc.} \qquad (1\text{-}117)$$

In the case of periodic motion an expansion in Fourier series yields:

$$\varphi(s) = \frac{2e}{\tau}\int_0^{\tau} \frac{e}{R}\, e^{is\omega_0(t+R/c)}\, dt \qquad (1\text{-}118)$$

and

$$\vec{A}(s) = \frac{2e}{\tau}\int_0^{\tau} \frac{\vec{\beta}e}{R}\, e^{is\omega_0(t+R/c)}\, dt \qquad (1\text{-}119)$$

where

$$\varphi(t) = \mathrm{Re}\left\{\sum_{s=1}^{\infty} \varphi(s)e^{-is\omega_0 t}\right\} \qquad (1\text{-}120)$$

and τ is the fundamental period of the motion $(= 2\pi/\omega_0)$.

To determine the electric and magnetic fields, we need to differentiate the potentials with respect to position and time (see equation (1-56)). To do this it is simpler to work with the integral expressions for the potentials (1-107) and (1-108). Using the relationships

$$\frac{d\hat{n}}{dt'} = c\left[\frac{\hat{n} \times (\hat{n} \times \beta)}{R}\right] \qquad (1\text{-}121)$$

and

$$\frac{1}{c}\frac{d}{dt'}(\kappa R) = \beta^2 - \vec{\beta}\cdot\hat{n} - \frac{R}{c}\,\hat{n}\cdot\dot{\vec{\beta}} \qquad (1\text{-}122)$$

one finds

$$\vec{E}(\vec{R}_o,t) = e\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\hat{n}}{\kappa^3 R}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\right]_{ret}$$

$$\vec{B} = \hat{n}\times\vec{E} \qquad (1\text{-}123)$$

The fields consist of two types. The first depends only on the velocity of the particle and not its acceleration and varies at large distances as $1/R^2$. The second depends on the acceleration and varies as $1/R$ at large distances. The ratio of the two types of fields is

$$|E_{vel}/E_{acc}| \sim \frac{c}{R_o}\frac{(1-\beta^2)}{\dot{\beta}} \sim \frac{c\tau}{R_o}(1-\beta^2) \sim \frac{\lambda}{R_o}(1-\beta^2) \qquad (1\text{-}124)$$

where τ is the characteristic time for changes in the system and λ is the characteristic wavelength of radiation from the system. Thus at distances R_o large compared to the wavelength of the radiation, $E_{vel}/E_{acc} << 1$, the electric and magnetic fields are given by

$$\vec{E}(\vec{R},t) = \frac{e}{c}\left[\frac{\hat{n}\times\{(n-\vec{\beta})\times\dot{\vec{\beta}}\}}{\kappa^3 R}\right]_{ret}$$

$$\vec{B}(\vec{R},t) = \hat{n} \times \vec{E} \tag{1-125}$$

Since the point of observation is assumed to be at a large distance compared with the region where the charge changes its direction, ($R_o >> R'$), the vector $\vec{R}$ is approximately in the same direction as $\vec{R}_o$, and

$$R \simeq R_o \left(1 - \frac{\hat{n}\cdot\vec{R}'}{R_o}\right) \tag{1-126}$$

To determine the potentials from equations (1-111) and (1-112) or the fields from equation (1-125), we can neglect $\hat{n}\circ\vec{R}'/R_o$ compared with unity in the denominator, but not in evaluating the expressions at the retarded time. Whether or not this term can be neglected in that calculation depends not on the relative values of $\vec{R}_o$ and $\hat{n}\cdot\vec{R}'$, but on how much the velocity and acceleration changes in the time $\hat{n}\cdot\vec{R}'/c$.

Thus when the point of observation is at a large distance compared with the region where the charge carries out its motion, the potentials can be written in the form

$$\varphi = \frac{e}{R_o}\left[\frac{1}{1 - \hat{n}\cdot\vec{\beta}(t')}\right]_{t'=t-\frac{R_o}{c}+\frac{\hat{n}\circ\vec{R}'}{c}} \tag{1-127}$$

and

$$\vec{A} = \frac{e}{R_o}\left[\frac{\vec{\beta}}{1 - \hat{n}\cdot\vec{\beta}(t')}\right]_{t'=t-\frac{R_o}{c}+\frac{\hat{n}\circ\vec{R}'}{c}} \tag{1-128}$$

The Fourier components become (see equations (1-115), (1-116))

$$\varphi(\omega) = \left(\frac{e}{2\pi R_o}\right) e^{ikR_o} \int e^{i[\omega t - \vec{k}\circ\vec{R}']} dt \qquad (1\text{-}129)$$

and

$$\vec{A}(\omega) = \left(\frac{e}{2\pi R_o}\right) e^{ikR_o} \int \vec{\beta}\, e^{i[\omega t - \vec{k}\circ\vec{R}']} dt \qquad (1\text{-}130)$$

1.9 Dipole Radiation

The term $\hat{n}\circ\vec{R}'/c$ in the retarded time can be neglected if the distribution of charge changes by a negligible amount during that time.

If r is the characteristic dimension of the system then

$$\hat{n}\circ\vec{R}'/c \sim r/c \qquad (1\text{-}131)$$

If the time scale for an appreciable change in position of the charge is τ, then the term $\hat{n}\cdot\vec{R}'/c$ will be small if

$$r \ll c\tau \qquad (1\text{-}132)$$

But τ is related to the frequency of the radiation from the system by $\tau \sim 1/\nu$, so the condition (1-132) can also be written as

$$r \ll c/\nu = \lambda \qquad (1\text{-}133)$$

That is, the dimensions of the system must be small compared to the wavelength of the radiation. From the definition of (1-70) of the wave number, this is also equivalent to

$$kR' << 1 \qquad (1\text{-}134)$$

The condition for the neglect of the term $\hat{n}\cdot\vec{R}'/c$ can also be expressed in terms of the velocity $\vec{v}$ of the charges, which must be of order r/τ, from which it follows, using (1-133) that

$$v << c \qquad (1\text{-}135)$$

i.e., the motion should be non-relativistic.

This approximation is called the dipole approximation and the radiation in this case is called dipole radiation. In this approximation we can set $\kappa = 1$, $R = R_0$ and evaluate all quantities at $t' = t-R_0/c$, which is a considerable simplification since R_0 is independent of t'. For all practical purposes we can drop the reference to the retarded time when working in the dipole approximation.

The expression for the electric field then takes the form

$$\vec{E} = \frac{e}{c}\,\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})/cR \qquad (1\text{-}136)$$

For a number of charges

$$\vec{E} = \hat{n} \times \frac{(\hat{n} \times \Sigma\, e\dot{\vec{\beta}})}{cR} = \hat{n} \times \frac{(\hat{n} \times \ddot{\vec{d}})}{c^2 R} \qquad (1\text{-}137)$$

where $\vec{d}$ is the dipole moment of the system (see equation (1-36)). Note that the radiation is determined by the second derivative of the dipole moment, hence the name "dipole approximation".

The power radiated is obtained from

Poynting's vector (see equation (1-15))

$$\vec{S} = c\,\frac{(\vec{E}\times\vec{B})}{4\pi} = (\ddot{d})^2 \sin^2\Theta\,\frac{\hat{n}}{4\pi c^3 R^2} \qquad (1\text{-}138)$$

where Θ is the angle between $\vec{\ddot{d}}$ and $\hat{n}$ and $d = |\vec{d}|$.

The power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} \equiv P(\Omega) = R^2\ \vec{S}\cdot\hat{n} = (\ddot{d})^2\,\frac{\sin^2\Theta}{4\pi c^3} \qquad (1\text{-}139)$$

Upon integration of equation (1-139) over solid angle $d\Omega = 2\pi\,\sin\Theta\,d\Theta$, the total power radiated is found to be

$$\frac{dW}{dt} \equiv P = \frac{2(\ddot{d})^2}{3c^3} \qquad (1\text{-}140)$$

This is Larmor's formula for the radiation from a non-relativistic charge. Note that the angular distribution of the radiation is symmetric about the direction of the acceleration of the charge and independent of its velocity (see equation (1-139) and Figure 1.2).

To obtain information about the spectrum of dipole radiation, we need the Fourier components of $dP/d\Omega$. We cannot calculate this, but we can calculate the energy $dW/d\Omega$ radiated per unit solid angle over the entire time during which the charge is accelerated:

$$\frac{dW}{d\Omega} = \int (dP/d\Omega)dt = (cR^2/4\pi)\int E^2\,dt \qquad (1\text{-}141)$$

by using Parseval's theorem:

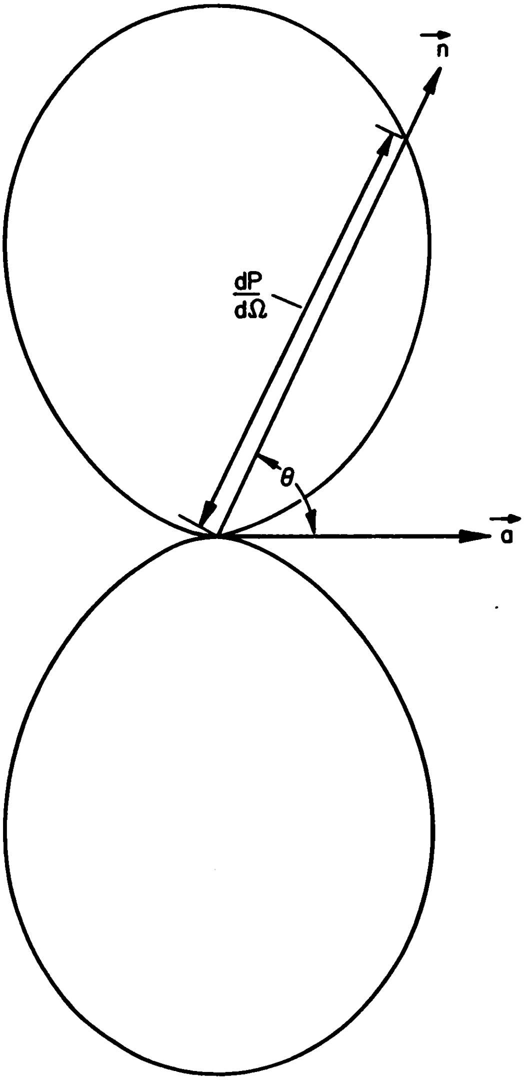

Figure 1.2. The angular distribution of the power radiated by a non-relativistic charge undergoing an acceleration a. The power unit solid angle radiated in the direction of the vector $\vec{n}$ is proportional to the radius vector as indicated in the figure.

$$\int_{-\infty}^{\infty} E^2(t)dt = 4\pi \int_0^{\infty} |E(\omega)|^2 \, d\omega \tag{1-142}$$

Therefore the amount of energy radiated per solid angle is

$$\frac{dW}{d\Omega} = cR^2 \int_0^{\infty} |E(\omega)|^2 \, d\omega \tag{1-143}$$

The energy per unit frequency interval is therefore

$$\frac{dW(\omega)}{d\Omega} = cR^2 \, |E(\omega)|^2 \tag{1-144}$$

where

$$E(\omega) = \hat{n} \times \frac{(\hat{n} \times \ddot{\vec{d}}(\omega))}{c^2 R} \tag{1-145}$$

Substituting (1-145) into (1-144) yields

$$dW(\omega)/d\Omega = [\ddot{d}(\omega)]^2 \, \sin^2\Theta/c^3$$

$$= \omega^4 \, d^2(\omega) \, \sin^2\Theta/c^3 \tag{1-146}$$

since $\ddot{d}(\omega) = \omega^2 d(\omega)$. Integrating (1-146) over all angles yields

$$W(\omega) = 8\pi(\ddot{d}(\omega))^2/3c^3 = 8\pi \, \omega^4 d^2(\omega)/3c^3 \tag{1-147}$$

For periodic motion Parseval's theorem takes the form

$$\frac{2}{T} \int_0^T E^2(t)dt = \sum_{s=1}^{\infty} |E_s|^2 \tag{1-148}$$

so

$$\tau^{-1}(dW_s/d\Omega) = (\ddot{d}_s)^2 \sin^2\Theta/8\pi c^3 \qquad (1\text{-}149)$$

The classical formula for dipole radiation can be used to calculate low frequency radiation resulting from the acceleration of a charged particle (see Chapters 4 and 5).

1.10 Radiation from a Relativistic Charged Particle

In the non-relativistic case we derived a simple expression for the total power radiated by an accelerated charge (see equation (1-140))

$$\frac{dW}{dt} = \frac{2}{3}\frac{e^2\dot{v}^2}{c^3} \qquad (1\text{-}150)$$

Of course this formula does not apply when the particle motion is relativistic. However in the frame of reference where the particle is at rest we certainly have $v << c$ and (1-150) applies. In this reference frame the particle radiates in time dt the energy

$$dW = \left(\frac{2e^2\dot{v}^2}{3c^3}\right) dt \qquad (1\text{-}151)$$

In this reference frame the momentum radiated is zero:

$$d\vec{p} = \int \overleftrightarrow{T} \cdot \hat{n}\, dA\, dt = 0 \qquad (1\text{-}152)$$

(see (1-19) and (1-136)). This is due to the symmetry of dipole radiation.

Writing equations (1-151) and (1-152) in four-vector notation we have

$$dp_i = (2e^2/3c)\ (du_k/ds)^2\ dx_i \qquad (1\text{-}153)$$

where

$$u_i = dx_i/ds = \gamma v_i/c \qquad (1\text{-}154)$$

$$ds = c\ dt/\gamma \qquad (1\text{-}155)$$

$$v_i = (\vec{v},\ ic) \qquad (1\text{-}156)$$

Since dp_i and dx_i are both four-vectors, the quantity relating them must be a scalar and therefore a Lorentz invariant:

$$(2\ e^2/3c)\ (du_k/ds)^2 = \text{invariant} \qquad (1\text{-}157)$$

The total power radiated in an arbitrary reference frame is found by noting that

$$dp_4 = i\ dW/c \quad ; \quad dx_4 = i\ c\ dt \qquad (1\text{-}158)$$

so

$$dp_4/dx_4 = (1/c^2)\ dW/dt$$

$$= (2\ e^2/3c^3)\gamma^2\,(du_i/dt)^2 \qquad (1\text{-}159)$$

Since this quantity is a Lorentz invariant we have that the total power radiated in any frame of reference (arbitrary velocity) is given by

$$dW/dt = (2\ e^2/3c)\gamma^2\,(du_i/dt)^2$$

$$= (2\ e^2/3c)\gamma^6\ \{\mathring{\beta}^2 - (\vec{\beta} \times \vec{\mathring{\beta}})^2\} \quad (1\text{-}160)$$

In terms of forces, note that for a particle of mass m:

$$d\vec{u}/ds = (\gamma/mc^2)d\vec{p}/dt \quad (1\text{-}161)$$

$$du_4/ds = (\gamma/mc^2)\ dW/dt = \gamma\ d\gamma/dt \quad (1\text{-}162)$$

In the case of a particle in electric and magnetic fields

$$d\vec{u}/ds = (e\gamma/mc^2)\ (\vec{E} + \vec{\beta} \times \vec{B}) \quad (1\text{-}163)$$

$$du_4/ds = e\gamma/mc^2)\ (\vec{\beta} \cdot \vec{E}) \quad (1\text{-}164)$$

(see equations (1-7) and (1-8)).

$$dW/dt = (2\ r_o^2\ c/3)\gamma^2\ \{|\vec{E} + \vec{\beta}\times\vec{B}|^2 - |\vec{\beta}\cdot\vec{E}|^2\} \quad (1\text{-}165)$$

where E, B refer to <u>external</u> fields, and

$$r_o = e^2/m\ c^2 \quad . \quad (1\text{-}166)$$

Equation (1-165) shows that the power radiated is inversely proportional to the square of the mass of the radiating particle, so that electrons radiate much more energy than protons in given electric and magnetic fields.

In the case of a particle moving parallel to the magnetic field and experiencing acceleration by an electric field which is parallel to the magnetic field,

$$dW/dt = (2\ r_o^2 c/3)E^2$$

$$= (2\ r_o^2 c/3e^2)(d\vec{p}/dt)^2 \qquad (1\text{-}167)$$

In the case of motion in a magnetic field, with the electric field equal to zero,

$$dW/dt = (2\ r_o^2 c/3)\gamma^2\ \beta_\perp^2\ B^2$$

$$= (2\ r_o^2 c/3e^2)\gamma^2 (d\vec{p}/dt)^2 \qquad (1\text{-}168)$$

where $\beta_\perp$ is the component of $\vec{\beta}$ perpendicular to the magnetic field. For relativistic particles these losses are proportional to the square of the energy and can become very large.

Thus for comparable forces, the power radiated by relativistic charges is a factor γ^2 less for acceleration parallel to the velocity than for acceleration perpendicular to the velocity.

In order to determine the angular distribution of the radiation, we must substitute the fields given by equation (1-125) into equation (1-15):

$$dP(t)/d\Omega = cE^2 R^2/4\pi$$

$$= (e^2/4\pi c)\ \left\{ |\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]^2/\kappa^6 \right\}_{t'+\frac{R}{c}=t} \qquad (1\text{-}169)$$

This is the energy per unit solid angle per unit time detected at the observer at time t due to radiation emitted by the charge at a time $t' = t - R/c$. To get the power radiated per unit solid angle in terms of the charge's time, we must multiply equation (1-169) by the factor (dt/dt') to take into account the relativistic effects caused by the charge's motion toward or away from the observer. From equation (1-126)

$$dt/dt' = 1 - \hat{n}\cdot\vec{\beta} = \kappa \tag{1-170}$$

In the ultrarelativistic case $\beta \simeq 1$, and $1 - \beta << 1$, so the terms in the denominator become small for $\hat{n}\circ\vec{\beta} \simeq \beta$ i.e., for radiation in the direction of $\vec{\beta}$. If θ denotes the angle between $\hat{n}$ and $\vec{\beta}$ then for small θ

$$\kappa = 1 - \beta \cos\theta \approx 1 - \beta + \frac{\beta\theta^2}{2} \tag{1-171}$$

For $\beta \simeq 1$, the expansion on the right will be small if the third term is of the order of the first two, i.e., if

$$\theta^2 \simeq 2(1-\beta)/\beta \simeq (1+\beta)(1-\beta) = 1/\gamma^2$$

or

$$\theta \simeq 1/\gamma \tag{1-172}$$

Thus most of the radiation is confined to a narrow cone of half-angle $\simeq \gamma^{-1}$ around the direction of the velocity of the particle.

When the velocity and acceleration of the particle are parallel, the intensity distribution is

$$dP(t')/d\Omega = (e^2/4\pi c)\dot{\beta}^2 \sin^2\theta/(1-\beta\cos\theta)^5 \tag{1-173}$$

For $\beta << 1$, this reduces to the Larmor result. As the speed of the charge approaches the speed of light

$$dP(t')/d\Omega \underset{\beta\to 1}{\to} (8e^2/\pi c)\gamma^8 \ \dot{\beta}^2 \ (\gamma\theta)^2/(1+\gamma^2\theta^2)^5 \tag{1-174}$$

As the velocity of the charge increases, the "figure eight" distribution characteristic of radiation from a non-relativistic charge is tipped forward and the peak intensity increases in magnitude proportional γ^8.

Integrating equation (1-174) over all angles, we obtain the result given in equation (1-160) for $\vec{\beta} \times \dot{\vec{\beta}} = 0$.

When the velocity and the acceleration are perpendicular

$$dP(t')/d\Omega = (e^2/4\pi c) \cdot \dot{\beta}^2 \left\{\frac{\gamma^2(1-\beta\cos\theta)^2 - \sin^2\theta\cos^2\varphi}{\gamma^2(1-\beta\cos\theta)^5}\right\} \tag{1-175}$$

where φ is the azimuthal angle of $\hat{n}$ relative to the plane passing through $\vec{\beta}$ and $\dot{\vec{\beta}}$. Again this reduces to the Larmor result for small β, since $1 - \sin^2\theta\ \cos^2\varphi = \sin^2\theta$. In the ultrarelativistic case (1-175) becomes

$$dP(t')/d\Omega = (2e^2/\pi c)\dot{\beta}^2 \cdot \gamma^6 \left\{\frac{(1+\gamma^2\theta^2)^2 - 4\gamma^2\theta^2\cos^2\varphi}{(1+\gamma^2\theta^2)^5}\right\} \tag{1-176}$$

The radiation pattern for the case $\varphi = 0$ and acceleration produced by the magnetic field (see Chapter 3) is shown below in Figure 1.3.

Figures 1.2 and 1.3, or equations (1-139), (1-173) and (1-176) illustrate how the radiation pattern from an accelerated charge changes as its velocity increases. For non-relativistic motion the angular distribution is independent of the velocity vector and is distributed over a wide angle. For relativistic motion the radiation is greatly enhanced in the direction of motion and is confined to a very narrow cone about that direction.

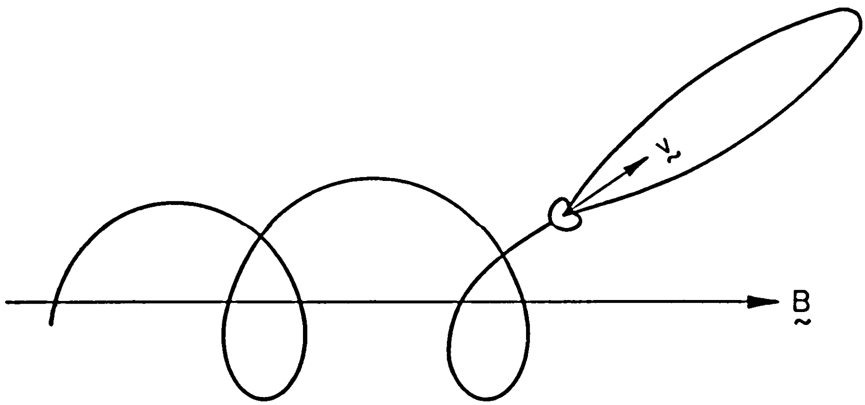

Figure 1.3. A relativistic particle spiraling in a magnetic field emitting synchrotron radiation with the angular pattern as indicated.

For a charged particle undergoing arbitrary ultra-relativistic motion the radiation emitted at any instant can be thought of as a coherent superposition of contributions coming from the components of acceleration parallel and perpendicular to the velocity. However, equations (1-167) and (1-168) show that, for comparable parallel and perpendicular forces the radiation from the parallel component is negligible (of order $1/\gamma^2$) compared to that from the perpendicular component. Therefore the radiation emitted by ultrarelativistic particles is very nearly the same as that emitted by a particle moving instantaneously along the arc of a circular path whose radius of curvature is given by

$$r_c = c^2/\dot{v}_\perp \qquad (1\text{-}177)$$

where $\dot{v}_\perp$ is the perpendicular component of the acceleration. As discussed above the radiation is concentrated primarily within a cone whose aperture angle $\Delta\theta$ is approximately equal to $2/\gamma$ about the direction of the instantaneous velocity of the particle. Within the limits of the angle $\Delta\theta$ the electron moves in the direction of the observer for a time

$$\Delta t' \simeq r_c\, \Delta\theta/c \simeq 2r_c/c\gamma \qquad (1\text{-}178)$$

During this time the electron moves a distance $v\Delta t'$ in the direction of the observer, so the radiation pulse contracts the length $v\Delta t'$ As a result, the observed length of the pulse is of the order

$$c\Delta t = (c-v)\Delta t' \qquad (1\text{-}179)$$

and its duration is

$$\Delta t = \Delta t'(1-\beta) \simeq \Delta t'/\gamma^2 \qquad (1\text{-}180)$$

The observed radiation spectrum will

therefore contain frequencies up to a maximum frequency ω_m:

$$\omega_m \simeq 1/\Delta t \simeq c\gamma^3/r_c \tag{1-181}$$

For frequencies much greater than this the exponential term in the Fourier transform of the fields oscillates rapidly and the slowly varying parts of the integral interfere destructively so that the integral becomes negligibly small (see equation (1-183)).

In the case of circular motion in a magnetic field

$$r_c = m_e c^2 \beta\ \gamma/eB \simeq c\gamma/\omega_B \tag{1-182}$$

where $\omega_B = eB/m_e c$ is the electron cyclotron frequency. The observed radiation spectrum will consist of harmonics of the frequency ω_B/γ extending up to $\omega_m \simeq \gamma^2 \omega_B$.

The equation describing the spectral distribution of the radiation from an accelerated charge can be obtained from equations (1-115), (1-125), (1-126), (1-144), and (1-170):

$$\frac{dW(\omega)}{d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \frac{\hat{n} \times [(\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}]}{\kappa^2} \cdot e^{i\omega(t' - \hat{n}\cdot R'/c)} dt' \right|^2 \tag{1-183}$$

This expression can be integrated by parts using the relationship

$$\frac{\hat{n} \times (\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}}{(1 - \hat{n}\cdot\vec{\beta})^2} = \frac{d}{dt'} \left\{\frac{\hat{n} \times (\hat{n}\times\vec{\beta})}{1 - \hat{n}\cdot\vec{\beta}}\right\}$$

The result is

$$dW(\omega)/d\Omega = (e^2\omega^2/4\pi^2 c)$$

$$\cdot \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n}\times\vec{\beta}) e^{i\omega(t' - \hat{n}\circ\vec{R}'/c)} dt' \right|^2 \qquad (1\text{-}183')$$

In the case of periodic motion

$$dW_s/\tau d\Omega = dP|d\Omega = (e^2\omega^2/8\pi^3 c)$$

$$\cdot \left| \int_0^T \hat{n} \times (\hat{n}\times\vec{\beta}) e^{is\omega_0(t - \hat{n}\circ\vec{R}'/c)} dt' \right|^2 \qquad (1\text{-}183'')$$

1.11 The Influence of Cosmic Plasma on the Propagation and Emission of Electromagnetic Waves

Up to this point it has been assumed that the radiation is emitted and propagated in a vacuum. Usually, this is a reasonable approximation to the actual situation. Sometimes, however, the medium radically influences the character of the electromagnetic radiation, with regard to both the emission and propagation of the waves.

Radiation processes in a dielectric medium can be discussed in terms of the formalism developed for radiation in a vacuum by making the substitutions

$$c \rightarrow c/n_r \qquad e \rightarrow e/n_r \qquad (1\text{-}184)$$

where n_r is the index of refraction, which for an isotropic plasma and for frequencies much greater than the frequency of collisions between particles is (Alfven and Falthammar, 1963, Ginzburg, 1964)

$$n_r^2(\omega) = 1 - (\omega_p^2/\omega^2) \tag{1-185}$$

$$\omega_p = (4\pi N_e e^2/m_e)^{\frac{1}{2}} = (3\times 10^9 N_e)^{\frac{1}{2}} \text{ rad/sec} \tag{1-186}$$

(N_e = electron number density).

That the transformation (1-184) is the correct one follows from the form of Maxwell's equations for a dielectric medium:

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{B} = 4\pi\vec{j}/c + \partial\vec{D}/c\partial t \tag{1-187}$$

Since $\vec{D} = n_r^2\,\vec{E}$, it is evident that the substitution (1-184) will cast the equations into a form identical with the form of Maxwell's equations for a vacuum.

The angular and frequency distribution of radiation emitted by a charged particle in motion is given by

$$dW(\omega)/d\Omega = (e^2\omega^2 n_r/4\pi^2 c) \cdot \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n}\times\vec{\beta}) e^{i\omega(t-n\,\hat{n}\cdot\vec{R}(t')/c)} dt \right|^2 \tag{1-188}$$

When $\omega < \omega_p$, n_r is imaginary and the radiation is exponentially damped.

The index of refraction can also have an imaginary component if absorption is occurring in the plasma. Denoting the imaginary component due to absorption by μ:

$$n_r^* = n_r + i\,\mu \qquad (1\text{-}189)$$

The electric field produced by an accelerated charge is

$$\vec{E} = \vec{E}_o\, e^{-\mu\omega z/c}\, e^{i\omega(t-n_r z/c)} \qquad (1\text{-}190)$$

for a wave propagated in the z-direction. In general the index of refraction and the absorption coefficient depend on the properties of the medium (density, temperature, magnetic field) and the frequency of the radiation. The determination of the exact nature of the dependence is a complex problem to which entire books are devoted (see e.g. Ginzburg, 1964). The general method of computing the absorption coefficient from the properties of the plasma is discussed in Chapter 2.

If the plasma has a magnetic field B, and if absorption is unimportant, index of refraction takes the form

$$n^2_{o,x} = 1 - \frac{2V(1-V)}{2(1-V) - U\sin^2\alpha \pm (U^2\sin^4\alpha + 4U(1-V)^2\cos^2\alpha)^{\frac{1}{2}}}$$

$$V = (\omega_p/\omega)^2 \qquad U = (\omega_B/\omega)^2 \qquad (1\text{-}191)$$

Here α is the angle between the direction of propagation of the wave and the magnetic field. The subscripts o and x refer to the two possible modes of propagation, the ordinary and the extraordinary modes, corresponding to taking the positive and negative square root, respectively, in the denominator of (1-191).

When $B = 0$, equation (1-191) reduces to the simple form (1-185) and there is no distinction between the o- and x-modes. For $B \neq 0$ and propagation along the magnetic field ($\alpha = 0$), (1-191) becomes

$$n^2_{o,x} = 1 - (\omega_p^2/\omega(\omega \pm \omega_B)) \qquad (1\text{-}192)$$

Using (1-192) in the wave equation shows that the o- and x-modes correspond to circularly polarized waves rotating clockwise (o) and counterclockwise (x) as they propagate along the field.

For waves propagating perpendicular to the magnetic field ($\alpha = \pi/2$),

$$n_o^2 = 1 - (\omega_p^2/\omega^2)$$

$$n_x^2 = 1 - (\omega_p^2(\omega^2 - \omega_p^2)/\omega^2(\omega^2 - \omega_p^2 - \omega_B^2)) \qquad (1\text{-}193)$$

The o-mode is polarized parallel to the magnetic field. Thus the magnetic field has no effect on the motion of the charges and the velocity of propagation is independent of the strength of the magnetic field. The x-mode is polarized perpendicular to the field so its propagation velocity does depend on the field strength.

For most cases of interest the frequency of the wave is much greater than the electron cyclotron frequency ($\omega >> \omega_B$) and the expression (1-191) can be simplified. In the limit where

$$1 - (\omega_p^2/\omega^2) >> \omega_B \sin^2\alpha/\omega \cos\alpha \tag{1-194}$$

(1-191) reduces to

$$n_{o,x}^2 = 1 - (\omega_p^2/\omega(\omega \pm \omega_L)) \tag{1-195}$$

where $\omega_L = \omega_B \cos\alpha$. Equation (1-195) is the same as equation (1-192) for propagation along the magnetic field (longitudingal propagation) with ω_B replaced by ω_L. Hence the propagation is called "quasi-longitudinal". This approximation is adequate to describe most of the situations encountered in astrophysics.

The propagation of waves in a magneto-active plasma generally depends strongly on the intensity and direction of the magnetic field. However, except in the vicinity of stars, the magnetic fields are sufficiently weak that the frequency of the radiation is much greater than the electron cyclotron frequency:

$$\omega >> \omega_B \tag{1-196}$$

In this limit the plasma can be considered practically isotropic, with the index of refraction given by (1-185). However, in the consideration of the polarization of the wave, even a small anisotropy can be important.

This is the basis of the Faraday effect, which is an important tool for measuring cosmic magnetic fields.

Consider a wave of amplitude A which is linearly polarized in the y-direction. This wave can be decomposed into two circularly polarized waves with opposite directions of rotation. At the origin we have

$$E = E_o + E_x = A\,e^{i\omega t} + Ae^{-i\omega t} \tag{1-197}$$

In the quasi-longitudinal approximation the two waves propagate at different velocities given by (1-195) so after propagation through a distance R the waves are described by

$$E_{o,x} = Ae^{\pm i\omega(t+n_{o,x}R/c)} \tag{1-198}$$

and the composite wave by

$$E = Ae^{i\Delta/2}\,(e^{i\omega(t+s)} + e^{-i\omega(t+s)}) \tag{1-199}$$

where

$$s = (n_o + n_x)(\omega R/c)$$

$$\Delta = (n_o - n_x)(\omega R/c) \tag{1-200}$$

Thus after traveling a distance R cm the wave is still linearly polarized but has been rotated through an angle

$$\begin{aligned}\psi = \Delta/2 &= \omega_p^2\omega_B R\cos\alpha/2\omega^2 c \\ &= 2.4\times10^4\ N_e B\ R\cos\alpha/\nu^2 \quad \text{rad.}\end{aligned} \tag{1-201}$$

($\nu = \omega/2\pi$).

In general, B, N_e and α will not be constant along the line of sight, so the product $N_e BR\cos\alpha$ must be replaced by the integral $\int N_e B\cos\alpha\, dR$, taken along the line of sight. If we express the frequency in terms of the wavelength of the radiation in meters, λ_m, and dR in parsecs, then

$$\psi = (8.1\times10^5 \int N_e B \cos\alpha\, dR)^2 \lambda_m^2 = R_m \lambda_m^2 \quad (1\text{-}202)$$

where R_m is called the <u>rotation measure</u>.

This integral cannot be determined from a single observation of the position angle of the plane of polarization because there is almost never any way to estimate the position angle of the plane of polarization at the source, and because there is no way to distinguish between values of ψ that differ by 180°. It is necessary to observe the source at several frequencies, and then to plot the observed position angles as a function of λ_m^2. The straight line fit to these points then gives the rotation measure.

Observations of the polarization of radio sources shows that the magnitudes of the Faraday rotation are on the average much smaller for high latitude extragalactic radio sources than for low latitude sources. Thus it seems that the major part of the rotation occurs within the galaxy rather than in the sources themselves or in the intergalactic medium. In addition it has been observed (Morris and Berge, 1964) that the sense of Faraday rotation changes sign from one side of the galactic plane to the other. This indicates that, in the neighborhood of the sun, the magnetic field changes sign when crossing the galactic plane. Magnetic fields calculated from the absolute value of the rotation measure range from 10^{-6} gauss to a few times 10^{-5} gauss, depending on the assumed values of N_e and R, the distribution of electron density over the field structures, and whether the field is predominantly uniform in direction, or is composed of a number of antiparallel filaments, or is rather irregular.

The radiation from cosmic radio sources can be de-polarized if the rotation measure is not the same for all the elements of the source within the observing beamwidth. These effects are discussed by Gardner and Whiteoak (1966) and Burn (1966).

References

Good general references for the material in this chapter are:

Jackson (1962) Chapters 6, 14, 15.

Landau and Lifshitz (1962) Chapters 4, 5, 6, 8 and 9.

Other references for particular topics:

Polarization:

Chandrasekhar (1960) Chapter 1.

Bekefi (1966) Chapter 1

Ginzburg and Syrovatskii (1969)

Influence of Plasmas and Magnetoactive Effects

Ginzburg (1964)

Landau and Lifshitz (1960) Chapter 11

Gardner and Whiteoak (1966)

Problems

1.1. The redshift $z = \Delta\lambda/\lambda = 2$ for receding galaxies corresponds to what value of v/c?

1.2. Show that, for a source with redshift z, the observed flux density $F(\nu)$ is related to the emitted flux density $F'(\nu)$ by

$$F(\nu) = F'[(1+z)\nu]/(1+z)$$

and the total fluxes are related by

$$F = F'(1+z)^2$$

1.3. A quarter-wave plate and a polarization filter are placed along the path of a beam of monochromatic light. Before entering the quarter-wave plate, the light has right-handed elliptical polarization; the ratio of the major to the minor axes is 4:1. No light is transmitted through the polarization filter. Show in a diagram the orientation of the axes of the plate and of the transmission axis of the filter with respect to the axes of the ellipse. Compute the angle formed by the transmission axis of the filter with the y-axis.

1.4. Magnetic dipole radiation is described by the same formulas as electric dipole radiation, with the electric dipole moment replaced by the magnetic dipole moment, and the electric vector rotated by 90°. Compute the radiation from a rotating magnetic star in which the magnetic moment is perpendicular to the axis of rotation. In particular if the magnetic moment is M and the angular velocity of rotation is ω, show that:

(a) the angular distribution of the radiation averaged over the period of the rotation is

$$dP/d\Omega = M^2 \omega^4 (1+\cos^2\theta)/8\pi c^3$$

where θ is the angle between the direction of observation and the axis of rotation;

(b) the total radiation is

$$P = 2M^2 \omega^4/3c^3$$

(c) the radiation along the axis of rotation is circularly polarized.

1.5. In some pulsar models (see, e.g., P. Sturrock, Ap. J. 164, 529, (1971) electrons are accelerated to ultrarelativistic speeds in a narrow cone near the surface of a neutron star and move away from the star along magnetic field lines. Since the lines are curved, they will emit "curvature radiation" as discussed in Section 1.10. Assuming a dipole configuration and considering only small angles near the pole, find the total power emitted and the peak frequency for the radiation from an electron of energy γmc^2.

1.6. Show that, if the index of refraction of the interstellar medium can be described by equation (1-185), then an infinitely sharp pulse of radiation emitted by a pulsar at a distance R from the earth will be smeared out at the receiver over a time

$$\Delta t = (R\omega_p^2/c\omega^3)\Delta\omega \qquad \text{sec}$$

where $\Delta\omega$ is the bandwidth of the receiver. Assume that $\omega >> \omega_p$, and the density is constant between the source and the observer.