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The Equivalence of Digital and Analog Signal Processing*

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A specific isomorphism is constructed via the transform domains between the analog signal space $L^2(-\infty, \infty)$ and the digital signal space l_2. It is then shown that the class of linear time-invariant realizable filters is invariant under this isomorphism, thus demonstrating that the theories of processing signals with such filters are identical in the digital and analog cases. This means that optimization problems involving linear time-invariant realizable filters and quadratic cost functions are equivalent in the discrete-time and the continuous-time cases, for both deterministic and random signals. Finally, applications to the approximation problem for digital filters are discussed.

LIST OF SYMBOLS

$f(t)$, $g(t)$	continuous-time signals
$F(j\omega)$, $G(j\omega)$	Fourier transforms of continuous-time signals
A	continuous-time filters, bounded linear transformations of $L^2(-\infty, \infty)$
$\{\mathbf{f}_n\}$, $\{\mathbf{g}_n\}$	discrete-time signals
$\mathbf{F}(z)$, $\mathbf{G}(z)$	z-transforms of discrete-time signals
A	discrete-time filters, bounded linear transformations of l_2
μ	isomorphic mapping from $L^2(-\infty, \infty)$ to l_2
$\mathfrak{F}L^2(-\infty, \infty)$	space of Fourier transforms of functions in $L^2(-\infty,\infty)$
$\mathfrak{z}l_2$	space of z-transforms of sequences in l_2
$\lambda_n(t)$	nth Laguerre function

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I. INTRODUCTION

The parallel between linear time-invariant filtering theory in the continuous-time and the discrete-time cases is readily observed. The theory of the z-transform, developed in the 1950's for the analysis of sampled-data control systems, follows closely classical Fourier transform theory in the linear time-invariant case. In fact, it is common practice to develop in detail a particular result for a continuous-time problem and to pay less attention to the discrete-time case, with the assumption that the derivation in the discrete-time case follows the one for continuous-time signals without much change. Examples of this can be found in the fields of optimum linear filter and compensator design, system identification, and power spectrum measurement.

The main purpose of this paper is to show, by the construction of a specific isomorphism between signal spaces $L^2(-\infty, \infty)$ and l_2, that the theories of processing signals with linear time-invariant realizable filters are identical in the continuous-time and the discrete-time cases. This will imply the equivalence of many common optimization problems involving quadratic cost functions. In addition, the strong link that is developed between discrete-time and continuous-time filtering theory will enable the data analyst to carry over to the digital domain many of the concepts which have been important to the communications and control engineers over the years. In particular, all the approximation techniques developed for continuous-time filters become available for the design of digital filters.

In the engineering literature, the term *digital filter* is usually applied to a filter operating on samples of a continuous signal. In this paper, however, the term *digital filter* will be applied to any bounded linear operator on the signal space l_2, and these signals will not in general represent samples of a continuous signal. For example if $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ are two sequences, the recursive filter

$$\mathbf{y}_n = \mathbf{x}_n - 0.5\,\mathbf{y}_{n-1}$$

will represent a digital filter whether or not the $\mathbf{x}_n$ are samples of a continuous signal. The important property is that a digital computer can be used to implement the filtering operation; the term *numerical filter* might in fact be more appropriate.

II. PRELIMINARIES

The Hilbert space $L^2(-\infty, \infty)$ of complex valued, square integrable, Lebesgue measurable functions $f(t)$ will play the role of the space of continuous-time signals. The Hilbert space l_2 of double-ended sequences of complex numbers $\{f_n\}_{n=-\infty}^{\infty}$ that are square summable will play the role of the space of discrete-time signals. A function in $L^2(-\infty, \infty)$ will be called an *analog* signal, and a sequence in l_2 will be called a *digital* signal. Similarly, a bounded linear transformation A of $L^2(-\infty, \infty)$ will be called an *analog filter,* and a bounded linear transformation A of l_2 will be called a *digital filter.* An analog filter A will be called *time-invariant* if

$$A: f(t) \to g(t), \qquad f(t), g(t) \in L^2(-\infty, \infty), \tag{1}$$

implies

$$A: f(t + \tau) \to g(t + \tau) \tag{2}$$

for every real number τ. Time-invariant analog filters can be represented by the convolution integral

$$g(t) = \int_{-\infty}^{\infty} f(\tau) a(t - \tau)\, d\tau, \tag{3}$$

where $a(t)$, the impulse response of the filter A, need not belong to $L^2(-\infty, \infty)$. Similarly, a digital filter A will be called *time-invariant* if

$$\mathrm{A}: \{f_n\} \to \{g_n\}, \qquad \{f_n\}, \{g_n\} \in l_2, \tag{4}$$

implies

$$\mathrm{A}: \{f_{n+\nu}\} \to \{g_{n+\nu}\} \tag{5}$$

for every integer ν. Time-invariant digital filters can be represented by the convolution summation

$$\{g_n\} = \left\{ \sum_{i=-\infty}^{\infty} f_i\, a_{n-i} \right\}, \tag{6}$$

where the sequence $\{a_n\}$, the impulse response of the filter A, need not belong to l_2.

Our program is to construct a specific isomorphism between the analog signal space and the digital signal space via their isomorphic transform domains. Hence, we now define the Fourier transform on the analog signal space, mapping $L^2(-\infty, \infty)$ to another space $L^2(-\infty, \infty)$ called the Fourier transform domain and denoted by $\mathfrak{F}L^2(-\infty, \infty)$. We need the following key results (Wiener, 1933; Titchmarsh, 1948):

THEOREM 1 (Plancherel). *If $f(t) \in L^2(-\infty, \infty)$, then*

$$F(s) = \underset{R\to\infty}{\text{l.i.m.}} \int_{-R}^{R} f(t) e^{-st}\, dt \tag{7}$$

exists for $s = j\omega$, *and* $F(j\omega) \in L^2(-\infty, \infty)$. *Furthermore,*

$$(f, f) = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} |F(s)|^2 \, ds, \tag{8}$$

and

$$f(t) = \underset{R\to\infty}{\text{l.i.m.}} \int_{-jR}^{jR} F(s) e^{st} \, ds. \tag{9}$$

Analytic extension of $F(j\omega)$ *to the rest of the s-plane (via* (7) *when it exists, for example) gives the two-sided Laplace transform.*

THEOREM 2 (Parseval). *If* $f(t), g(t) \in L^2(-\infty, \infty)$, *then*

$$(f, g) = \int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s) G^*(t) \, ds. \tag{10}$$

The theory required for the analogous construction of a z-transform domain for digital signals is really no more than the theory of Fourier series. Consider the digital signal as a sequence of Fourier coefficients, and consider the periodic function with these Fourier coefficients as the z-transform evaluated on the unit circle in the z-plane. The Riesz-Fischer Theorem (Wiener, 1933) then reads:

THEOREM 3 (F. Riesz-Fischer). *If* $\{\mathbf{f}_n\} \in l_2$, *then*

$$\mathbf{F}(z) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{n=-N}^{N} \mathbf{f}_n z^{-n} \tag{11}$$

exists for $z = e^{j\omega T}$, *and* $\mathbf{F}(e^{j\omega T}) \in L^2(0, 2\pi/T)$, *where* ω *is the independent variable of* $L^2(0, 2\pi/T)$, *and this* ω *is unrelated to the* ω *used in the s-plane. Furthermore,*

$$(\{\mathbf{f}_n\}, \{\mathbf{f}_n\}) = \sum_{n=-\infty}^{\infty} |\mathbf{f}_n|^2 = \frac{1}{2\pi j} \oint |\mathbf{F}(z)|^2 \frac{dz}{z}, \tag{12}$$

and

$$\mathbf{f}_n = \frac{1}{2\pi j} \oint \mathbf{F}(z) z^n \frac{dz}{z}, \tag{13}$$

where integrals in the z-plane are around the unit circle in the counterclockwise direction.

As in the analog case, the analytic extension of $\mathbf{F}(e^{j\omega T})$ to the rest of the z-plane will coincide with the ordinary z-transform, which is usually defined only for digital signals of exponential order.

THEOREM 4 (Parseval). *If* $\{\mathbf{f}_n\}, \{\mathbf{g}_n\} \in l_2$, *then*

$$(\{\mathbf{f}_n\}, \{\mathbf{g}_n\}) = \sum_{n=-\infty}^{\infty} \mathbf{f}_n \mathbf{g}_n^* = \frac{1}{2\pi j} \oint \mathbf{F}(z)\mathbf{G}^*(z) \frac{dz}{z}. \tag{14}$$

We denote the space $L^2(0, 2\pi/T)$ *of z-transforms of digital signals by* $_{\mathfrak{Z}}l_2$.

III. A SPECIFIC ISOMORPHISM BETWEEN THE ANALOG AND DIGITAL SIGNAL SPACES

Intuitively, if we wish to connect the space of analog signals with the space of digital signals in such a way as to preserve the time-invariance and realizability of filters, we should somehow connect the $j\omega$-axis in the s-plane with the unit circle in the z-plane. The natural correspondence provided by the instantaneous sampling of analog signals matches e^{sT} with z, but is not one-to-one and hence cannot be an isomorphism. The next natural choice is the familiar bilinear transformation

$$s = \frac{z-1}{z+1}, \qquad z = \frac{1+s}{1-s}. \tag{15}$$

There is an additional factor required so that the transformation will preserve inner products. Accordingly, the image $\{\mathbf{f}_n\} \in l_2$ corresponding to $f(t) \in L^2(-\infty, \infty)$ will be defined as the sequence with the z-transform

$$= \frac{\sqrt{2}}{z+1} F\left(\frac{z-1}{z+1}\right). \tag{16}$$

Thus the mapping $L^2(-\infty, \infty) \to l_2$ is defined by a chain which goes from $L^2(-\infty, \infty)$ to $\mathfrak{F}L^2(-\infty, \infty)$ to $_{\mathfrak{Z}}l_2$ to l_2 as follows:

$$\mu: f(t) \to F(s) \to \frac{\sqrt{2}}{z+1} F\left(\frac{z-1}{z+1}\right) = \mathbf{F}(z) \to \{\mathbf{f}_n\}. \tag{17}$$

The inverse mapping is easily defined, since each of these steps is uniquely reversible:

$$\mu^{-1}: \{\mathbf{f}_n\} \to \mathbf{F}(z) \to \frac{\sqrt{2}}{1-s} \mathbf{F}\left(\frac{1+s}{1-s}\right) = F(s) \to f(t). \tag{18}$$

We then have

THEOREM 5. *The mapping*

$$\mu: L^2(-\infty, \infty) \to l_2$$

defined by (17) *and* (18) *is an isomorphism.*

Proof: μ is obviously linear and onto. To show that it preserves inner product, let $z = (1 + s)/(1 - s)$ in Parseval's relation (10), yielding

$$(f, g) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s)G^*(s)\, ds = \frac{1}{2\pi j} \oint \mathrm{F}(z)\mathrm{G}^*(z) \frac{dz}{z} = (\{\mathrm{f}_n\}, \{\mathrm{g}_n\}). \tag{19}$$

We can show that μ is one-to-one in the following way: if $f \neq g$, then $(f - g, f - g) = (\{\mathrm{f}_n\} - \{\mathrm{g}_n\}, \{\mathrm{f}_n\} - \{\mathrm{g}_n\}) \neq 0$; which implies that $\{\mathrm{f}_n\} \neq \{\mathrm{g}_n\}$, and hence that μ is one-to-one.

We note here that under the isomorphisms μ and μ^{-1} signals with rational transforms are always matched with signals with rational transforms, a convenience when dealing with the many signals commonly encountered in engineering problems with transforms which are rational functions of s or z.

IV. THE ORTHONORMAL EXPANSION ATTACHED TO μ

The usual way of defining an isomorphism from $L^2(-\infty, \infty)$ to l_2 is to map an arbitrary function in $L^2(-\infty, \infty)$ to the sequence in l_2 of its coefficients in some orthonormal expansion. It comes as no surprise, then, that the isomorphism μ could have been so generated. This section will be devoted to finding this orthonormal expansion.

We start with the z-transform of the digital signal $\{\mathrm{f}_n\}$ which is the image under μ of an arbitrary analog signal $f(t)$:

$$\mathrm{F}(z) = \frac{\sqrt{2}}{z + 1} F\left(\frac{z - 1}{z + 1}\right) = \sum_{n=-\infty}^{\infty} \mathrm{f}_n z^{-n}. \tag{20}$$

By (13), the formula for the inverse z-transform, we have

$$\mathrm{f}_n = \frac{1}{2\pi j} \oint \frac{\sqrt{2}}{z + 1} F\left(\frac{z - 1}{z + 1}\right) z^n \frac{dz}{z}. \tag{21}$$

Letting $z = (1 + s)/(1 - s)$, this integral becomes

$$\mathrm{f}_n = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s) \frac{\sqrt{2}}{1 + s} \left(\frac{1 + s}{1 - s}\right)^n ds. \tag{22}$$

By Parseval's relation (10) this can be written in terms of time functions as

$$\mathrm{f}_n = \int_{-\infty}^{\infty} f(t)\lambda_n(t)\, dt, \tag{23}$$

where the $\lambda_n(t)$ are given by the following inverse two-sided Laplace transform

$$\lambda_n(t) = \mathfrak{L}^{-1}\left[\frac{\sqrt{2}}{1-s}\left(\frac{1-s}{1+s}\right)^n\right]. \tag{24}$$

We see immediately that, depending on whether $n > 0$ or $n \leqq 0$, $\lambda_n(t)$ vanishes for negative time or positive time. By manipulating a standard transform pair involving Laguerre polynomials we find:

$$\lambda_n(t) = \begin{cases} (-1)^{n-1}\sqrt{2}e^{-t}L_{n-1}(2t)u(t), & n = 1, 2, 3, \cdots \\ (-1)^{-n}\sqrt{2}e^{t}L_{-n}(-2t)u(-t), & n = 0, -1, -2, \cdots \end{cases} \tag{25}$$

where $u(t)$ is the Heaviside unit step function, and $L_n(t)$ is the Laguerre polynomial of degree n, defined by

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 0, 1, 2, \cdots \tag{26}$$

The set of functions $\lambda_n(t)$, $n = 1, 2, 3, \cdots$, is a complete orthonormal set on $(0, \infty)$ and are called Laguerre functions. They have been employed by Lee (1931–2), Wiener (1949), and others for network synthesis; and are tabulated in Wiener (1949), and, with a slightly different normalization, in Head and Wilson (1956). The functions $\lambda_n(t)$, $n = 0, -1, -2, \cdots$, are similarly complete and orthonormal on $(-\infty, 0)$, so that the orthonormal expansion corresponding to (23) is

$$f(t) = \sum_{n=-\infty}^{\infty} \mathbf{f}_n\lambda_n(t). \tag{27}$$

We see that the values of the digital signal $\{\mathbf{f}_n\}$ for $n > 0$ correspond to the coefficients in the Laguerre expansion of $f(t)$ for positive t; and that the values of $\{\mathbf{f}_n\}$ for $n \leqq 0$ correspond to the coefficients in the Laguerre expansion of $f(t)$ for negative t.

V. THE INDUCED MAPPING FOR FILTERS

Thus far, we have explicitly defined four isomorphic Hilbert spaces as follows

$$\begin{array}{ccc} L^2(-\infty, \infty) & \text{———} & l_2 \\ | & & | \\ \mathfrak{F}L^2(-\infty, \infty) & \text{———} & {}_{\mathfrak{F}}l_2 = L^2(0, 2\pi/T) \end{array} \tag{28}$$

Therefore an analog or a digital filter as a bounded linear transformation has image transformations induced on the remaining three spaces.

A time-invariant analog filter A, defined by the convolution integral (3), has an image in $\mathfrak{F}L^2\ (-\infty,\ \infty)$, in l_2, and in $_{\mathfrak{z}}l_2$. Its image in $\mathfrak{F}L^2\ (-\infty,\ \infty)$ is multiplication by $A(s)$, the Fourier transform of $a(t)$. Its image A in l_2 can be found in the following way: let $\mathbf{x}$ be any digital signal. There corresponds to $\mathbf{x}$ a unique analog signal $\mu^{-1}(\mathbf{x})$. The result of operating on this analog signal by the analog filter A, $A\mu^{-1}(\mathbf{x})$, is also uniquely defined. This new analog signal can then be mapped by μ into a unique digital signal $\mu A\mu^{-1}(\mathbf{x})$, which we designate as the result of operating by A on $\mathbf{x}$. Thus we define A to be the composite operator

$$\mathrm{A} = \mu A\mu^{-1}(\quad). \tag{29}$$

To find the image of the analog filter A in $_{\mathfrak{z}}l_2$, notice that the Fourier transform of the analog signal Af is $A(s)F(s)$ and the z-transform of the digital signal μAf is

$$\frac{\sqrt{2}}{z+1}\, A\left(\frac{z-1}{z+1}\right) F\left(\frac{z-1}{z+1}\right). \tag{30}$$

Therefore, the image in $_{\mathfrak{z}}l_2$ of A and hence of A is multiplication by

$$\mathrm{A}(z) = A\left(\frac{z-1}{z+1}\right). \tag{31}$$

Similarly, a time-invariant digital filter A has an image in $_{\mathfrak{z}}l_2$ given by multiplication by $\mathrm{A}(z)$, the z-transform of the impulse response $\{\mathrm{a}_n\}$; an image in $L^2\ (-\infty,\ \infty)$ given by

$$A = \mu^{-1}\mathrm{A}\mu(\quad); \tag{32}$$

and an image in $\mathfrak{F}L^2\ (-\infty,\ \infty)$ given by multiplication by

$$A(s) = \mathrm{A}\left(\frac{1+s}{1-s}\right). \tag{33}$$

We have therefore proved

THEOREM 6. *The isomorphism μ always matches time-invariant analog filters A with time-invariant digital filters* A. *Furthermore,*

$$\mathrm{A}(z) = A\left(\frac{z-1}{z+1}\right), \tag{34}$$

and

$$A(s) = \mathrm{A}\left(\frac{1+s}{1-s}\right). \tag{35}$$

Those time-invariant filters which are physically realizable in the sense that they are nonanticipatory are of great importance in many fields. A time-invariant analog filter A will be called *realizable* if $Af = 0$ for $t < 0$ whenever $f = 0$ for $t < 0$. Similarly, a time-invariant digital filter A will be called *realizable* if $\mathrm{A}\{\mathbf{f}_n\} = 0$ for $n \leqq 0$ whenever $\{\mathbf{f}_n\} = 0$ for $n \leqq 0$. It is an important property of the mapping μ that it preserves the realizability of time-invariant filters. To see this, suppose first that A is a time-invariant realizable analog filter. Let $\{\mathbf{f}_n\}$ be any digital signal for which $\{\mathbf{f}_n\} = 0$ for $n \leqq 0$. Then its analog image $f(t)$ is such that $f(t) = 0$ for $t < 0$, by (27). Thus $Af = 0$ when t is negative, which implies that $\mathrm{A}\{\mathbf{f}_n\} = 0$ for $n \leqq 0$, by (23). Hence A is a realizable digital filter. The same argument works the other way, and this establishes:

THEOREM 7. *The mapping μ always matches time-invariant realizable analog filters with time-invariant realizable digital filters.*

VI. OPTIMIZATION PROBLEMS FOR SYSTEMS WITH DETERMINISTIC SIGNALS

We are now in a position to see how some optimization problems can be solved simultaneously for analog and digital signals. Suppose, for example, that a certain one-sided analog $r(t)$ is corrupted by a known additive noise $n(t)$, and that we are required to filter out the noise with a stable, realizable time-invariant filter H whose Laplace transform is, say, $H(s)$. If we adopt a least integral-square-error criterion, we require that

$$\int_0^\infty [r \quad H(r+n)]^2 \, dt = \min. \tag{36}$$

As described by Chang (1961), this can be transformed by Parseval's relation to the requirement

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [R - H(R+N)][R - H(R+N)]^* \, ds = \min., \tag{37}$$

where R, H, and N are functions of s, and $(\quad)^*$ means that s is replaced by $-s$. It can be shown, using an adaptation of the calculus of variations, that the realizable solution for $H(s)$, say $H_0(s)$, is given by

$$H_0(s) = \frac{1}{Y}\left[\frac{(R+N)^*R}{Y^*}\right]_{\text{LHP}}, \tag{38}$$

where

$$YY^* = (R+N)(R+N)^* \tag{39}$$

Y has only left-half plane poles and zeros, and Y^* has only right-half plane poles and zeros. The notation $[\ \ ]_{\text{LHP}}$ indicates that a partial fraction expansion is made and only the terms involving left-half plane poles are retained.

The fact that a least integral-square-error criterion is used means that the optimization criterion (36) can be expressed within the axiomatic framework of Hilbert space. Thus, in $L^2(-\infty, \infty)$, (36) becomes

$$\| r - H(r+n) \| = \min. \tag{40}$$

If we now apply the isomorphism μ to the signal $r - H(r+n)$, we have

$$\| r - H(r+n) \| = \| \mu[r - H(r+n)] \| = \| \mathbf{r} - \mathrm{H}(\mathbf{r}+\mathbf{n}) \|, \tag{41}$$

since μ preserves norm. Hence H_0 is the solution to the optimization problem

$$\| \mathbf{r} - \mathrm{H}(\mathbf{r}+\mathbf{n}) \| = \min. \tag{42}$$

Furthermore, since μ matches one-sided analog signals with one-sided digital signals and realizable time-invariant analog filters with realizable time-invariant digital filters, we see that H_0 is the solution to a digital problem that is completely analogous to the original analog problem. Thus

$$\mathrm{H}_0(z) = H_0\left(\frac{z-1}{z+1}\right) = \frac{z}{\mathbf{Q}}\left[\frac{(\mathbf{R}+\mathbf{N})^*\mathbf{R}}{z\mathbf{Q}^*}\right]_{\text{in}}, \tag{43}$$

where

$$\mathbf{Q}\mathbf{Q}^* = (\mathbf{R}+\mathbf{N})(\mathbf{R}+\mathbf{N})^*. \tag{44}$$

Now $\mathbf{R}$, $\mathbf{N}$, and H_0 are functions of z; $(\ \ )^*$ means that z is replaced by z^{-1}; $\mathbf{Q}$ and $\mathbf{Q}^*$ have poles and zeros inside and outside the unit circle respectively; and the notation $[\ \ ]_{\text{in}}$ indicates that only terms in a partial fraction expansion with poles inside the unit circle have been retained.

In other optimization problems we may wish to minimize the norm of some error signal while keeping the norm of some other system signal within a certain range. In a feedback control system, for example, we may want to minimize the norm of the error with the constraint that the norm of the input to the plant be less than or equal to some prescribed number. Using Lagrange's method of undetermined multipliers, this problem can be reduced to the problem of minimizing a quantity of the form

$$\| e \|^2 + k \| i \|^2, \tag{45}$$

where e is an error signal, i is some energy limited signal, and both e and i depend on an undetermined filter H. Again, if $H_0(s)$ is the time-invariant realizable solution to such an analog problem, then $\mathrm{H}_0(z)$ is the time-invariant realizable solution to the analogous digital problem determined by the mapping μ.

More generally, we can state

THEOREM 8. *Let ν be an isomorphism between L^2 $(-\infty, \infty)$ and l_2. Further, let the following optimization problem be posed in the analog signal space L^2 $(-\infty, \infty)$: Find analog filters $H_1, H_2, \cdots, H_n$ which minimize some given function of some norms in a given analog signal transmission system and which are in a class of filters $\mathcal{K}$. Then if the class of filters $\mathcal{K}$ is invariant under ν, the corresponding digital problem is equivalent to the original analog problem in the sense that, whenever one can be solved, the other can be also. In particular, when ν is μ, $\mathcal{K}$ can be taken as the class of time-invariant filters or the class of time-invariant realizable filters. In this situation, the optimum filters are related by*

$$\mathrm{H}_i(z) = H_i\left(\frac{z-1}{z+1}\right), \qquad i = 1, 2, 3, \cdots, n. \tag{46}$$

VII. RANDOM SIGNALS AND STATISTICAL OPTIMIZATION PROBLEMS

While the consideration of systems with deterministic signals is important for many theoretical and practical reasons, it is often the case that the engineer knows only the statistical properties of the input and disturbing signals. For this reason the design of systems on a statistical basis has become increasingly important in recent years. The method of connecting continuous-time theory with discrete-time theory described above can be extended to the random case in a natural way if we restrict ourselves to random processes which are stationary with zero mean, ergodic, and have correlation functions of exponential order. For our purposes, such processes will be characterized by their second order properties. In the analog case these are the correlation function $\phi_{xy}(t)$ and its Fourier transform $\Phi_{xy}(s)$. In the digital case these are the correlation sequence $\boldsymbol{\phi}_{xy}(n)$ and its z-transform $\boldsymbol{\Phi}_{xy}(z)$.

We define the mapping μ for correlation functions in the following way, motivated by mapping the signals in the ensembles by the isomorphism μ for signals:

$$\mu: \phi_{xy}(t) \to \Phi_{xy}(s) \to \frac{2z}{(z+1)^2} \Phi_{xy}\left(\frac{z-1}{z+1}\right) = \boldsymbol{\Phi}_{xy}(z) \to \boldsymbol{\phi}_{xy}(n). \quad (47)$$

The inverse mapping is

$$\mu^{-1}: \boldsymbol{\phi}_{xy}(n) \to \boldsymbol{\Phi}_{xy}(z) \to \frac{2}{1-s^2} \boldsymbol{\Phi}_{xy}\left(\frac{1+s}{1-s}\right) = \Phi_{xy}(s) \to \phi_{xy}(t). \quad (48)$$

The important invariants under μ are the quantities

$$\phi_{xy}(0) = E[x(t)y(t)], \quad (49)$$

and

$$\boldsymbol{\phi}_{xy}(0) = E[\mathbf{x}_n\mathbf{y}_n], \quad (50)$$

which correspond to the inner products in the deterministic case. As before, time-invariant filters are matched with time-invariant filters, and time-invariant realizable filters are matched with time-invariant realizable filters. Hence, we have

THEOREM 9. *Let the following optimization problem be posed for random analog signals: Find analog filters* $H_1, H_2, \cdots, H_n$ *which minimize some given function of the mean-square values of some signals in an analog signal transmission system and which are in a class of filters* $\mathcal{K}$. *Then if* $\mathcal{K}$ *is the class of time-invariant filters, or the class of time-invariant realizable filters, the corresponding digital problem is equivalent to the original analog problem in the sense that, whenever one can be solved, the other can be also. If the correlation functions and power spectral densities are related by* μ, *the optimum filters are again related by* (46).

In summary, we have shown that in the time-invariant case the theory of processing analog signals and the theory of processing digital signals are the same.

VIII. THE APPROXIMATION PROBLEM FOR DIGITAL FILTERS

The mapping μ can be used to reduce the approximation problem for digital filters to that for analog filters (Steiglitz, 1962; Golden and Kaiser, 1964). Suppose that we wish to design a digital filter with a rational transform and a desired magnitude or phase characteristic as a function of ω, $-\pi/T \leqq \omega \leqq \pi/T$. For real frequencies the transformation μ relates the frequency axes by

$$\omega = \tan \omega T/2. \tag{51}$$

We can therefore transform the desired characteristic to a function of ω simply by stretching the abscissa according to (51). This new characteristic can be interpreted as the frequency characteristic of an analog filter, and we can approximate this with the rational analog filter $A(s)$. $\text{A}(z) = A((z - 1)/(z + 1))$ will then be a rational function digital filter with the appropriate frequency characteristic. Many of the widely used approximation criteria, such as equal-ripple or maximal flatness, are preserved under this compression of the abscissa. Also, by Theorems 6 and 7, the time-invariant or the time-invariant realizable character of the approximant is preserved. Applications to the design of windows for digital spectrum measurement are discussed elsewhere (Steiglitz, 1963).

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