The Random Function of Time and Phase

I wish to discuss methods of handling a random process and, particularly, methods of handling the spectrum of the process, although these methods have a much more general application. In order to do this, I wish to bring in a discussion of the Brownian motion. By this I mean a Brownian motion in one dimension. Furthermore, for this hour perhaps, I shall discuss the Brownian motion over an interval of time 0 to 1 (Figure 1.1).

Let us consider the wandering of a particle. As time goes on, I want this particle to wander in a random way, so that the amount that it has

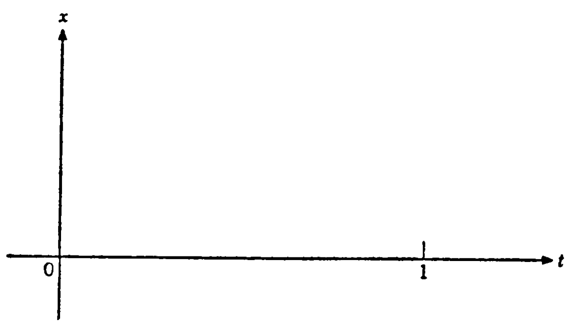

Figure 1.1

departed from a given position at a given time has a Gaussian distribution at another time, and so that in nonoverlapping intervals of time these Gaussian distributions are independent. I shall first discuss these motions at fixed times, and then, on the basis of the discussion of this motion at fixed times, I shall discuss the distribution of curves of this motion—first, over a finite time, and then over an infinite time. Then I am going to consider processes that depend upon this process in linear and in nonlinear ways.

A formula that we want is the formula of composition of Gaussian distributions that are independent over independent intervals of time. Consider a quantity x that has a Gaussian distribution. The probability that this quantity is between x_1 and x_2 is given by

$$P = \int_{x_1}^{x_2} \frac{1}{(2\pi a)^{1/2}} \exp\left(-\frac{x^2}{2a}\right) dx \tag{1.1}$$

I have not yet said how a depends upon time, but in order to get a reasonable way of doing this, I shall first discuss the composition of two such motions. That is, I want a particle that at the end of a certain time has a Gaussian distribution. Starting from where it is, the new departure has a Gaussian distribution. What is the distribution at the end of the two times?

So, I start with the joint distribution function

$$\left[\frac{1}{(2\pi a)^{1/2}} \exp\left(-\frac{x^2}{2a}\right)\right]\left\{\frac{1}{(2\pi b)^{1/2}} \exp\left[-\frac{(y-x)^2}{2b}\right]\right\} \tag{1.2}$$

Note that it is $y - x$ rather than y which is the parameter of the second probability density in Expression 1.2. Now, I want the probability distribution for y when x goes to all possible values. What I am interested in is

$$dy \int_{-\infty}^{\infty} \frac{1}{(2\pi a)^{1/2}} \exp\left(-\frac{x^2}{2a}\right) \frac{1}{(2\pi b)^{1/2}} \exp\left(-\frac{(y-x)^2}{2b}\right) dx \tag{1.3}$$

It is easy to compute the integral of Expression 1.3. It will not be necessary for me to go through it here. It's quite trivial. The answer is given by Equation 1.4:

$$dy \int_{-\infty}^{\infty} \frac{1}{(2\pi a)^{1/2}} \exp\left(-\frac{x^2}{2a}\right) \frac{1}{(2\pi b)^{1/2}} \exp\left[-\frac{(y-x)^2}{2b}\right] dx$$
$$= dy \frac{1}{[2(a+b)]^{1/2}} \exp\left(-\frac{y^2}{2(a+b)}\right) \tag{1.4}$$

That is the law of composition of Gaussian distributions. Notice that this parameter a adds up when we compound two Gaussian distributions.

If we then consider that the wandering in nonoverlapping intervals is Gaussian and independent, and that the amount of wandering is dependent only on the time interval and not on the original time, then we see that, because a and b add, the distribution starting from a certain time and ending at a certain other time is given by

$$\frac{1}{(2\pi k)^{1/2}} \exp\left(-\frac{x^2}{2k}\right) \tag{1.5}$$

where k depends linearly on the time difference t; and I shall normalize it so that it is t itself; thus

$$\frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{x^2}{2t}\right) \tag{1.6}$$

Notice that if I compound Gaussian distributions, then I add the time parameters for the new Gaussian distributions.

For the moment, I am going to consider trying to map all paths of particles on the variable α when α goes from 0 to 1. I want to set up this mapping in detail. The first thing is that all random motions of particles end up somewhere; that is, they will all map on this interval 0 to 1 of α. At the present stage, I have not separated the different ranges of α; I have taken the whole range from 0 to 1 and assigned it to all of the curves.

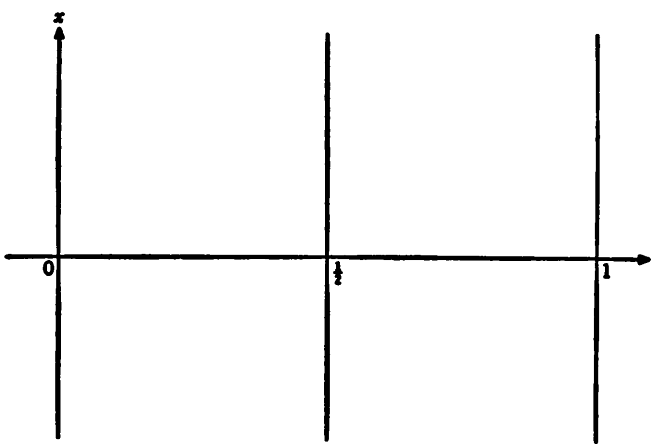

Figure 1.2

Let us consider the Brownian motion at two times, $\frac{1}{2}$ and 1 (Figure 1.2). I shall introduce four classes of Brownian motion. Notice that at time $\frac{1}{2}$ we have two possibilities: below the axis and above the axis. The probability of it landing on the axis is 0. It is like tossing a coin, where the probability of standing on end is 0. We toss the coin twice: once at time $\frac{1}{2}$ and once at time 1. Given that it is either heads or tails, there is a possibility of heads-heads, heads-tails, tails-heads, and tails-tails. What are these possibilities of heads-heads, tails-tails, and so forth? The two values of a and b in the case of Equation 1.4 corresponding to the two tossings are both $\frac{1}{2}$. The distribution is given by

$$\frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{x^2}{2(\frac{1}{2})}\right] \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{(y-x)^2}{2(\frac{1}{2})}\right] dx\, dy \quad (1.7)$$

Expression 1.7 is the probability that the particle at time $\frac{1}{2}$ lies between x and $x + dx$, and at time 1 lies between y and $y + dy$. On this basis, the probability that the particle lies, say, below the axis at time $\frac{1}{2}$ (that x be negative at time $\frac{1}{2}$) and x be negative at time 1 is given by

$$\int_{-\infty}^{0} \int_{-\infty}^{0} \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{x^2}{2(\frac{1}{2})}\right] \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{(y-x)^2}{2(\frac{1}{2})}\right] dx\, dy \quad (1.8)$$

This expression is a quantity that has a definite value. Now let us start from 0 and lay out an interval AB of that length on the line of α (Figure 1.3). We shall say that AB corresponds to curves ending below both

Figure 1.3

axes. Then we shall take x between 0 and ∞, and take y between $-\infty$ and 0; and that probability will be given by

$$\int_{x=0}^{\infty} \int_{y=-\infty}^{0} \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{x^2}{2(\frac{1}{2})}\right] \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{(y-x)^2}{2(\frac{1}{2})}\right] dx\, dy \quad (1.9)$$

Let us then lay off another interval, BC, of α, that will correspond to Expression 1.9 (Figure 1.4). (Notice that the first two probabilities will

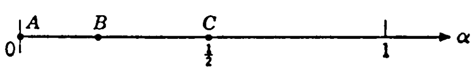

Figure 1.4

add up to the probabilities of all the curves that end below the axis, and that probability will be $\frac{1}{2}$.) Then we integrate Expression 1.7 as follows:

$$\int_{x=-\infty}^{0} \int_{y=0}^{\infty} \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{x^2}{2(\frac{1}{2})}\right] \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{(y-x)^2}{2(\frac{1}{2})}\right] dx\, dy \quad (1.10)$$

That will give us a third segment CD of the line (Figure 1.5). Finally, Expression 1.11 gives the fourth segment DE (Figure 1.6).

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{x^2}{2(\frac{1}{2})}\right] \frac{1}{[2\pi(\frac{1}{2})]^{1/2}} \exp\left[-\frac{(y-x)^2}{2(\frac{1}{2})}\right] dx\, dy_1 \tag{1.11}$$

We now have four segments of the line α, that is, four probabilities that add up to 1.

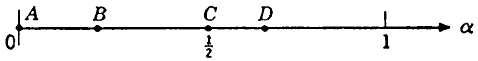

Figure 1.5

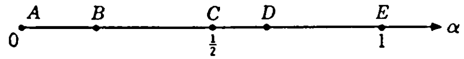

Figure 1.6

Let us take a finer subdivision of the Brownian motions; we shall take the subdivisions where, as you will easily see, every possibility lies in one and only one of the previous possibilities, and where a certain fixed number of these possibilities will add up to give the previous complete probability. We shall get a finer subdivision of the α line; it will correspond to regions of wandering of the Brownian motion. The first thing that I shall do, is to divide the time finer. That is, I shall work by quarter

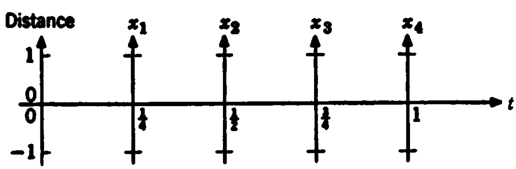

Figure 1.7

times (Figure 1.7). The probability that a particle lies in a small region specified at each of the four times is given by the integral of

$$\left\{\frac{1}{[2\pi(\frac{1}{4})]^{1/2}}\right\}^4 \exp\left[-\frac{x^2}{2(\frac{1}{4})}\right] \exp\left[-\frac{(x_2-x_1)^2}{2(\frac{1}{4})}\right]$$
$$\times \exp\left[-\frac{(x_3-x_2)^2}{2(\frac{1}{4})}\right] \exp\left[-\frac{(x_4-x_3)^2}{2(\frac{1}{4})}\right] dx_1\, dx_2\, dx_3\, dx_4 \tag{1.12}$$

over the appropriate region.

I now consider the probability that the particle moves through a hole (region) specified at each of the four times. I divide the time into regions

$\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1, and the regions will now be 4 for each time. At each one of these times, not only do I distinguish positive and negative values but I distinguish values from $-\infty$ to -1, from -1 to 0, from 0 to 1, and from 1 to ∞. How many types of curves will I have? I will have four possible regions for each time, but there will be four successive times. That is 4^4 regions, which is 256. How do I get the probability for each one of those regions? I simply integrate Expression 1.12 with respect to the different x's over that region. I will get 256 quantities that add up to 1; moreover, there will be 64 of these quantities adding up to each of the previous intervals. They will not be the same length, but I will be able to get, with a definite arrangement, 256 subregions, each adding up by 64's to the previous subregions, and I will have a mapping of ranges of my random curve on smaller ranges of α.

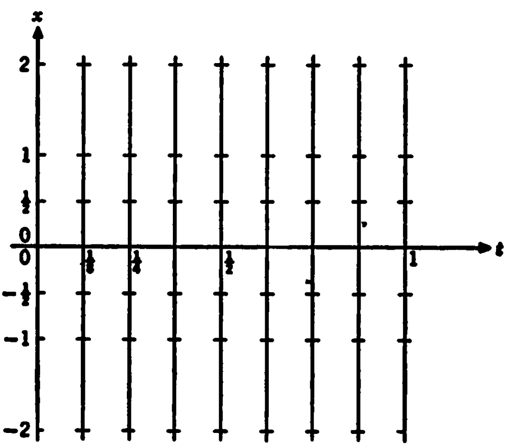

Figure 1.8

Having done this, I take a finer subdivision (Figure 1.8). In the first place, the times will now go by eights, so that for the probability density we shall have

$$\left\{\frac{1}{[2\pi(\frac{1}{8})]^{\frac{1}{2}}}\right\}^8 \exp\left[-\frac{x^2}{2(\frac{1}{8})}\right]\cdots\exp\left[\frac{(x_8-x_7)^2}{2(\frac{1}{8})}\right] \tag{1.13}$$

I shall make the integration over regions in the following way (and this trick I shall continue): I have now eight times. I take all of the previous subdivisions, not only for the times I had before but also for these other intermediate times. Then, for any of these intervals that have limits at both ends, I introduce a new subdivision halfway up. There is no virtue to a half, but using a half is a perfectly good way of doing it. For the regions that run to $+\infty$, the new subdivision will be one unit up; for the regions that run to $-\infty$, the new subdivision will be one unit down.

Now we have not 4^4 regions but 8^8 of these curves. Each one of these will have a length; each one of these will fit into one of the regions of the previous subdivisions, and have finer divisions of the 0-to-1 line of α. Thus, as we continue cutting the motion finer and finer, both in time and space, we shall have a larger number of steps of α. With this trick, or with a much more general trick of subdivision, it is easy to show that these intervals of α will all go to 0 in length. If we consider any value of α that is not one of this denumerable set of subdivisions, this value of α will lie in one of the four regions that we got at the first stage, and in one of the 256 regions that we got at the second stage, and so on. It will be uniquely determined (except for the boundaries that do not count) by where it is at the different stages of subdivisions. In other words, I am able to box in these wanderings of our point more and more finely.

It will not be true, in general, that as I do this I close down on a particular curve that is continuous; but I want to show you that I can get around this difficulty. I can introduce a certain quantity, which I shall call the straightness of a series of subdivisions. There is no virtue in the 2^n at present, so that I just say that I have a series of holes—one hole at each of an arbitrary number of times (Figure 1.9).

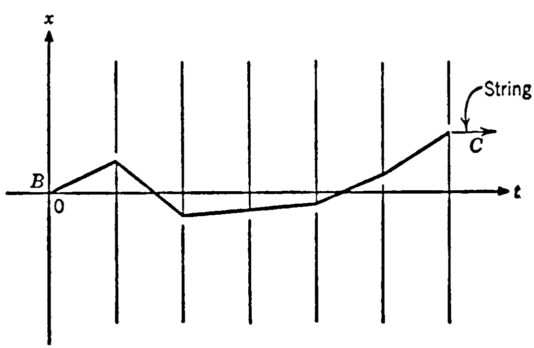

Figure 1.9

Suppose that we take a string BC fastened at 0 and thread it through the holes. Let me pull that string taut. Then there will be at least one part of the curve with maximum slope. This maximum slope for some taut string will be less than the maximum slope of any other curve passing through the same series of holes. We can call this taut string the straightest string that lies in the series of holes. This string will give us the smoothness of the series of holes in the following manner. I call this straightest string through a series of holes, $x = s(t)$, for that series of holes; a particular s will belong to each series of holes. Now consider

$$\max \frac{|s(t + \tau) - s(t)|}{|\tau|^{1/4}} \tag{1.14}$$

That maximum value I call the roughness of the series of holes. There is no virtue in one fourth, except this: It is a definite number less than one half. Now I have a series of holes at each stage of subdivision, and at each stage of subdivision I throw away all of the series of holes for which the roughness is greater than A; that is,

$$\max \frac{|s(t + \tau) - s(t)|}{|\tau|^{1/4}} > A \tag{1.15}$$

I can now prove the following: The sum of the lengths of the α mappings of the series of holes that I have thrown away at each stage will be finite. At each stage when I throw away this series of holes, I will have an expression $l(A)$, the sum of the lengths that I have thrown away. There will be only a denumerable number of holes in the series (a denumerable number of intervals); the sum of these lengths will converge and will form what we call a measurable set. The measure of that set will be less than $l(A)$.

Although I shall not spend the time tō do it here, I can prove that

$$\lim_{A \to \infty} l(A) = 0 \tag{1.16}$$

That is, the sum of the lengths of all of the series of holes at any stage which are less smooth than a certain amount is finite; and if A increases, this sum goes to 0. Let me discard the series of holes that are less than a certain smoothness. That means that I discard a certain set of values of α of measure less than some small amount. Let us take the remaining series of holes. The remaining series of holes will contain at each stage a continuous curve; moreover, all of these continuous curves will satisfy the same condition of equicontinuity. That is, Δx goes to 0 faster than a certain function of Δt.

We invoke the following mathematical theorem: Suppose that I take a set of curves that pass through a series of holes and satisfy a certain condition of equicontinuity. Suppose, also, that I make the holes narrow down to zero, and that I increase the density of the time instants at which the holes are defined. Our condition of equicontinuity binds my curves tighter and tighter at more and more points. The curves then tend uniformly to a limit that satisfies the same condition of equicontinuity.

That is, if I throw away a certain set of values of α and s such that the measure of s is less than ϵ, then to all of the remaining values of α there will be assigned a limit curve for all the series of holes that correspond successively to this value of α. The limit curve will be continuous; not only that, it will also satisfy the same condition of equicontinuity. Thus, by this process, I have assigned to all values of α (except for a set of zero measure) a curve, which I call $x(t, \alpha)$, that satisfies some condition of equicontinuity; and what is more, to all of these curves except a set of measure zero I have assigned a value of α. These limit curves will prove to be unique. Therefore I will have assigned, except for a set of zero measure, to every value of α, one and only one continuous curve. This I call $x(t, \alpha)$, as I have said. It is a well-defined function of t for almost all values of α. It can easily be shown to be a measurable, bounded function of t and α, and a continuous function of t for almost all α. Furthermore, if the process at each stage of arrangement of the holes is given definitely, which can be done, this is a well-defined function of t and α—as well-defined as any mathematical function. So, I have now introduced what we call the stochastic function $x(t, \alpha)$.

Are there any other things that I can say about $x(t, \alpha)$? I have said that for almost all values of α this is continuous. Is it differentiable? I shall not go into the proof—it is shown by merely taking the contrary cases, adding them, and counting—but I can say that the following thing can be proved.

Consider the set of curves for which

$$\frac{\Delta_t\, x(t, \alpha)}{\Delta t} \tag{1.17}$$

has a limit as Δt tends to 0 for at least one value of t. This set of curves has zero measure. That is, almost all of the curves $x(t, \alpha)$ are nowhere differentiable. This is important. We shall have to use nondifferentiable continuous curves in the work that we are doing. Not only that; the limit of Expression 1.18 will exist for no t for almost all α if λ is greater than $\frac{1}{2}$, and will exist for every t for almost all α uniformly if λ is less than $\frac{1}{2}$.

$$\frac{\Delta_t\, x(t, \alpha)}{(\Delta t)^\lambda} \tag{1.18}$$

(I am leaving out the $\frac{1}{2}$ case. I am stating the facts here rather than proving.)

The function that I want to use in our further work is $x(t, \alpha)$. I assume that we have established $x(t, \alpha)$. We shall call it the stochastic function. Now I want to build up some of the integral properties of $x(t, \alpha)$.

Let us consider the following integral:

$$I = \int_0^1 [x(t_2, \alpha) - x(t_1, \alpha)]^n \, d\alpha \qquad [t_1 < t_2] \tag{1.19}$$

Now, $[x(t_2, \alpha) - x(t_1, \alpha)]$ has a Gaussian distribution. That can be proved very easily from our definition. With $[t_2 - t_1]$ as the parameter of the Gaussian distribution, Equation 1.20 follows.

$$I = \frac{1}{[2\pi(t_2 - t_1)]^{1/2}} \int_{-\infty}^{\infty} u^n \exp\left[-\frac{u^2}{2(t_2 - t_1)}\right] du \qquad [t_1 < t_2] \tag{1.20}$$

This equation is certainly true for binary intervals, and by continuity we can extend it easily to nonbinary intervals. That is, these distributions are Gaussian. Equation 1.20 can be computed as follows. Let

$$v = \frac{u}{(t - t_1)^{1/2}} \tag{1.21}$$

Then

$$I = (t_2 - t_1)^{n/2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} v^n \exp\left(-\frac{v^2}{2}\right) dv \tag{1.22}$$

Note that $\exp(-v^2/2)$ is an even function. If n is odd, then

$$(t_2 - t_1)^{n/2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} v^n \exp\left(-\frac{v^2}{2}\right) dv = 0 \tag{1.23}$$

because the integral from $-\infty$ to $+\infty$ of the product of an odd and an even function is zero.

If n is even, we can compute Equation 1.22 by an integration by parts. Note that

$$-v \exp\left(-\frac{v^2}{2}\right) dv = d \exp\left(-\frac{v^2}{2}\right) \tag{1.24}$$

Then we have

$$I = (t_2 - t_1)^{n/2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} (-1) v^{n-1} \, d \exp\left(-\frac{v^2}{2}\right) \tag{1.25}$$

Integrating by parts, we get

$$I = (t_2 - t_1)^{n/2} \frac{1}{(2\pi)^{1/2}} \left\{ \left[-v^{n-1} \exp\left(-\frac{v^2}{2}\right) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (n-1) v^{n-2} \exp\left(-\frac{v^2}{2}\right) dv \right\} \tag{1.26}$$

$$I = (t_2 - t_1)^{n/2} \frac{1}{(2\pi)^{1/2}} (n-1) \int_{-\infty}^{\infty} v^{n-2} \exp\left(-\frac{v^2}{2}\right) dv \tag{1.27}$$

Continuing this method, we get

$$I = (t_2 - t_1)^{n/2}(n - 1)(n - 3) \cdots (1) \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{v^2}{2}\right) dv \quad (1.28)$$

The series $\{n - 1 - 2k\}$ goes down to 1 because n is even. At the 1 stage, we can evaluate

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{v^2}{2}\right) dv = 1 \quad (1.29)$$

Hence

$$\int_0^1 [x(t_2, \alpha) - x(t_1, \alpha)]^n \, d\alpha = \begin{cases} 0, & n \text{ odd} \\ (t_2 - t_1)^{n/2}(n - 1)(n - 3) \cdots (1), & n \text{ even} \end{cases} \quad (1.30)$$

The right-hand side of Equation 1.30 is interesting. Suppose that I have n terms and that I want to consider the number of ways in which I can divide n terms into pairs. If n is odd, there is no way of separating n terms into pairs. If n is even, let us see how many ways there are. We take any term. There are $n - 1$ possible terms we can match it with. When we have done that, there are $n - 2$ terms. Then we take any term, and we can match that with $n - 3$ possibilities. Hence, in all cases, odd or even, the following equation is true:

$$\int_0^1 [x(t_2, \alpha) - x(t_1, \alpha)]^n \, d\alpha = (t_2 - t_1)^{n/2} \times \text{(the number of ways of separating } n \text{ terms into pairs)} \quad (1.31)$$

This is the beginning of the calculus of random functions.

Now let us consider the following integral:

$$\int_0^1 f(t) \, dx(t, \alpha) \quad (1.32)$$

Notice that as an ordinary Stieltjes integral this integral will not exist because $x(t, \alpha)$ is almost never differentiable. We can get around this. For the moment, let us suppose that $f(t)$ is differentiable and that $f'(t)$ is bounded. (I confine myself at present to that case.) First, we shall define Expression 1.32 by integration by parts.

$$\int_0^1 f(t) \, dx(t, \alpha) = f(1)\, x(1, \alpha) - f(0)\, x(0, \alpha) - \int_0^1 f'(t)\, x(t, \alpha) \, dt \quad (1.33)$$

Now $x(0, \alpha)$ is 0. Remember that the strings that I pick pass through the origin. Therefore

$$\int_0^1 f(t)\, dx(t, \alpha) = f(1)\, x(1, \alpha) - \int_0^1 f'(t)\, x(t, \alpha)\, dt \tag{1.34}$$

Furthermore, the right-hand side of Equation 1.34 is a well-defined expression for almost all values of α. The function $x(t, \alpha)$ is bounded, f' is bounded, and hence the integral of Equation 1.34 exists. Moreover, if I change $x(t, \alpha)$ into $-x(t, \alpha)$, you will see that the distribution of the Brownian motions is not changed at all. If I just simply turn $x(t, \alpha)$ into $-x(t, \alpha)$, not only will Equation 1.34 remain valid, but its integral with respect to α will be 0.

The next expression that I want to evaluate is

$$\int_0^1 d\alpha \left[\int_0^1 f(t)\, dx(t, \alpha)\right]^2 \tag{1.35}$$

Integrating by parts and expanding, we have

$$\int_0^1 d\alpha \left[\int_0^1 f(t)\, dx(t, \alpha)\right]^2 = \int_0^1 d\alpha \left[f(1)\, x(1, \alpha) - \int_0^1 f'(t)\, x(t, \alpha)\, dt\right]^2 \tag{1.36}$$

This yields

$$\begin{aligned}\int_0^1 d\alpha \left[\int_0^1 f(t)\, dx(t, \alpha)\right]^2 &= \int_0^1 d\alpha\, f^2(1)\, x^2(1, \alpha) \\ &\quad - 2\int_0^1 d\alpha\, f(1)\, x(1, \alpha) \int_0^1 f'(t)\, x(t, \alpha)\, dt \\ &\quad + \int_0^1 d\alpha \int_0^1 f'(t)\, x(t, \alpha)\, dt \int_0^1 f'(s)\, x(s, \alpha)\, ds\end{aligned} \tag{1.37}$$

We now consider the following expression:

$$\int_0^1 x(t_1, \alpha)\, x(t_2, \alpha)\, d\alpha = \int_0^1 x(t_1, \alpha)\, [x(t_1, \alpha) + x(t_2, \alpha) - x(t_1, \alpha)]\, d\alpha \tag{1.38}$$

I assume that t_1 is less than t_2. By adding and subtracting the same thing, we are able to check the right-hand side of Equation 1.38. Note that $x(t_1, \alpha)$ and $[x(t_2, \alpha) - x(t_1, \alpha)]$ are independent in distribution. So, when one is plus, the other is equally likely to be plus or minus; and the average of the product will be zero. Thus, the only quantity that survives is

$$\int_0^1 x(t_1, \alpha)\, x(t_2, \alpha)\, d\alpha = \int_0^1 x^2(t_1, \alpha)\, d\alpha = t_1 \tag{1.39}$$

Now consider the three terms of Equation 1.37. The first term is given by

$$\int_0^1 d\alpha\, f^2(1)\, x^2(1, \alpha) = f^2(1) \tag{1.40}$$

Let us suppose that we can interchange the order of integration for the remaining two terms of Equation 1.37. (It can be fairly easily proved that this interchange is justified.) Hence, for the second term, we have

$$-2\int_0^1 d\alpha\, f(1)\, x(1, \alpha) \int_0^1 f'(t)\, x(t, \alpha)\, dt = -2f(1) \int_0^1 t f'(t)\, dt \tag{1.41}$$

Equation 1.41 is true, because t is less than 1.

Now consider the remaining term of Equation 1.37.

$$\int_0^1 d\alpha \int_0^1 f'(t)\, x(t, \alpha)\, dt \int_0^1 f'(s)\, x(s, \alpha)\, ds$$
$$= \int_0^1 f'(t)\, dt \int_0^1 f'(s)\, ds \int_0^1 x(t, \alpha)\, x(s, \alpha)\, d\alpha \tag{1.42}$$

Recall that

$$\int_0^1 x(t, \alpha)\, x(s, \alpha)\, d\alpha = \begin{cases} s, & s \leq t \\ t, & t \leq s \end{cases} \tag{1.43}$$

Equation 1.43 means that we can divide the integral of Equation 1.42 into two ranges; in one of them, s is smaller; in one, t is smaller. But that means that the two integrals are the same integral, because s and t are completely interchangeable. I substitute Equation 1.43 in Equation 1.42, keeping s less than t, and I multiply by 2 to account for the two cases. The right-hand side of Equation 1.42 becomes

$$\int_0^1 f'(t)\, dt \int_0^1 f'(s)\, ds \int_0^1 x(t, \alpha)\, x(s, \alpha)\, d\alpha$$
$$= 2\int_0^1 f'(t)\, dt \int_0^t s f'(s)\, ds \tag{1.44}$$

Substitution of Equations 1.40, 1.41, and 1.44 in Equation 1.37 gives

$$\int_0^1 d\alpha \left[\int_0^1 f(t)\, dx(t, \alpha)\right]^2 = f^2(1) - 2f(1)\int_0^1 t f'(t)\, dt$$
$$+ 2\int_0^1 f'(t)\, dt \int_0^t s f'(s)\, ds \tag{1.45}$$

Now

$$2\int_0^1 f'(t)\,dt\int_0^t sf'(s)\,ds = 2\left[\int_0^t sf'(s)\,ds\,f(t)\right]_0^1 - 2\int_0^1 tf(t)f'(t)\,dt \tag{1.46}$$

$$2\int_0^1 f'(t)\,dt\int_0^t sf'(s)\,ds = 2f(1)\int_0^1 sf'(s)\,ds - 2\int_0^1 tf(t)f'(t)\,dt \tag{1.47}$$

Therefore

$$\int_0^1 d\alpha\left[\int_0^1 f(t)\,dx(t,\alpha)\right]^2 = f^2(1) - 2\int_0^1 tf(t)f'(t)\,dt \tag{1.48}$$

Now

$$\frac{d}{dt}[tf^2(t)] = f^2(t) + 2tf(t)f'(t) \tag{1.49}$$

Therefore

$$2\int_0^1 tf(t)f'(t)\,dt = [tf^2(t)]_0^1 - \int_0^1 f^2(t)\,dt \tag{1.50}$$

$$2\int_0^1 tf(t)f'(t)\,dt = f^2(1) - \int_0^1 f^2(t)\,dt \tag{1.51}$$

Thus

$$\int_0^1 d\alpha\left[\int_0^1 f(t)\,dx(t,\alpha)\right]^2 = \int_0^1 f^2(t)\,dt \tag{1.52}$$

Notice what we have if we start with the assumption that $f(t)$ is differentiable and belongs to L^2 and we go from that to the function of α given by Equation 1.52. We have a unitary transformation. This allows us to extend the definition of the integral to any function $F(\alpha)$ that belongs to the Lebesgue class L^2 by the following trick.

Suppose that we have a sequence $f_n(t)$ of real functions which belong to L^2 such that Condition 1.53 is satisfied:

$$\int_0^1 [f_n(t) - f(t)]^2\,dt \to 0 \tag{1.53}$$

Clearly, given any function f in L^2, I can find such a sequence. Every function of L^2, Lebesgue-measurable and Lebesgue-integrable-square, can be approximated by functions of bounded derivatives. There is no problem about that.

Then I form $F_n(\alpha)$ defined by Equation 1.54.

$$F_n(\alpha) = \int_0^1 f_n(t)\,dx(t,\alpha) \tag{1.54}$$

It follows at once that

$$\int_0^1 [F_n(\alpha) - F_m(\alpha)]^2 \, d\alpha = \int_0^1 [f_n(t) - f_m(t)]^2 \, dt \tag{1.55}$$

and that

$$\int_0^1 [f_n(t) - f_m(t)]^2 \, dt \to 0 \tag{1.56}$$

as m and n tend to ∞ independently.

Now we use the Riesz-Fischer theorem. If we have a sequence $\{F_n(\alpha)\}$ belonging to L^2 such that

$$\int_0^1 [F_n(\alpha) - F_m(\alpha)]^2 \, d\alpha \to 0 \tag{1.57}$$

then there is a function of α to which they converge in the mean.

$$F(\alpha) = \underset{n \to \infty}{\text{l.i.m.}} \, F_n(\alpha) \tag{1.58}$$

It can be proved that $F(\alpha)$ does not depend on the sequence $\{F_n(\alpha)\}$ that approximates it but that it will be the same for any such sequence.

I now define

$$F(\alpha) = \int_0^1 f(t) \, dx(t, \alpha) \tag{1.59}$$

Equation 1.59 applies to almost all values of α. $F(\alpha)$ is a function of L^2. There is no problem in verifying that

$$\int_0^1 F^2(\alpha) \, d\alpha = \int_0^1 f^2(t) \, dt \tag{1.60}$$

So, we have extended our integral to all functions belonging to the Lebesgue class L^2.

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