

Shot Noise from Thermionic Cathodes

C. F. Quate

Introduction

In the majority of RF amplifiers that make use of electron beams, the random emission of electrons at the cathode surface is the primary source of noise. In these amplifiers the cathode may be operated either in the "temperature-limited" region, wherein all emitted electrons are drawn to the anode, or in the "space-charge-limited" region, wherein only a small fraction of the emitted electrons are drawn to the anode. In the temperature-limited type the noise properties of the beam consist of a noise current which is pure shot noise and a noise velocity which is calculated as the deviation from the mean value of the emission velocities, described by the Maxwellian distribution. In space-charge-limited operation the velocity and current noises are modified by the action of the potential minimum region. A large part of the following discussion will be devoted to this phenomenon.

We will treat only the problem of a one-dimensional diode consisting of two parallel planes of infinite extent—one plane being the cathode which emits the electrons and the second plane being the anode which is maintained at a positive potential with respect to the cathode. The d-c, or steady-state, conditions for space-charge-limited flow for

this one-dimensional diode will first be considered by following the work of Fry and others.¹⁻⁴ This work, together with a discussion of several fluctuation parameters associated with the Maxwellian distribution function, will serve as an introduction to the noise calculations. For simplicity, we will first discuss the noise in space-charge-limited flow by using an approximate "single-velocity" model in order to give some insight into the low-frequency region where the presence of the potential minimum must result in noise current that is less than that of shot noise. After this we will consider the actual multivelocity flow problem with the aid of North's⁵ solution, which again is limited to the low-frequency case of short-transit angles from cathode to anode.

At higher frequencies, where the transit angles are no longer small, we make use of the equations for one-dimensional flow as given by Llewellyn and Peterson.⁶ These equations are limited to problems of electron flow with only a single velocity at one given point in the beam at a given instant of time. In the region near the potential minimum, the spread in transit time between electrons of different initial velocities may be an appreciable part of an RF cycle and the single-velocity approximations are severely strained. Because of the multivelocity character of the beam of this critical region, no rigorous analysis of the noise has yet been obtained. The single-velocity equations are valid, however, if we accept as the input boundary a plane beyond the potential minimum where the electrons have reached an average velocity several times the value of their initial velocity owing to thermal energy. The problem then becomes one of determining the noise fluctuations in velocity and current at this input plane beyond which the single-velocity equations can be used. Some approximations for the input conditions have been obtained by Robinson,⁷ Watkins,⁸ and Whinery.⁹ We will discuss part of this work and show that it leads to answers that are reasonably close to those obtained from experiments. It is believed, however, that the rigorous solution must be obtained with the aid of a computer. Tien has worked out one problem with this approach, using the "Monte Carlo" method, and a summary of his results will be presented as the conclusion to this chapter.

1. Steady-State Conditions for Space-Charge-Limited Flow

Single-Velocity Approximation

The simplest model that can be visualized for space-charge-limited flow in a plane-parallel diode is that used by Child. He neglected the initial velocities of the electrons and used the "single-velocity" model

to obtain the three-halves power law of current versus voltage. We shall write down the steps leading to Child's law, since the more rigorous treatment, which includes the multivelocity flow, will proceed in a similar fashion. It will also be evident that in many cases the equation is a sufficiently good approximation for the potential distribution in a diode.

We solve the one-dimensional Poisson equation by specifying that both the potential and the potential gradient are zero at the input which is located at the cathode surface.

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0} = \frac{J_0}{\epsilon_0 v} \quad (1.1)$$

where J_0 is the current density, ρ is the charge density given by $-J/v$, ϵ_0 is the dielectric constant of free space, and v is the velocity of the electron at point x . The energy equation tells us that

$$v^2 = -2 \frac{e}{m} V \quad (1.2)$$

where e , a negative number, represents the charge of the electron and m denotes the mass.

If we multiply Eq. 1.1 by $2(dV/dx)$ and use Eq. 1.2, we obtain, after one integration,

$$\frac{dV}{dx} = \left(\frac{4J_0}{\epsilon_0 \sqrt{-2e/m}} \right)^{1/2} V^{1/4} \quad (1.3)$$

This can be written after a second integration:

$$J_0 = \frac{4\epsilon_0}{9} \sqrt{\frac{-2e}{m}} \frac{V^{3/2}}{x^2} = 2.33 \times 10^{-6} \frac{V^{3/2}}{x^2} \quad (1.4)$$

Equation 1.4 is the familiar expression for Child's law, which describes the variation of current in a model of a diode in which the electrons have only a single velocity at a given point, and in which they leave the cathode surface with zero initial velocity.

Multivelocity Flow

In an actual diode the thermal energy of the electrons at the cathode surface imparts initial velocities to the electrons that are distributed according to Maxwell's law, and, as a consequence of this multivelocity type of flow, there must be a negative potential gradient at

the cathode. If it were otherwise, all the emitted electrons would reach the anode, and the beam current would not be controlled by the anode voltage. The potential profile in the diode must therefore be somewhat as pictured in Fig. 1.1. There is a negative gradient of potential between the cathode and the minimum which is located at x_m and has a potential V_m .

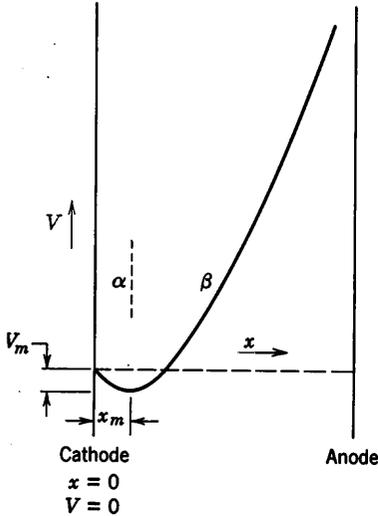


Fig. 1.1. Potential distribution in parallel-plane diode with finite emission velocity.

We designate the initial velocity of emission at the cathode, which is normal to the cathode surface, as v_s , and the velocity at an arbitrary point x by v . The value of v is a function of emission velocity v_s . We shall denote by v_m the velocity at the cathode surface of that group of electrons which just reach the potential minimum. The number of electrons emitted per unit area and unit time in the velocity range from v_s to $v_s + dv_s$ may be called $n(v_s) dv_s$. We see that $n(v_s)$ is the distribution function giving the number of electrons in each velocity class. As one simple application of this distribution function we may calculate the total number of electrons per unit area per unit time N from the relation

$$N = \int_0^{\infty} n(v_s) dv_s \quad (1.5)$$

For our diode problem we must evaluate the charge density in order to apply Poisson's equation. At point x where the electrons that left the cathode with velocity v_s travel at a velocity v we have

$$d\rho = \frac{e n(v_s) dv_s}{v} \quad (1.6)$$

For the total charge density we must integrate Eq. 1.6 between the proper limits. In the region between the minimum and the anode, which will be called the β region, we will find only those electrons that had an initial velocity great enough to overcome the retarding field of the potential minimum. Thus we must integrate Eq. 1.6 from v_m through ∞ to obtain in the β region

$$\rho_\beta = e \int_{v_m}^{\infty} \frac{n(v_s) dv_s}{v} \quad (1.7)$$

In the region between the cathode and potential minimum, called the α region, at point x the electrons moving away from the cathode contain all velocity groups v_s from v_x to ∞ , where v_x is the velocity at the cathode of an electron that just reaches point x ($0 < x < x_m$). The group that has been returned and is now moving toward the cathode had initial velocities from v_x to v_m . We can then write

$$\rho_\alpha = e \int_{v_x}^{\infty} \frac{n(v_s) dv_s}{v} + e \int_{v_x}^{v_m} \frac{n(v_s) dv_s}{v}$$

which in turn can be written

$$\rho_\alpha = 2e \int_{v_x}^{v_m} \frac{n(v_s) dv_s}{v} + e \int_{v_m}^{\infty} \frac{n(v_s) dv_s}{v} \quad (1.8)$$

We now use Poisson's equation, and with Eq. 1.7 we can write for the potential in the β region

$$\frac{d^2V}{dx^2} = -\frac{e}{\epsilon_0} \int_{v_m}^{\infty} \frac{n(v_s) dv_s}{v} \quad (1.9)$$

If we multiply both sides of Eq. 1.9 by $2(dV/dx)$, we have

$$2 \frac{dV}{dx} \frac{d^2V}{dx^2} = -\frac{2e}{\epsilon_0} \int_{v_m}^{\infty} \frac{1}{v} \frac{dV}{dx} n(v_s) dv_s$$

Since $V = -mv^2/2e + \text{constant}$, where the constant represents the electron energy at the cathode surface, we can write

$$\frac{d}{dx} \left(\frac{dV}{dx} \right)^2 = \frac{m}{\epsilon_0} \int_{v_m}^{\infty} \frac{1}{v} \frac{d(v^2)}{dx} n(v_s) dv_s$$

or, by integrating with respect to x from x_m to x ,

$$\left(\frac{dV}{dx} \right)_\beta^2 = \frac{2m}{\epsilon_0} \int_{v_m}^{\infty} n(v_s)(v - v_{ms}) dv_s \quad (1.10)$$

where v_{ms} is the velocity at the potential minimum of an electron which left the cathode with a velocity v_s .

Similarly for the α region we can integrate from x to x_m to obtain from Eq. 1.8 (noting that the limits of integration are variable)

$$\left(\frac{dV}{dx}\right)_\alpha^2 = \frac{2m}{\epsilon_0} \left[\int_{v_m}^{\infty} n(v_s)(v - v_{ms}) dv_s + 2 \int_{v_x}^{v_m} v n(v_s) dv_s \right] \quad (1.11)$$

Properties of the Maxwellian Distribution

It is now necessary to discuss the distribution function $n(v_s)$. This is simply Maxwell's distribution law and can be written

$$n(v_s) = \frac{mN}{kT_c} v_s \exp\left(-\frac{mv_s^2}{2kT_c}\right) \quad (1.12)$$

Here k is Boltzmann's constant, and T_c is the cathode temperature in degrees Kelvin.

We wish to compute with this particular distribution of electrons the average velocity \bar{v} at point x in the α region for all electrons moving in the forward direction. It is a quantity that will recur somewhat later, and it can be written

$$\bar{v} = \frac{\int_{v_x}^{\infty} v v_s \exp\left(-\frac{mv_s^2}{2kT_c}\right) dv_s}{\int_{v_x}^{\infty} v_s \exp\left(-\frac{mv_s^2}{2kT_c}\right) dv_s} \quad (1.13)$$

The velocity at point x , which is at a potential V , is related to the initial velocity v_s by

$$v^2 = v_s^2 - 2\frac{e}{m}V \quad (1.14)$$

or

$$v dv = v_s dv_s \quad (1.15)$$

and Eq. 1.13 becomes

$$\bar{v} = \frac{\int_0^{\infty} v^2 \exp\left[-\frac{m}{2kT_c}\left(v^2 + 2\frac{e}{m}V\right)\right] dv}{\int_0^{\infty} v \exp\left[-\frac{m}{2kT_c}\left(v^2 + 2\frac{e}{m}V\right)\right] dv} \quad (1.16)$$

which is evaluated with the aid of the integrals

$$\int_0^\infty x e^{-x^2} dx = \frac{1}{2} \tag{1.17}$$

and

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \tag{1.18}$$

to give

$$\bar{v} = \left(\frac{\pi k T_c}{2m} \right)^{1/2} \tag{1.19}$$

We wish to stress the significance of Eq. 1.19. It tells us that the average velocity of the electrons moving away from the cathode remains constant as we move from the cathode to the potential minimum. This is a consequence of the Maxwellian distribution, together with the fact that the field at the surface of the cathode is negative. This second condition is necessary in order that we may always integrate from 0 to ∞ in the numerator of Eq. 1.16.

It is not difficult to see why the average, Eq. 1.19, should remain constant: Although each electron loses velocity as it moves from the cathode, the more slowly moving electrons are continually sorted and returned to the cathode. Later we shall use this property to show that fluctuations in the magnitude of the potential minimum may smooth out the fluctuations in emitted current but will not change the velocity fluctuations.

The distribution function may equally well be written in terms of the average velocity \bar{v} , and in this form it will be useful later in a physical interpretation of some of our noise equations. Thus with Eq. 1.19 we can write

$$n(v_s) = \frac{\pi N}{2 \bar{v}^2} \exp \left(- \frac{\pi v_s^2}{4 \bar{v}^2} \right) \tag{1.20}$$

We can easily see from Eqs. 1.20 or 1.12 that the normalizing factor N is the total number of electrons emitted per unit area per unit time and is given by Eq. 1.5.

Potential Distribution for the Multivelocity Diode

Now, to return to the problem of solving Poisson's equation for the potential distribution, we can combine Eqs. 1.10, 1.11, and 1.12. First, it will prove simpler to change the variable from v_s to v with the

aid of Eqs. 1.14 and 1.15. This gives

$$\begin{aligned} \left(\frac{dV}{dx}\right)^2 = & \frac{2m^2N}{\epsilon_0kT_c} \left\{ \int_0^\infty v^2 \exp\left[-\frac{m}{2kT_c}\left(v^2 + 2\frac{e}{m}V\right)\right] dv \right. \\ & - \int \frac{v v_m}{\sqrt{2\frac{e}{m}(V_m - V)}} \exp\left[-\frac{m}{2kT_c}\left(v^2 + 2\frac{e}{m}V\right)\right] dv \\ & \left. \pm \int_0^{\sqrt{2\frac{e}{m}(V_m - V)}} v^2 \exp\left[-\frac{m}{2kT_c}\left(v^2 + 2\frac{e}{m}V\right)\right] dv \right\} \quad (1.21) \end{aligned}$$

The upper sign is to be used in the α region, and the lower sign in the β region. We note that V_m is a negative number.

We wish to convert to normalized parameters which serve to measure both voltage and distance from the potential minimum. We therefore define

$$\eta = -\frac{e(V - V_m)}{kT_c} = 11,605 \frac{V - V_m}{T_c} \quad (1.22)$$

and

$$\xi = \frac{(2m\pi)^{1/4}}{(kT_c)^{3/4}} \left| \frac{eJ_0}{\epsilon_0} \right|^{1/2} (x - x_m) = 9.19 \times 10^5 \frac{J_0^{1/2}}{T_c^{3/4}} (x - x_m) \quad (1.23)$$

where J_0 = current density beyond the potential minimum.

We define the error function by the relation

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx \quad (1.24)$$

and note that, for $x \gg 1$, $\operatorname{erf}(x) = 1$.

Equation 1.21 can now be evaluated with Eqs. 1.22 and 1.23 to be

$$\left(\frac{d\eta}{d\xi}\right)^2 = e^\eta - 1 \pm e^\eta \operatorname{erf} \sqrt{\eta} \pm \frac{2}{\sqrt{\pi}} \sqrt{\eta} \quad (1.25)$$

Here again the upper signs apply to the α region between 0 and x_m , and the lower signs apply to the β region beyond x_m .

Equation 1.25 cannot be integrated explicitly, but it has been tabulated,¹⁻⁴ with the latest and most complete tables being given in reference 10. In Fig. 1.2 there is shown a plot of η vs. ξ . We should note that at the cathode ξ_c approaches a limit of -2.55 for large values of η_c . In order to determine what is meant by large values of

η_c , let us evaluate the current which is given by

$$J_0 = - e \int_{v_m}^{\infty} n(v_s) dv_s \tag{1.26}$$

or

$$J_0 = J \exp\left(-\frac{eV_m}{kT_c}\right) \tag{1.27}$$

where J = current density at cathode. Therefore

$$V_m = -\frac{kT_c}{e} \ln \frac{J_0}{J} = \frac{T_c}{11,605} \ln \frac{J_0}{J} \tag{1.28}$$

and

$$\eta_c = \frac{eV_m}{kT_c} = -\ln \frac{J_0}{J} \tag{1.29}$$

Approximate Solutions of the Multivelocity Flow

In Eq. 1.29 we find that large values of η_c correspond to small values of J_0/J . This then states that the limiting value of $\xi_c = -2.55$ is determined by the requirement that the current density passing through the potential minimum is a small fraction of the emitted current density.

Thus, since η_c is very large for small values of J_0/J , we can replace erf (x) by 1, and Eq. 1.25 has the following approximate form:

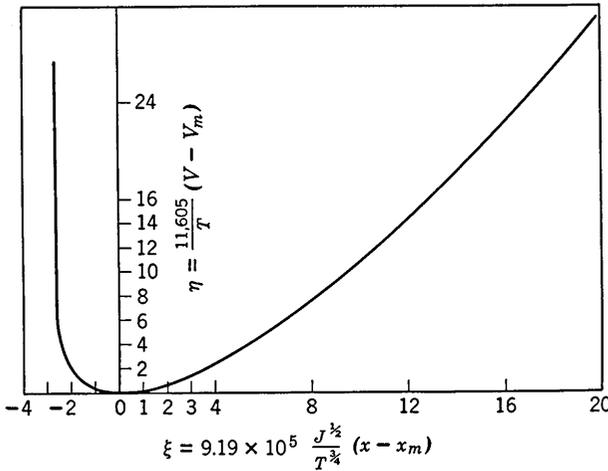


Fig. 1.2. Normalized potential distribution in multivelocity diode.

In the α region at the cathode surface,

$$\left(\frac{d\eta_c}{d\xi_c}\right)^2 = e^{\eta_c} - 1 + e^{\eta_c} \operatorname{erf} \sqrt{\eta_c} - \frac{2}{\sqrt{\pi}} \sqrt{\eta_c} \approx 2e^{\eta_c} \quad (1.30)$$

In the β region at the anode (if we assume that V_a is large) η_a will be large, and Eq. 1.25 becomes

$$\begin{aligned} \left(\frac{d\eta_a}{d\xi_a}\right)^2 &= e^{\eta_a} - 1 - e^{\eta_a} \operatorname{erf} \sqrt{\eta_a} + \frac{2}{\sqrt{\pi}} \sqrt{\eta_a} \\ &\approx \frac{2}{\sqrt{\pi}} \sqrt{\eta_a} - 1 \end{aligned} \quad (1.31)$$

For positive ξ , and for values of η greater than 8, the solution of Eq. 1.25 can be expressed in the form

$$\xi = 1.255\eta^{3/4} + 1.688\eta^{1/4} - 0.51 - 0.1677\eta^{-1/4} + \quad (1.32)$$

If we use the first term of Eq. 1.32, we find from Eqs. 1.22 and 1.23 that

$$J_0 = 2.33 \times 10^{-6} \frac{(V - V_m)^{3/2}}{(x - x_m)^2} \quad (1.33)$$

Equation 1.33 is the familiar Child's law, Eq. 1.4, for the diode which is derived on a single-velocity basis. However, the potential difference and the diode spacing are now measured from the potential minimum rather than from the cathode.

If we use the first two terms of Eq. 1.32 and note that we must square ξ to obtain the current density, we obtain

$$J_0 = 2.33 \times 10^{-6} \frac{(V - V_m)^{3/2}}{(x - x_m)^2} \left(1 + \frac{2.66}{\sqrt{\eta}}\right) \quad (1.34)$$

or

$$J_0 = 2.33 \times 10^{-6} \frac{(V - V_m)^{3/2}}{(x - x_m)^2} \left[1 + \frac{0.025T_e^{1/2}}{(V - V_m)^{1/2}}\right] \quad (1.35)$$

Langmuir discusses this approximation in reference 4, page 244, and points out that it is an accurate representation of the actual η vs. ξ curve (Fig. 1.2) down to values of $\eta = 1$ with an error in ξ of about 2%. Therefore, in most practical cases it is sufficient to use Eq. 1.35 together with the limiting value of $-\xi_c = 2.55$.

It may give us more insight into Eq. 1.34 and will be helpful in a later discussion of noise if we re-express this equation in terms of the average velocity rather than in terms of cathode temperature.

We can write from Eq. 1.19

$$\frac{kT_c}{e} = \frac{2}{\pi} \frac{m}{e} \bar{v}^2$$

and Eq. 1.35 becomes

$$J_0 = 2.33 \times 10^{-6} \frac{(V - V_m)^{3/2}}{(x - x_m)^2} \left[1 + \frac{2.66 \left| \frac{2m}{\pi e} \right|^{1/2} \bar{v}}{(V - V_m)^{1/2}} \right] \quad (1.36)$$

Thus we see from Eq. 1.36 that a finite velocity of emission causes an increase of the current by an amount expressed by the second term on the right.

Dependence of Anode Current on Changes in Emission Current

In later sections, as we discuss noise, we will be interested in the change in anode current caused by fluctuations in the cathode current. With the foregoing equations we can¹¹ easily derive an expression for the variation of anode current under the following restrictions. We limit ourselves to small values of J_0/J so that Eq. 1.30 is valid, and to large values of anode voltage so that Eq. 1.31 holds. From Eq. 1.23 evaluated at the anode, $x = x_a$,

$$\begin{aligned} \xi_a &= K J_0^{1/2} (x_a - x_m) \\ K = \text{constant} &= \frac{9.19 \times 10^5}{T_c^{3/4}} \end{aligned} \quad (1.37)$$

At the cathode, $x = 0$, we have

$$\xi_c = -K J_0^{1/2} x_m \quad (1.38)$$

These two equations can be combined to give

$$\xi_a - \xi_c = K x_a J_0^{1/2} \quad (1.39)$$

and

$$J_0 = \frac{(\xi_a - \xi_c)^2}{(K x_a)^2} \quad (1.40)$$

If we differentiate Eq. 1.40 with respect to the emitted current density J , we obtain

$$\frac{\partial J_0}{\partial J} = \frac{2(\xi_a - \xi_c)}{(K x_a)^2} \left(\frac{\partial \xi_a}{\partial J} - \frac{\partial \xi_c}{\partial J} \right)$$

and, from Eq. 1.22,

$$\frac{\partial \xi_a}{\partial J} = \frac{\partial \xi_a}{\partial \eta_a} \frac{\partial \eta_a}{\partial J} - \frac{\partial \xi_a}{\partial \eta_a} \frac{\partial V_m}{\partial J} \frac{e}{kT_c}$$

and

$$\frac{\partial \xi_c}{\partial J} = \frac{\partial \xi_c}{\partial \eta_c} \frac{\partial \eta_c}{\partial J} = \frac{\partial \xi_c}{\partial \eta_c} \frac{\partial V_m}{\partial J} \frac{e}{kT_c}$$

Therefore, we have

$$\frac{\partial J_0}{\partial J} = \frac{2eJ_0}{kT_c(\xi_a - \xi_c)} \left(\frac{\partial \xi_a}{\partial \eta_a} - \frac{\partial \xi_c}{\partial \eta_c} \right) \frac{\partial V_m}{\partial J} \quad (1.41)$$

Now we see, from Eq. 1.30, that

$$\frac{\partial \xi_c}{\partial \eta_c} = \frac{1}{2} \exp \left(-\frac{1}{2} \eta_c \right)$$

and, from Eq. 1.31,

$$\frac{\partial \xi_a}{\partial \eta_a} = \frac{1}{\left(\frac{2}{\sqrt{\pi}} \sqrt{\eta_a} - 1 \right)^{1/2}}$$

Since η_c is large under our assumptions, we see that

$$\frac{\partial \xi_c}{\partial \eta_c} \ll \frac{\partial \xi_a}{\partial \eta_a}$$

and Eq. 1.41 can be written

$$\frac{\partial J_0}{\partial J} = \frac{2eJ_0}{kT_c(\xi_a - \xi_c)} \frac{\partial \xi_a}{\partial \eta_a} \frac{\partial V_m}{\partial J} \quad (1.42)$$

We can eliminate $\partial V_m/\partial J$ through the use of Eq. 1.27:

$$\frac{\partial J_0}{\partial J} = \exp \left(-\frac{eV_m}{kT_c} \right) - \frac{Je}{kT_c} \exp \left(-\frac{eV_m}{kT_c} \right) \frac{\partial V_m}{\partial J}$$

or

$$\frac{\partial V_m}{\partial J} = \frac{kT_c}{eJ_0} \left(\frac{J_0}{J} - \frac{\partial J_0}{\partial J} \right) \quad (1.43)$$

Equation 1.42 becomes

$$\frac{\partial J_0}{\partial J} \left(1 + \frac{2}{\xi_a - \xi_c} \frac{\partial \xi_a}{\partial \eta_a} \right) = \frac{2J_0}{J(\xi_a - \xi_c)} \frac{\partial \xi_a}{\partial \eta_a} \quad (1.44)$$

The term $\partial\xi_a/\partial\eta_a$ is not convenient to use, and we can easily express this in terms of g , which is defined by

$$g = \frac{\partial J_0}{\partial V_a} = \frac{2(\xi_a - \xi_c)}{(Kx_a)^2} \left[\frac{\partial\xi_a}{\partial\eta_a} - \left(\frac{\partial\xi_a}{\partial\eta_a} - \frac{\partial\xi_c}{\partial\eta_c} \right) \frac{\partial V_m}{\partial V_a} \right] \frac{|e|}{kT_c}$$

$$\approx \frac{2J_0}{\xi_a - \xi_c} \left(\frac{\partial\xi_a}{\partial\eta_a} \right) \left(1 - \frac{\partial V_m}{\partial V_a} \right) \frac{|e|}{kT_c} \quad (1.45)$$

From Eq. 1.27 we know that

$$g = \frac{eJ_0}{kT_c} \frac{\partial V_m}{\partial V_a} \quad (1.46)$$

and Eq. 1.45 becomes

$$g = \frac{2J_0}{\xi_a - \xi_c} \left(\frac{\partial\xi_a}{\partial\eta_a} \right) \left(1 - \frac{kT_c}{e} \frac{g}{J_0} \right) \frac{|e|}{kT_c}$$

Since $\partial V_m/\partial V_a \ll 1$, we find for the above, with the aid of Eq. 1.46,

$$g \left[1 + \frac{2}{\xi_a - \xi_c} \left(\frac{\partial\xi_a}{\partial\eta_a} \right) \right] = \frac{2J_0}{\xi_a - \xi_c} \left(\frac{\partial\xi_a}{\partial\eta_a} \right) \frac{|e|}{kT_c} \quad (1.47)$$

Equation 1.47 can be combined with Eq. 1.44 to give

$$\frac{\partial J_0}{\partial J} = \frac{kT_c}{|e|J} g \quad (1.48)$$

Numerical Example for a Typical Diode

To conclude this section we will consider a numerical example to determine the order of magnitude of some of the quantities that we have been discussing. Consider a cathode operating at 1100°K with a cathode emission of 2 amperes per cm² and an anode-current density of 0.1 ampere per cm². Thus $J/J_0 = 20$, and we can use the asymptotic value of $-\xi_c = 2.55$.

We find with Eq. 1.23 that

$$x_m = \frac{2.55}{9.19 \times 10^5} \frac{T_c^{3/4}}{J_0^{1/2}} = 0.0017 \text{ cm}$$

From Eq. 1.28,

$$V_m = \frac{1100}{11,605} \ln 20 = 0.29 \text{ volt}$$

Now for a diode operating at 10 volts we have for the last factor of Eq. 1.35, which is a correction to Child's law,

$$1 + \frac{0.025T_c^{3/2}}{(V - 0.28)^{3/2}} = 1.26$$

whereas, if $V = 100$ volts, the correction factor becomes 1.08.

Thus for the steady-state conditions the corrections to the single-velocity (Child's law) expression are of importance only for low-voltage diodes. However, we shall find this multivelocity treatment of great importance and indeed the starting point for the succeeding calculations on noise in the presence of the potential minimum.

2. Noise Properties of an Electron Stream at the Surface of the Cathode

We will discuss in this section the noise current and noise velocity in a temperature-limited diode. The term temperature-limited is used to describe the condition of operation wherein all the emitted electrons are drawn to the anode. The resulting relations will be valid for the input conditions at the surface of the cathode for space-charge-limited flow which will be studied later. The diode will be assumed to have a small cathode-anode distance so that transit time effects can be neglected.

Shot Noise

The noise current in a temperature-limited diode, called shot noise, was first described by Schottky.^{12,13} Here we wish to give two arguments which have been used by Pierce^{14,15} that may or may not make the shot-noise formula more plausible.

The first example considers a short diode consisting of two emitting surfaces opposing each other. Both surfaces are assumed to be identical, and the entire diode is held at the same temperature. If V is the voltage of cathode 2 with respect to cathode 1, we then have, from Eq. 1.27, that the current flowing to cathode 2, which is held at $-|V|$ with respect to cathode 1, is given by

$$I_0 = I \exp\left(-\frac{eV}{kT}\right) \quad (1.49)$$

(note: e is a negative quantity) where I is the total emitted current of cathode 1. The diode conductance is defined

$$g \equiv \frac{\partial I_0}{\partial V} = -\frac{eI}{kT_c} \exp\left(-\frac{eV}{kT_c}\right) \quad (1.50)$$

We now short-circuit the diode, and, since there are no external sources of power, the electric-energy flow in the lead connecting cathode 1 to cathode 2 must be the Johnson noise attributable to the resistance of the diode.

Johnson noise is given in the equation

$$\overline{i^2} = 4kTg \Delta f \tag{1.51}$$

where g is the conductance of the noise element, k is Boltzmann's constant, T is the temperature in degrees Kelvin, and Δf is the bandwidth of the measuring system in cycles per second. From Eq. 1.50, when $V = 0$,

$$g = \frac{|e|I}{kT_c}$$

so that we may write for the short-circuited diode, using Eq. 1.51, and the fact that the diode is at equilibrium temperature T_c ,

$$\overline{i^2} = 4kT_c \frac{|e|I}{kT_c} \Delta f$$

$$\overline{i^2} = 4|e|I \Delta f$$

If $V = 0$, a current flows from cathode 2 to 1 equal to that from cathode 1 to 2, so that the noise associated with a current I_0 flowing from cathode 1 is given by

$$\overline{i^2} = 2|e|I \Delta f \tag{1.52}$$

This is recognized as the shot-noise formula.

The second argument, which may make Eq. 1.52 appear reasonable, begins by considering a periodic procession of electrons, one electron every T sec. These electrons (carrying a charge e) form a series of current impulses, and the spectrum of current flow is made up of a number of harmonics spaced by

$$\Delta f = \frac{1}{T} \tag{1.53}$$

The peak amplitude of each harmonic is given by

$$i_x = \frac{2|e|}{T} \tag{1.54}$$

We assume there to be many such periodic processions of electrons, each producing its own set of harmonics, and this large number of electrons produces a net harmonic component at a given frequency of

peak amplitude i_m . To this we add one more periodic procession of electrons carrying a current with a peak harmonic amplitude i_x and phase θ with respect to i_m .

The square of the total current I is given by

$$\begin{aligned} |I^2| &= \frac{1}{2}(i_m + i_x e^{j\theta})(i_m + i_x e^{-j\theta}) \\ &= \frac{1}{2}(i_m^2 + i_x^2 + 2i_m i_x \cos \theta) \end{aligned} \quad (1.55)$$

If θ is chosen at random, the result of a large number of additions of i_x will just as likely have negative values of θ as positive values, and the third term of Eq. 1.55 does not contribute to the total current. Therefore, by adding a current i_x at a random phase, we increase $|I|^2$ by an amount i_x^2 .

Thus, for p sets of electrons per sec, the mean-square value \bar{i}^2 for each harmonic will be, from Eq. 1.54,

$$\bar{i}^2 = \frac{p}{2} \left(\frac{2e}{T} \right)^2 = 2p \frac{e^2}{T^2} \quad (1.56)$$

The average current is given by

$$I_0 = p \frac{|e|}{T} \quad (1.57)$$

and therefore, from Eqs. 1.53, 1.56, and 1.57, we obtain

$$\bar{i}^2 = 2|e|I_0 \Delta f \quad (1.58)$$

which is again the shot-noise formula.

Noise-Velocity Parameters of the Temperature-Limited Diode

Let us now turn our attention to a second noise parameter which will appear frequently in our discussions; namely, the mean square deviation of the velocity, $\overline{\delta v^2}$.

Just as in Section 1, where we calculated the average velocity from the Maxwellian distribution, we can now calculate the square of the deviation from this average. Let

$$\delta v^2 \equiv (v - \bar{v})^2 \quad (1.59)$$

and thus

$$\overline{\delta v^2} = \frac{\int_0^\infty (v - \bar{v})^2 v \exp\left(-\frac{mv^2}{2kT_c}\right) dv}{\int_0^\infty v \exp\left(-\frac{mv^2}{2kT_c}\right) dv} \quad (1.60)$$

or

$$\overline{\delta v^2} = \frac{1}{\int_0^\infty v \exp\left(-\frac{mv^2}{2kT_c}\right) dv} \left[\int_0^\infty v^2 v \exp\left(-\frac{mv^2}{2kT_c}\right) dv - 2\bar{v} \int_0^\infty v v \exp\left(-\frac{mv^2}{2kT_c}\right) dv + \bar{v}^2 \int_0^\infty v \exp\left(-\frac{mv^2}{2kT_c}\right) dv \right] \quad (1.61)$$

With Eq. 1.13 this can be written

$$\overline{\delta v^2} = \overline{v^2} - \bar{v}^2 \quad (1.62)$$

From Eq. 1.19 we have

$$\bar{v}^2 = \frac{\pi k T_c}{2m} \quad (1.19)$$

and further

$$\overline{v^2} = \frac{\int_0^\infty v^3 \exp\left(-\frac{mv^2}{2kT_c}\right) dv}{\int_0^\infty v \exp\left(-\frac{mv^2}{2kT_c}\right) dv}$$

$$\overline{v^2} = \frac{2kT_c}{m} \quad (1.63)$$

Thus we have

$$\overline{\delta v^2} = \frac{kT_c}{2m} (4 - \pi) \quad (1.64)$$

Now we must consider how this is related to the mean-square fluctuation of velocity which is created by shot noise in the emission current. Suppose we have a current I_s which forms part of a current I and leaves the cathode with a velocity v_s . Then the average velocity \bar{v} will be

$$\bar{v} = \frac{\Sigma I_s v_s}{I} \quad (1.65)$$

We can find the fluctuation in \bar{v} due to the fluctuation $\Delta I_s = i_s$ in the current I_s as

$$\Delta \bar{v}_s = \left(-\frac{1}{I^2} \frac{dI}{dI_s} \Sigma I_s v_s + \frac{v_s}{I} \right) i_s \quad (1.66)$$

Since a change in I_s by an amount ΔI_s also represents a change in I

by ΔI_s , we have

$$\frac{dI}{dI_s} = 1 \quad (1.67)$$

and therefore

$$\Delta \bar{v}_s = \left(-\frac{1}{I^2} \bar{v} I + \frac{v_s}{I} \right) i_s$$

or

$$\Delta \bar{v}_s = (v_s - \bar{v}) \frac{i_s}{I} \quad (1.68)$$

Since each velocity class is emitted independently, we may assume that the fluctuation i_s is a shot-noise current, $(2|e|I_s \Delta f)^{1/2}$, independent of the fluctuation in any other stream, and the total mean-square fluctuation in velocity will be given by

$$\begin{aligned} \overline{(\Delta \bar{v}_s)^2} &= \sum_s (v_s - \bar{v})^2 \frac{\overline{i_s^2}}{I^2} \\ &= \sum_s (v_s - \bar{v})^2 \frac{2|e|I_s \Delta f}{I^2} \\ &= \frac{2|e|}{I^2} \Delta f \sum_s (v_s - \bar{v})^2 I_s \end{aligned} \quad (1.69)$$

but

$$I_s = K v_s \exp\left(-\frac{mv_s^2}{2kT_c}\right) \Delta v_s \quad (1.70)$$

where Δv_s is the width of the velocity class and

$$I = K \sum_s v_s \exp\left(-\frac{mv_s^2}{2kT_c}\right) \Delta v_s \quad (1.71)$$

Using Eqs. 1.70 and 1.71 in Eq. 1.69, and replacing the summation by integration, we obtain

$$\overline{\Delta \bar{v}_s^2} = \frac{2|e| \Delta f}{I} \frac{\int_0^\infty (v_s - \bar{v})^2 v_s \exp\left(-\frac{mv_s^2}{2kT_c}\right) dv_s}{\int_0^\infty v_s \exp\left(-\frac{mv_s^2}{2kT_c}\right) dv_s} \quad (1.72)$$

and, from Eq. 1.72

$$\overline{\Delta \bar{v}_s^2} = \frac{2|e|\Delta f}{I} (\bar{v}^2 - \bar{v}^2) = \frac{2|e|\Delta f}{I} \delta v^2 \quad (1.73)$$

which then gives, for the mean-square fluctuation within the frequency band Δf ,¹⁶

$$\overline{\Delta \bar{v}_s^2} = \frac{|e|\Delta f k T_c}{mI} (4 - \pi) \quad (1.74)$$

In the next section we will treat the problem of noise in a space-charge-limited diode and find that fluctuations in the potential minimum will act to smooth the shot-noise current of emission. However, from Eq. 1.19 we see that the average velocity is not changed by fluctuations in the potential minimum, and therefore we shall assume that the fluctuation velocity given by Eq. 1.74 does not change with changes in the potential minimum.

The above quantity is of sufficient importance to introduce a separate symbol for it. We shall denote it hence simply by \bar{v}^2 . Although v has been used before to denote the velocity of single electron groups, we believe that there will be no occasion for confusion.

3. Space-Charge Reduction of Noise Using the Single-Velocity Approximation

In the next section we will present a more general treatment of the low-frequency noise in a space-charge-limited diode and show that this noise is reduced much below shot noise. Since this treatment is complicated by the multivelocity character of the beam, we will consider first some approximate derivations of this reduction factor which are based on simplified models. The model will be assumed to have an electron stream which has only a single velocity at a given plane, x , and the mean-square fluctuation in this velocity will be assumed to be equal to the mean-square fluctuation in velocity which was calculated for the multivelocity beam in Section 2. The space-charge reduction of noise based on a single-velocity theory was first demonstrated by Rack¹⁶ to give results that were in good agreement with the multivelocity treatment. Rack's work was based on the Llewellyn-Peterson equations, and we will take this up somewhat later. First, however, we can illustrate rather easily from the equations of Section 1 that a given noise fluctuation at the cathode associated with shot noise must result in a noise current at the anode which is reduced below shot noise by a "smoothing factor" Γ^2 .

Simple Expression for Smoothing Factor Γ^2

In Eq. 1.36 we have an approximate expression which relates the anode current to the finite velocity of emission at the cathode which can be written (D. O. North pointed out this method)

$$I_0 = I_{00} \left[1 + \frac{2.66\bar{v}}{\left[\frac{\pi}{2} \left| \frac{e}{m} \right| (V_a - V_m) \right]^{1/2}} \right] \quad (1.36a)$$

where I_{00} is the diode current found from Eq. 1.33. From the numerical calculation in Section 1 (p. 14), we concluded that this is a good approximation to the average current density for voltages above 20 volts. If there is a fluctuation $\Delta\bar{v}_s$ of the average emission velocity, caused by excess current in the velocity group v_s , the resulting change in anode current ΔI is given from Eq. 1.36a as

$$\Delta I = \frac{I_{00} 2.66}{\left[\frac{\pi}{2} \left| \frac{e}{m} \right| (V_a - V_m) \right]^{1/2}} \Delta\bar{v}_s \quad (1.75)$$

and the mean square of the fluctuation current is

$$\overline{(\Delta I)^2} = \frac{(2.66 I_{00})^2}{\frac{\pi}{2} \left| \frac{e}{m} \right| (V_a - V_m)} \overline{(\Delta\bar{v}_s^2)} \quad (1.76)$$

If we now associate the fluctuation of velocity $\overline{\Delta\bar{v}_s^2}$ at the cathode with shot noise, the mean-square fluctuation in velocity is, from Eq. 1.74,

$$\overline{\Delta\bar{v}_s^2} \equiv \bar{v}^2 = \frac{|e| \Delta f k T_c}{m I_0} (4 - \pi) \quad (1.74)$$

Setting $I_{00} \cong I_0$, the mean-square fluctuation in anode current is, from Eq. 1.76,

$$\begin{aligned} \overline{(\Delta I)^2} &= \frac{(2.66)^2}{\pi} (4 - \pi) \frac{2|e| I_0 \Delta f}{(|e|/k T_c) (V_a - V_m)} \\ &\cong 2|e| I_0 \Delta f \frac{9}{4} \left(\frac{4 - \pi}{\eta_a} \right) \end{aligned} \quad (1.77)$$

The factor which relates the reduced noise current to shot noise is denoted by Γ^2 , and from Eq. 1.77 we see that

$$\Gamma^2 = \frac{9}{4} \left(\frac{4 - \pi}{\eta_a} \right) \quad (1.78)$$

which can be written from Eq. 1.22, assuming $T_c = 1100^\circ\text{K}$

$$\begin{aligned}\Gamma^2 &\approx \left(1 - \frac{\pi}{4}\right) \frac{1}{V_a - V_m} \\ &\approx \frac{1}{5V_a} \quad (\text{with } V_a \text{ in volts})\end{aligned}\tag{1.79}$$

Thus we find Γ^2 considerably less than unity. The relation of Eq. 1.78 is valid only for large values of η_a since it is based on Eq. 1.36a. Also, it neglects the effect of the electrons which were returned to the cathode because of low initial velocities. We will see from the next section, which treats the problem in a more rigorous fashion, that the effect of these returning electrons is small and the limiting expression for large values of η_a is just that given by Eq. 1.78.

Smoothing Factor Γ^2 Derived from Another Simplified Model

In reference 14, Pierce discusses the noise in space-charge-limited single-velocity flow along the following lines: In an actual diode, the relation between the anode current I_0 and the emitted current I is given (from Section 1) as

$$I_0 = I \exp\left(-\frac{eV_m}{kT_c}\right)\tag{1.27}$$

If we now assume V_m to be held constant, the electrons returning to the cathode are independent of those going on to the anode, and hence the anode current must contain pure shot noise, given by

$$\overline{i^2} = 2|e|I_0 \Delta f\tag{1.80}$$

Consider an *a-c open-circuited diode*. In order to keep the fluctuation current zero, V_m must change in such a way as to create a current equal and opposite to that given in Eq. 1.80. Therefore at x_m there must be a fluctuating voltage

$$\overline{V_m^2} = 2|e|I_0 \Delta f R_m^2\tag{1.81}$$

where

$$R_m = \frac{dV_m}{dI_0}$$

and, from Eq. 1.27,

$$R_m = \frac{kT_c}{|e|I_0}\tag{1.82}$$

Therefore Eq. 1.81 becomes

$$\overline{V_m^2} = \frac{1}{2} 4kT_c R_m \Delta f \quad (1.83)$$

If there were no change in $V_a - V_m$, the fluctuation in anode voltage would also be given by Eq. 1.83. However, if we compare this to the equivalent result, Eq. 1.77 multiplied by $1/g^2$, we find that Eq. 1.83 gives a value that is much too small. We must therefore look for fluctuations in the space between the potential minimum and the anode. These variations in voltage between the potential minimum and the anode are related to the fluctuations in average velocity of the electrons in the region.

We will neglect the effect of the thermal velocities of emission, and this in turn implies that we are neglecting the effect of the electrons that return to the cathode. The assumption can be verified through comparison with the results of the next section. With these assumptions the steady-state conditions are given by Child's law

$$I_0 = 2.33 \times 10^{-6} \frac{V_a^{3/2}}{x^2} \quad (1.4)$$

From Eq. 1.4 we obtain for the conductance of the diode

$$g \equiv \frac{\partial I_0}{\partial V_a} = \frac{3}{2} \frac{I_0}{V_a}$$

and the diode resistance is

$$R \equiv \frac{1}{g} = \frac{2}{3} \frac{V_a}{I_0} \quad (1.84)$$

Consider now an electron which crossed the potential minimum at $x = 0$ at a time $t = 0$. The charge between the electron and the potential minimum is given by $-I_0 t$. Since the field at the minimum is zero, the potential gradient at x is given from Gauss's theorem as

$$\frac{\partial V}{\partial x} = \frac{I_0 t}{\epsilon_0} \quad (1.85)$$

and the acceleration is

$$\ddot{x} = \left| \frac{e}{m} \right| \frac{I_0 t}{\epsilon_0} \quad (1.86)$$

which gives

$$\dot{x} = \left| \frac{e}{m} \right| \frac{I_0}{2\epsilon_0} t^2 + \dot{x}_0 \quad (1.87)$$

and

$$x = \left| \frac{e}{m} \right| \frac{I_0}{\epsilon_0} t^3 + \dot{x}_0 t = \left| \frac{e}{m} \right| \frac{I_0}{6\epsilon_0} t^3 + \dot{x}_0 t \quad (1.88)$$

where \dot{x}_0 is the velocity at $t = 0$, $x = 0$.

Now the voltage between the potential minimum and x (where henceforth in this computation x is the position of the anode) is given by

$$\dot{x}^2 - \dot{x}_0^2 = -2 \frac{e}{m} V_a \quad (1.89)$$

or

$$V_a = \frac{1}{2 \left| \frac{e}{m} \right|} \left(\frac{\frac{e}{m} I_0}{2\epsilon_0} \right)^2 t^4 + \frac{I_0}{2\epsilon_0} \dot{x}_0 t^2 \quad (1.90)$$

If at constant x we now vary \dot{x}_0 by a small amount, we find, from Eq. 1.88,

$$\frac{dt}{d\dot{x}_0} = \frac{-t}{\left(\frac{|e|I_0}{m\epsilon_0} t^2 + \dot{x}_0 \right)} \quad (1.91)$$

and, from 1.90,

$$dV_a = \frac{I_0 t}{\epsilon_0} \left(\frac{|e|I_0}{2\epsilon_0 m} t^2 + \dot{x}_0 \right) dt + \frac{I_0}{2\epsilon_0} t^2 d\dot{x}_0 \quad (1.92)$$

and, using Eq. 1.91,

$$dV_a = - \frac{I_0}{2\epsilon_0} t^2 d\dot{x}_0 \quad (1.93)$$

We will now evaluate t . Since the major portion of the thermal velocities at the potential minimum are small compared with the velocities in the rest of the region, we can take the value of t for $\dot{x}_0 = 0$. Thus, from Eqs. 1.87 and 1.88, we can write

$$t^2 = \left| \frac{eI_0}{2m\epsilon_0} \right|^{-1} \left(\frac{2|e|V_a}{m} \right)^{1/2} \quad (1.94)$$

and, from Eqs. 1.93 and 1.94,

$$dV_a = -2^{-1/2} \left| \frac{e}{m} \right|^{-1/2} V_a^{1/2} d\dot{x}_0 \quad (1.95)$$

If $\overline{(d\dot{x}_0)^2}$ is the mean-square fluctuation in velocity, the mean-square fluctuation in voltage $\overline{V_a^2}$ will be, from Eq. 1.95,

$$\overline{V_a^2} = 2 \left| \frac{m}{e} \right| V_a \overline{d\dot{x}_0^2} \quad (1.96)$$

or, with Eq. 1.75, we can write

$$\overline{V_a^2} = 3 \left| \frac{m}{e} \right| I_0 R \overline{d\dot{x}_0^2} \quad (1.97)$$

From Section 2, Eq. 1.74, we have

$$\overline{v^2} = \overline{d\dot{x}_0^2} = \frac{|e| \Delta f k T_c}{I_0 m} (4 - \pi)$$

and Eq. 1.97,

$$\begin{aligned} \overline{V_a^2} &= 3(4 - \pi) k T_c R \Delta f \\ \overline{V_a^2} &= (0.644) 4 k T_c R \Delta f \end{aligned} \quad (1.98)$$

This is the fluctuation in voltage between the anode and potential minimum for an open-circuited diode. It is also the fluctuation in anode-cathode voltage for an a-c open-circuited diode under the assumption that we can neglect the effect of those electrons that return to the cathode. We see here the well-known result that the noise from the space-charge-limited diode is two thirds of that from a thermal resistor with resistance equal to the diode resistance.

Smoothing Factor from the Llewellyn-Peterson Equations

We will now present the approach used by Rack.¹⁶ It will be assumed that the reader is familiar with the Llewellyn-Peterson equations⁶ which can be written in the form†

$$V_b - V_a = A^* I + B^* J_a + C^* v_a \quad (1.99)$$

where $V_b - V_a$ is the alternating voltage between two planes in a diode and J_a and v_a are the a-c convection-current density and a-c velocity at plane a . I is the total a-c current density in the diode.

If we consider the case where we have no fluctuations in the diode, i.e., $J_a = 0$ and $v_a = 0$, Eq. 1.99 reduces to

$$V_b - V_a = A^* I \quad (1.100)$$

from which we identify A^* with the a-c impedance of the diode.

† See also Chapter 3 but note changes in notation.

Therefore we associate the last two terms with the voltage produced in the diode by fluctuations in the electron stream, and we write for the a-c open-circuited diode

$$V_b - V_a = B^*J_a + C^*v_a \tag{1.101}$$

The coefficients B^* and C^* are given in reference 6 as

$$B^* = j \frac{1}{\epsilon_0} \frac{T^2}{\theta^3} u_a (2P - \beta Q) \tag{1.102}$$

$$C^* = - \frac{2m}{e} (u_a + u_b) \frac{P}{\theta^2} \tag{1.103}$$

In these equations

- u_a = average velocity at plane a
- u_b = average velocity at plane b
- θ = transit angle from plane a to plane b
- T = transit time from plane a to plane b

and

$$P = 1 - e^{-j\theta} - j\theta e^{-j\theta} \tag{1.104}$$

and

$$Q = 1 - e^{-j\theta}$$

respectively. For our noise problems we will consider the a plane to be just slightly beyond the potential minimum, so that we encounter no electrons returning to the cathode, but close enough to the minimum so that the d-c acceleration may be taken equal to zero. Under these conditions u_a will be considered to be zero, and, from Eq. 1.102, B^* is zero. This tells us that fluctuations in current density at the potential minimum produce no significant effect on fluctuations at the anode. We are left with the equation

$$V_b - V_a = C^*v_a \tag{1.105}$$

And from Eq. 1.104 we have, for $\theta \rightarrow 0$,

$$\frac{P}{-\theta^2} = \frac{1}{2} \tag{1.106}$$

If b represents the anode plane, Eq. 1.106 is applicable to the short-

transit-angle diode problem. Equation 1.105 becomes†

$$\begin{aligned} V_b - V_a &= \frac{mu_b}{e} v_a \\ &= - \left(2 \frac{m}{|e|} V_{ob} \right)^{1/2} v_a \end{aligned} \quad (1.107)$$

and the mean-square fluctuation in anode voltage is given by

$$\overline{V^2} = 2 \frac{m}{|e|} V_{ob} \overline{v_a^2} \quad (1.108)$$

This will be recognized as identical to Eq. 1.96, and therefore we can write again

$$\overline{V^2} = \frac{3}{4}(4 - \pi) 4kT_c R \Delta f \quad (1.109)$$

In this section we have derived several consistent expressions for the noise-smoothing factor Γ^2 , using models that are limited to single-velocity flow. We must now develop the problem of multivelocity flow, and we will find that the above approximations are valid for small values of J_0/J and relatively large values of V_a ($V_a > 20$ volts). These are familiar approximations and apply for most operating conditions in the low-frequency region.

4. Space-Charge-Limited Noise for Diodes with Short Transit Angles

The problem of reduced shot noise with multivelocity flow that is encountered in space-charge-limited diodes when the transit time from cathode to anode is short compared to an RF cycle will be treated by following the discussion presented by Thompson, North, and Harris in reference 5. As pointed out in Section 1, the mechanism of space-charge-limited flow is such that part of the emitted electrons are turned back to the cathode by the negative gradient prior to the potential minimum. Since this gradient is established by the space charge of the electrons, it is not difficult to understand that an instantaneous increase of emitted electrons over and above the average number would result in a lowering of the potential minimum, and hence a larger number of electrons will be returned to the cathode. It is this gating action of the potential minimum that reduces the noise below the level of pure shot noise.

When additional electrons are emitted which have velocities suffi-

† Note that in the Llewellyn-Peterson notation V_a corresponds to the input plane, and, in this context, to the potential minimum, not the anode!

cient to pass the potential minimum, the number of electrons passing through to the anode is momentarily increased, thus lowering the value of V_m owing to the added space charge. The more negative potential minimum turns back some electrons which would have otherwise passed to the anode. Similarly, if the number of emitted high-velocity electrons is less, V_m becomes less negative, and an additional number of the lower-velocity electrons are allowed to pass to the anode. It is this critical relation between the fluctuations in emission velocities and the corresponding fluctuations in potential minimum that we wish to study.

Formulation of Smoothing Factor

The analysis will proceed along the following lines: Assume that steady-state conditions exist within the diode. We now inject into this diode a small current which contains electrons with velocities of emission from v_s to $v_s + \Delta v_s$. The fluctuation current is given by the shot-noise formula $(2eI_s \Delta f)^{1/2}$, with I_s being the current carried by the velocity group v_s to $v_s + \Delta v_s$. From this we shall determine the resulting fluctuation in the potential minimum. This in turn will allow us to calculate the corresponding fluctuations in anode current. An integration over all the velocity classes will then give the total fluctuation in the anode current.

We shall consider only fluctuations of long enough duration so that they act as a succession of equilibrium states. We assume the cathode to be at zero potential and define V_s by

$$-2 \frac{e}{m} V_s = v_s^2 \quad (1.110)$$

where v_s is the velocity of emission of a small steady increment of current, $i_s(v_s)$. We further define the parameter

$$\lambda = \frac{-e(V_s + V_m)}{kT} \quad (1.111)$$

where V_m is a negative number. Thus we can use λ rather than v_s to designate the emission velocity of electrons which comprise i_s . If we choose v_s such that the element of current in this velocity class crosses the potential minimum to the anode, λ is positive. For smaller values of v_s , such that the current is turned back, λ is negative and must lie in the range $-\eta_c \leq \lambda \leq 0$, where η_c is defined in Eq. 1.29. We see that the value $\lambda = -\eta_c$ corresponds to a value of v_s equal to zero. Also, from the η vs. ξ plot of Fig. 1.2, the point at which the electrons stop and return to the cathode is given by $\eta = -\lambda$.

We now consider the noise fluctuations in the anode current. For every value of $i_s(\lambda)$ that we inject into the diode, we will find the new equilibrium current \hat{I} . Thus, if I_0 represents the steady-state anode current which flows before the admission of $i_s(\lambda)$, we are able to determine the net increase in I_0 . The ratio of this net increase in anode current to the increment of current $i_s(\lambda)$ will be a function only of λ . This follows from the previous argument which indicated that the change in anode current due to an incremental change in emission current would be a function of the velocity of emission and hence λ . The ratio of the net increase in anode current to i_s will be denoted by $\gamma(\lambda)$. It represents the factor by which a change in emission is converted into a change in plate current.

The fluctuation Δi_s in the incremental current is a true shot fluctuation and can be written

$$\overline{\Delta i_s^2} = 2|e| \Delta I_s \Delta f \quad (1.112)$$

where ΔI_s is the emission current containing electrons with emission velocities between λ and $\lambda + \Delta\lambda$. From Eq. 1.27 we can write

$$\Delta I_s = I e^{-\lambda} \Delta\lambda \quad (1.113)$$

or

$$\overline{\Delta i_s^2} = 2|e| I \Delta f (e^{-\lambda} \Delta\lambda) \quad (1.114)$$

Now we have stated that the fluctuations in anode current will be changed by the factor $\gamma(\lambda)$ from those in the emitted current. We can then write the fluctuations in an incremental element of anode current as

$$\overline{\Delta i^2} = 2|e| I \Delta f [\gamma^2(\lambda) e^{-\lambda} \Delta\lambda] \quad (1.115)$$

Since the fluctuations of Eq. 1.112 associated with one velocity class are independent of other velocity classes, we can obtain the total fluctuations in anode current by integrating Eq. 1.115 to give

$$\overline{i^2} = 2|e| I \Delta f \int_{-\eta_c}^{\infty} \gamma^2(\lambda) e^{-\lambda} d\lambda \quad (1.116)$$

which we can write as

$$\overline{i^2} = 2|e| I \Delta f \Gamma^2 \quad (1.117)$$

Γ^2 will be recognized as the space-charge reduction factor and can be written as

$$\Gamma^2 = \Gamma_\alpha^2 + \Gamma_\beta^2 \quad (1.118)$$

where

$$\Gamma_\alpha^2 = \int_{-\eta_c}^0 \gamma^2(\lambda) e^{-\lambda} d\lambda \quad (1.119)$$

and

$$\Gamma_{\beta}^2 = \int_0^{\infty} \gamma^2(\lambda) e^{-\lambda} d\lambda \tag{1.120}$$

where α refers to the group of electrons that do not have initial velocities at the cathode sufficient to overcome the potential minimum and hence are returned to the cathode. The subscript β denotes that group of electrons which pass the potential minimum and hence reach the anode.

Evaluation of Smoothing Factor

The factor $\gamma(\lambda)$ for the α and β groups can be expressed as follows. For values of λ pertaining to the α group the total anode current after injection of i_s is \hat{I} , whereas before injection it was I_0 . We can then write

$$\gamma(\lambda) = \frac{\hat{I} - I_0}{i_s(\lambda)} \quad (\alpha \text{ group}) \tag{1.121}$$

For the β group the total anode current after injection is $\hat{I} + i_s$, for i_s now passes on to the anode, and therefore

$$\gamma(\lambda) = 1 + \frac{\hat{I} - I_0}{i_s(\lambda)} \quad (\beta \text{ group}) \tag{1.122}$$

Let us now calculate \hat{I} when we inject a small additional current i_s from the cathode. For the β group we can write, in place of Eq. 1.9,

$$\left. \frac{d^2V}{dx^2} \right|_{\beta} = -\frac{e}{\epsilon_0} \int_{v_m}^{\infty} \frac{n(v_s) dv_s}{v} - \frac{i_s}{\epsilon_0 v} \tag{1.123}$$

Following the procedure of Section 1, we can make a first integration of Eq. 1.123 by multiplying both sides by $2(dV/dx)$ and, since

$$v = \sqrt{2 \left| \frac{e}{m} \right| \sqrt{V + V_s}}$$

we obtain, in place of Eq. 1.10,

$$\begin{aligned} \left[\frac{dV}{dx} \right]_{\beta}^2 &= \frac{-2m}{e} \int_{v_m}^{\infty} n(v_s)(v - v_m) dv_s \\ &\quad - \frac{i_s 2 \sqrt{2}}{\epsilon_0 \sqrt{|e/m|}} (\sqrt{V + V_s} - \sqrt{V_m + V_s}) \end{aligned} \tag{1.124}$$

If we now change from x and V to ξ and η as given in Eqs. 1.22 and

1.23, we obtain, in place of Eq. 1.124,

$$\left(\frac{d\eta}{d\xi}\right)_\beta^2 = \frac{i_s}{\hat{I}_0} \frac{2}{\sqrt{\pi}} (\sqrt{\eta + \lambda} - \sqrt{\lambda}) + e^\eta - 1 - e^\eta \operatorname{erf} \sqrt{\eta} + \frac{2}{\sqrt{\pi}} \sqrt{\eta} \quad (1.125)$$

which we can write as

$$\left(\frac{d\eta}{d\xi}\right)_\beta^2 = \frac{i_s}{\hat{I}} F(\eta, \lambda) + \Phi_\beta(\eta) \quad (1.126)$$

$$\text{where } F(\eta, \lambda) = \frac{2}{\sqrt{\pi}} (\sqrt{\eta + \lambda} - \sqrt{\lambda})$$

and $\Phi_\beta(\eta)$ follows from Eq. 1.125.

Now, since we have assumed $i_s \ll \hat{I}$, we shall neglect all the higher powers of i_s/I_0 and write

$$d\xi = \left[1 - \frac{1}{2} \frac{i_s}{\hat{I}} \frac{F(\eta, \lambda)}{\Phi_\beta(\eta)} \right] \frac{d\eta}{\Phi_\beta(\eta)^{1/2}} \quad (1.127)$$

If we denote the value of ξ and η at the anode by the subscript a , we can write

$$\xi_a = \int_0^{\hat{\eta}^a} \frac{d\eta}{\Phi_\beta(\eta)^{1/2}} - \frac{1}{2} \frac{i_s}{\hat{I}} \int_0^{\hat{\eta}^a} \frac{F(\eta, \lambda)}{\Phi_\beta(\eta)^{3/2}} d\eta \quad (1.128)$$

where $\hat{\xi}$ and $\hat{\eta}$ represent the perturbed values, whereas ξ represents the unperturbed value and is given from Eq. 1.128 with $i_s = 0$:

$$\xi_a = \int_0^{\eta^a} \frac{d\eta}{\Phi_\beta(\eta)^{1/2}} \quad (1.129)$$

Now since the quantities $(\hat{\eta} - \eta)$ and $(\hat{\xi} - \xi)$ are also first-order infinitesimals, we can write for Eq. 1.128

$$\hat{\xi}_a - \xi_a = \frac{\hat{\eta}^a - \eta^a}{\Phi_\beta(\eta^a)^{1/2}} - \frac{1}{2} \frac{i_s}{\hat{I}} \int_0^{\eta^a} \frac{F(\eta, \lambda)}{\Phi_\beta(\eta)^{3/2}} d\eta \quad (1.130)$$

A similar treatment for the α group gives

$$\hat{\xi}_c - \xi_c = \frac{\hat{\eta}_c - \eta_c}{\Phi_\beta(\eta_c)^{1/2}} + \frac{1}{2} \frac{i_s}{\hat{I}} \int_0^{\eta_c} \frac{F(\eta, \lambda)}{\Phi_\alpha(\eta)^{1/2}} d\eta \quad (1.131)$$

where the subscript c denotes the cathode surface. Now we will treat the problem wherein the anode potential is held constant as in an

a-c short-circuited diode, and, from Eqs. 1.22, 1.23, and 1.29, we have

$$\hat{\eta}_a - \eta_a = \hat{\eta}_c - \eta_c = \frac{e}{kT} (V_m - \hat{V}_m) = \ln \frac{I_0}{\hat{I}} \cong - \frac{\hat{I} - I_0}{I_0} \quad (1.132)$$

and

$$\begin{aligned} (\hat{\xi}_a - \hat{\xi}_c) - (\xi_a - \xi_c) &= (\xi_a - \xi_c) \left(\sqrt{\frac{\hat{I}}{I_0}} - 1 \right) \\ &= \frac{1}{2} (\xi_a - \xi_c) \frac{\hat{I} - I_0}{I_0} \end{aligned} \quad (1.133)$$

If we subtract Eq. 1.131 from Eq. 1.130 and use Eqs. 1.132 and 1.133, we obtain

$$\frac{\hat{I} - I_0}{i_s} = - \frac{1}{2D} \left[\int_0^{\eta_c} \frac{F(\eta, \lambda)}{\Phi_\alpha(\eta)^{3/2}} d\eta + \int_0^{\eta_a} \frac{F(\eta, \lambda)}{\Phi_\beta(\eta)^{3/2}} d\eta \right] \quad (1.134)$$

where

$$D = \frac{1}{2}(\xi_a - \xi_c) + \Phi_\beta(\eta_a)^{-1/2} + \Phi_\alpha(\eta_c)^{-1/2} \quad (1.135)$$

Therefore, from Eq. 1.122, we write for the β region

$$\gamma(\lambda) = 1 - \frac{1}{2D} \left[\int_0^{\eta_c} \frac{F(\eta, \lambda)}{\Phi_\alpha(\eta)^{3/2}} d\eta + \int_0^{\eta_a} \frac{F(\eta, \lambda)}{\Phi_\beta(\eta)^{3/2}} d\eta \right] \quad (1.136)$$

where

$$\lambda \geq 0$$

It is now necessary to resort to numerical integration in order to evaluate Eq. 1.136. This is done for the case of complete space-charge-limited flow, which means that we can assume $I_0/I \ll 1$. This in turn means that we may use the values $\eta_c = \infty$, $\xi_c = -2.55$ and $\Phi_\alpha(\eta_c) = \infty$. Thus, in Eq. 1.136, $\gamma(\lambda)$ becomes a function of η_a and λ . The resulting values of $\gamma(\lambda)$ were obtained by North and others in the range $5 \leq \eta_a \leq 100$ and $0.05 \leq \lambda \leq 5$. Near $\lambda = 0$, $\gamma(\lambda)$ has a logarithmic discontinuity and must be evaluated separately. In Fig. 1.3 there is shown a plot of $\gamma(\lambda)$ vs. λ as taken from reference 5 for $\eta_a = 30$. The value of Γ_β^2 for $\eta_a = 30$ is now obtained from Fig. 1.3 with the aid of Eq. 1.119. This method then allows a plot of Γ_β^2 vs. η_a , which is given in Fig. 1.4.

Also shown in Fig. 1.4 is a plot of Γ_α^2 vs. η_a , which is obtained in a manner similar to the preceding calculation for Γ_β^2 . It is significant to note that Γ_α^2 is always much smaller than Γ_β^2 . This is to be expected, since the major portion of the electrons in the α group are returned to the cathode before they approach very near the potential minimum. Therefore the action of the α electrons is confined to a very

short region. We can thus neglect their contribution to the noise and set $\Gamma_\alpha^2 \approx 0$. This assumption, together with single-velocity equations, allowed us to evaluate Γ_β^2 in the approximate but explicit forms of Section 3.

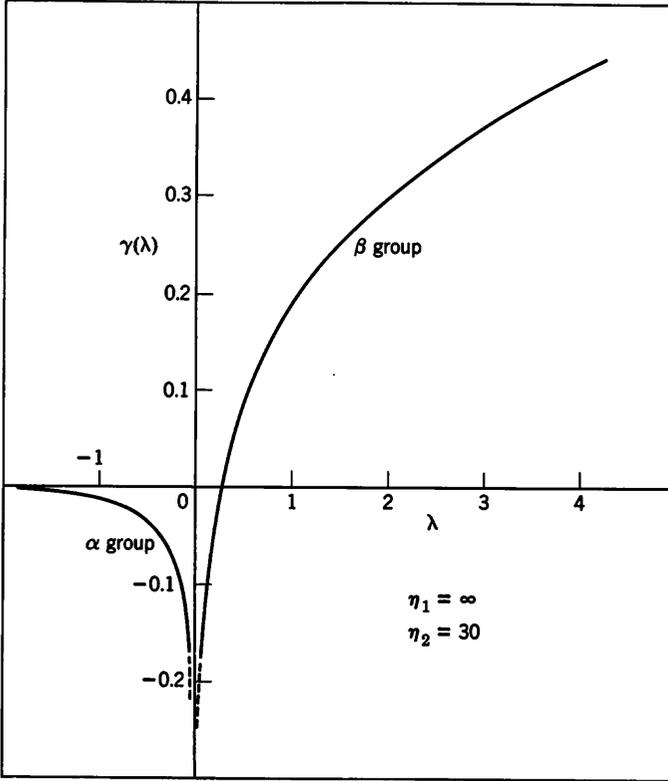


Fig. 1.3. Shot-effect reduction factor γ as a function of the velocity parameter λ .

Smoothing Factor in Terms of the Diode Transconductance

Before concluding this section, we will discuss Eq. 1.117 in a different form in terms of the diode conductance g . Thus Eq. 1.51 can be written

$$\overline{i^2} = \theta \cdot 4kTg \Delta f \quad (1.137)$$

The diode conductance g as defined by

$$g = \frac{\partial I_0}{\partial V_a}$$

was derived in Section 1, Eqs. 1.45 through 1.47. In Eq. 1.47 we set the multiplier approximately to one and replace $\xi_a - \xi_c$ by Eq. 1.135, where

$$\phi_a^{-1/2} \cong 0$$

We obtain

$$g = \frac{2I_0}{2D - \phi_\beta(\eta_a)^{-1/2}} \frac{\partial \xi_a}{\partial \eta_a} \frac{|e|}{kT_c}$$

Further, using Eqs. 1.31, 1.125, and 1.126, we can replace $\partial \xi_a / \partial \eta_a$ by $\phi_\beta(\eta_a)^{-1/2}$ and obtain

$$g = \frac{2I_0}{2D\phi_\beta(\eta_a)^{1/2} - 1} \frac{|e|}{kT_c}$$

At large η_a we can neglect the one in the denominator, and finally obtain

$$\frac{|e|}{kT_c} \frac{I_0}{g} = D\Phi_\beta(\eta_a)^{1/2} \tag{1.138}$$

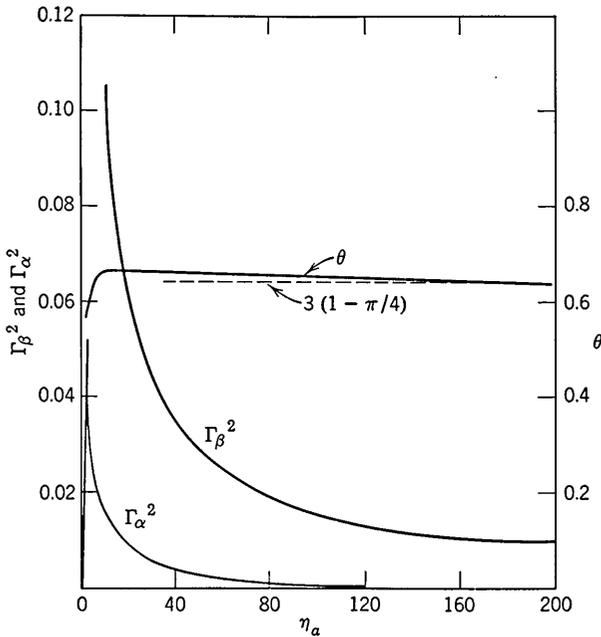


Fig. 1.4. Shot-noise reduction factor for complete space-charge-limited flow ($I_s/I \gg 1$).

With Eqs. 1.117 and 1.137 we can write for θ

$$\theta = \frac{1}{2}\Gamma^2 \frac{|e| I_0}{kT_c g} = \frac{1}{2}\Gamma^2 D\Phi_\beta(\eta_a)^{1/2} \quad (1.139)$$

This value of θ is also plotted in Fig. 1.4, and we see that for $\eta_a \rightarrow \infty$ we have a limiting value for θ given by

$$\theta \sim 3 \left(1 - \frac{\pi}{4}\right) = 0.644 \quad (1.140)$$

This familiar result agrees with that found by the approximate methods of Section 3. Again it can be stated that a space-charge-limited diode generates a mean-square noise power equal to the thermal noise in a resistor whose resistance is equal to two thirds of the a-c resistance of a diode.

5. Discussion of Emission Noise at High Frequencies When the Transit Time Is Not Small

There is no adequate treatment of electron streams with Maxwellian distribution of velocities for high frequencies, and therefore the analysis of the noise problem must be based on the single-velocity approximation. Rack¹⁶ has extended the analysis of Section 4 to include finite transit angles. Peterson¹⁷ has used the Llewellyn-Peterson equations for calculating the noise in a high-frequency tetrode. Pierce¹⁸ used this method in calculating the noise in the stream of a traveling-wave tube. In applying this method, he assumed that at the input plane near the potential minimum the average velocity was zero, and hence any fluctuation in current density at this point produced no effect at a later point in the stream. The only source of noise was the fluctuation in velocity which was taken to be equal to the mean-square deviation as calculated for the multiveLOCITY stream. This analysis is subject among other things to the defect that the average velocity at the potential minimum is not zero but finite as calculated in Section 2, and hence the velocity fluctuation is not the only source of noise. There is a second source of noise in the current fluctuation at the potential minimum. The question that remains to be answered relates to the magnitude of these current fluctuations. Is it equal to shot noise or reduced shot noise, and is it correlated or uncorrelated with the noise velocity?

α Plane Beyond Potential Minimum Considered as Input

The first attempt to answer these questions was made by Robinson.⁷ The single-velocity equations break down because the spread in transit angles between the potential minimum and the anode is not small. In fact, MacDonald¹⁹ has given an expression for the anode transit time $\tau(E)$ of an electron with an initial energy E at the minimum to be

$$\tau(E) = \tau(0) \left(1 - 0.85 \left| \frac{E}{eV_a} \right|^{3/4} \right) \tag{1.141}$$

$\tau(0)$ = transit time of electron with zero initial energy

Thus for a typical microwave tube $\tau(0)$ might be 4 cycles and $V_a = 1000$ volts. If we use $E = kT$, the average energy of the minimum, we arrive at a value of $\omega[\tau(0) - \tau(E)] = 2\pi/3$ radians, which is not small. Robinson points out that this spread takes place largely in a region very close to the minimum, and hence, if we consider our input plane to be a given distance beyond the potential minimum, the single-velocity theory should be valid at the high frequencies. That such a plane exists can be seen from Eq. 1.141, for, if we choose 4 electrons with initial energies of 0, kT , $2kT$, and $3kT$, their transit times in the above example would be $\tau(0)$, $0.915\tau(0)$, $0.898\tau(0)$, and $0.888\tau(0)$. Thus we see that the spread in transit angle between the third and fourth electrons of 0.08π is certainly small. If we choose our input plane, called the α plane, at a point beyond the minimum where the potential is $\alpha(kT/e)$, we see that the slowest electron will have an energy of αkT at this plane. If α is sufficiently large, the spread in transit angle beyond this plane will be small.

At the α plane Robinson assumes the current fluctuations to be equal to shot noise, and the velocity fluctuations to be given by the mean-square velocity deviation for the multivelocity stream at the α plane, which is calculated in a manner similar to that used in Section 2.

Estimate of Smoothing Factor at High Frequencies for Noise Current

We see that the approach just considered would be very good if we knew the true value of the current and velocity fluctuations at the α plane. Robinson assumed pure shot noise with no correlation between velocity and current. Following Watkins,⁸ we may argue that the current fluctuations at the α plane should be somewhat less than true shot noise.† Consider the diode to be divided into two diodes in series: the first between the cathode and the potential minimum and

† Compare Sec. 2, page 229, Chapter 5.

the second between the potential minimum and the α plane. The first diode prior to the minimum is a retarding field diode of area A , which has, for small transit angles, an a-c admittance given by

$$Y = \left(\frac{|e|J_0}{kT_c} + j \frac{\omega\epsilon}{d} \right) A \quad (1.142)$$

where J_0 is the current density through the diode, and d is the distance between the cathode and the potential minimum. If we consider the diode to be a-c short-circuited, the current fluctuation through the diode will be shot noise, as argued in Section 3, and we can write

$$i_{sc} = |2eI_0 \Delta f|^{1/2} \quad (1.143)$$

If we assume a linear system, Eqs. 1.142 and 1.143 can be used with Thévenin's theorem to give the open-circuit noise voltage as

$$V_{oc} = \frac{i_{sc}}{Y} = \frac{|2eI_0 \Delta f|^{1/2}}{\left(\frac{|e|J_0}{kT_c} + j \frac{\omega\epsilon_0}{d} \right) A} \quad (1.144)$$

We now consider this diode to be in series with the second diode, and, as a result of the large transit angle beyond the potential minimum, the latter diode can be considered to be open-circuited for alternating current. With this condition, the total alternating current must be zero, or

$$i + j\omega\epsilon_0 A E = 0 \quad (1.145)$$

where i is the alternating convection current.

$$i = j\omega\epsilon_0 A \frac{V_{oc}}{d} = \frac{j\omega\epsilon_0}{d} \frac{|2eI_0 \Delta f|^{1/2}}{|e|J_0/kT_c + j\omega\epsilon_0/d} \quad (1.146)$$

or, for the mean-square current fluctuation, we can write

$$\overline{i^2} = |2eI_0 \Delta f| \Gamma^2 \quad (1.147)$$

where

$$\Gamma^2 = \frac{1}{1 + (eJ_0 d / \omega\epsilon_0 kT_c)^2} \quad (1.148)$$

We can write d in Eq. 1.148 in terms of the parameter ξ_c defined by Eq. 1.23. Using a new parameter a ,

$$a = \frac{\omega}{2\pi^{1/4} (m/2kT_c)^{1/4} \left| \frac{eJ_0}{m\epsilon_0} \right|^{1/2}} = \frac{\omega}{\omega_{pm}} \quad (1.149)$$

where ω_{pm} may be defined as a plasma frequency at the potential minimum. Equation 1.148 can be written

$$\Gamma^2 = \frac{1}{1 + \xi_c^2/4\pi a^2} \tag{1.150}$$

Now, recalling that, for complete space charge $J_0/J \ll 1$, we have as the limiting value of $\xi_c = 2.55$. With this value Eq. 1.150 is written

$$\Gamma^2 = \frac{1}{1 + 0.52/a^2} \tag{1.151}$$

For a numerical example we will use the following values:

$$T = 1020^\circ \text{ K}, \quad f = 3000 \text{ Mc}, \quad J_0 = 0.1 \text{ ampere/cm.}$$

in which case $a = 0.665$ and $\Gamma^2 = 0.46$. The mean-square fluctuation in current is reduced by a factor of 0.46 below shot noise.

Watkins points out that the assumption of small transit angles from cathode to potential minimum is not justified as it is found to be about 1.4 radians in the above example. Whinnery⁹ has discussed some aspects of the noise at the potential minimum at high frequencies by using the physical picture of Thompson, North, and Harris as in Section 4. He discusses the effects of using different values of η_c in numerically evaluating the integrals of Section 4. He considers the perturbation of the potential minimum as one injects an excess of charge at the cathode and finds that the potential minimum "over-compensates." It oscillates back and forth at a frequency corresponding to the plasma frequency calculated at the potential minimum. He further points out that this plasma frequency for typical tubes occurs in the microwave region from about 2000 to 4000 Mc.

Treatment of Multivelocity Problem with the Use of a Computer

In each of the approaches that has just been presented it was necessary to make rather severe assumptions; so far it has proved difficult to assess their validity. Therefore, these solutions have limitations, for they do not give a complete answer to the multivelocity flow problem near the potential minimum. In Watkins' analysis a short transit time from cathode to potential minimum is assumed, an assumption which is not generally true. In Whinnery's work the electrons that return to the cathode are all assumed to return at the point of the potential minimum, $x = x_m$, whereas in reality the point of return is distributed between the cathode and x_m . In an analysis of this sort one must make use of a linear theory. Since the d-c velocity is small

in the region of interest, the a-c velocities can be of comparable amplitude, which violates the assumptions necessary for the linear theory.

Because of these difficulties Tien and Moshman²⁰ attacked the problem by using numerical integration to trace individual electrons through the potential minimum of a typical diode. We can see in principle what is required for this task. The d-c conditions of the diode can be established from the equations of Section 1. Then one electron injected at the cathode can be traced step by step through the minimum to the anode. By injecting a sufficient number of electrons and computing their cumulative effect, it should be possible to find the noise current and noise velocity at the α plane just beyond the potential minimum.

The number of electrons that are injected at the cathode is necessarily limited when a computer is used, and the question immediately arises as to how these electrons should be initiated so that the emission noise is properly simulated. It is necessary to know the number of electrons injected, their time of injection, and the velocity at which they are injected. These initial properties were determined by Tien and Moshman by the "Monte Carlo method" of statistics, which is characterized by the use of random numbers.

We will first discuss briefly random numbers so as to illustrate how they were used in obtaining the initial conditions of the injected electrons. Consider the simple problem of finding the area under the curve of Fig. 1.5. We would begin by dividing the interval into n equal spaces (Δx) and sampling the heights of the curve at each inter-

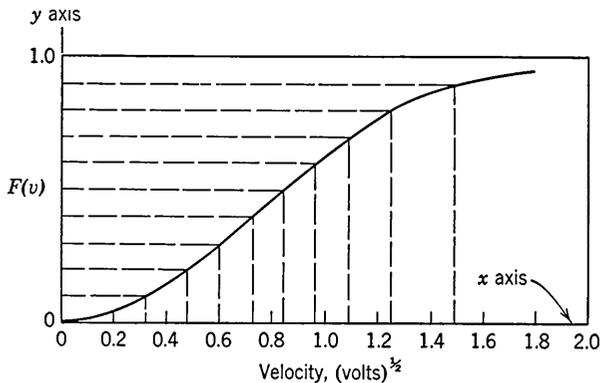


Fig 1.5. Plot showing transformation from uniform probability distribution to xe^{-ax^2} .

val. If we desired to increase the accuracy of the computations, we would subdivide the interval and use $2n$ steps. Thus we are limited to a discrete number of steps. With the use of random numbers we would not use the uniform interval but choose n random numbers which were distributed with uniform probability in the desired interval. If we chose 100 intervals, for example, the random numbers might be 1.1, 2.9, 3.1, 3.8, etc. However, with random numbers we can increase the number that we use without regard to the discrete steps as before, and we can equally well use 99 numbers or 101 or 138, and they would still have the same distribution. As the number of samples increases, the accuracy increases.

Random numbers with a uniform distribution are readily obtainable from tables, or, if a computing machine is involved, it may be faster to generate the numbers. However, there are many instances when one wishes to use a distribution other than uniform, and this is obtained by a simple transformation. Let us consider how we might transform a set of uniformly distributed numbers to a distribution which is expressed by a probability density function

$$xe^{-ax^2}$$

This is illustrated in Fig. 1.5, where we see the evenly spaced points along the vertical axis and the transformed points along the horizontal axis. It can be seen that the points along the horizontal axis are obtained from the intersection of the equispaced vertical intervals with the curve R , where

$$R = \int_0^x xe^{-ax^2} dx = 1 - e^{-ax^2}$$

This is the integral of the desired probability density function. Since the first factor is constant, the function $R = e^{-ax^2}$ may equally well be used to obtain the desired distribution.

Now let us present a physical picture of the problem which Tien and Moshman studied. The d-c potential profile shown in Fig. 1.6 is computed from the equations of Section 1. The computer must memorize every electron in transit since interaction is considered. Therefore, for an assumed cathode-current density we must limit the area of the diode under consideration. If it is too large, we cannot handle the required number of electrons in the computer. On the other hand, if the area chosen is too small, the problem loses its random character. Therefore, an area of $(\pi/4)x_m^2$ was chosen since it was felt that all electrons emitted within this area had an equally important effect at the potential minimum. Bear in mind that this is a one-

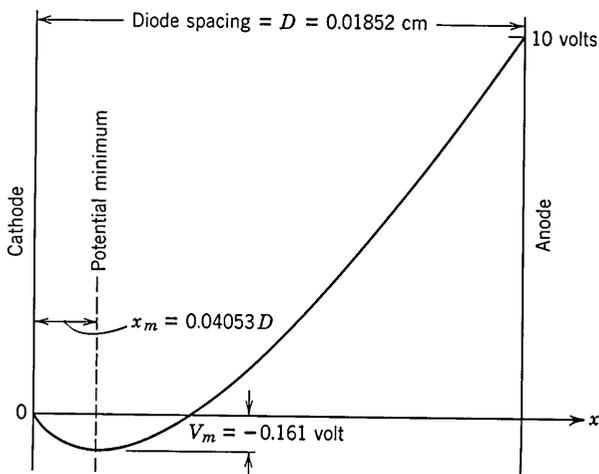


Fig. 1.6. D-c potential distribution used in Tien-Moshman computations.

dimensional analysis, and so it contains none of the effects of sideways displacement and velocities, etc.

The velocity distribution used is as shown in Figure 1.7. For the calculation the time is quantized into intervals of 2×10^{-12} sec, and during this interval an average of 8.152 electrons are emitted in the area $(\pi/4)x_m^2$.

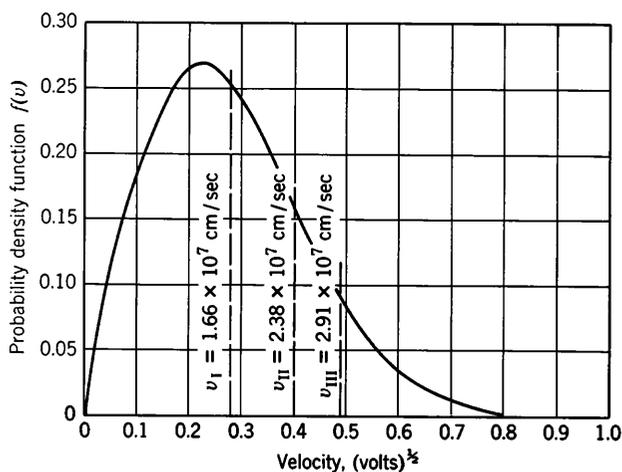


Fig. 1.7. Maxwellian velocity distributions.

In the time interval from t_r to $t_r + \Delta t$, we must know the number of electrons emitted, the time of emission, and the velocity of emission. These quantities are found with the use of random numbers. For the number of electrons emitted, a Poisson distribution normalized about the average of 8.152 was used. The function

$$f(s) = \frac{e^{-8.1525} \times 8.1525^s}{s!}$$

is used to transform the random numbers, generated with a uniform distribution, to a set with a Poisson distribution.

Another set of random numbers with a uniform distribution is generated and used for the time of emission, since the emission would normally occur at a uniform rate throughout the velocity distribution.

For the velocity of emission a set of uniform numbers is transformed according to the Maxwellian distribution

$$f(v) = \frac{mv}{kT_c} \exp\left(-\frac{mv^2}{2kT_c}\right)$$

and

$$F(v) = \int_0^v f(v) dv = \left[1 - \exp\left(-\frac{mv^2}{2kT_c}\right)\right]$$

which gives

$$R_i = 1 - \exp\left(-\frac{mv^2}{2kT_c}\right) \quad \text{or} \quad R_i = \exp\left(-\frac{mv^2}{2kT_c}\right)$$

The velocity of emission corresponding to the uniformly distributed random number R_i is given by

$$v_i = \left(\frac{2kT_c}{m}\right)^{1/2} (-\log R_i)^{1/2}$$

This establishes the initial conditions, and we turn the crank until the diode is filled with 363 electrons. The process is then repeated 2000 times to obtain the final data on the noise current and velocity at the α plane, which is taken to be at $x = 1.2x_m$.

If we transform the current as a function of time to a plot of current as a function of frequency with the use of the autocorrelation function, we obtain a plot of Γ^2 versus frequency, as in Fig. 1.8. This curve answers the question which we initially posed, for we now have a picture of the noise current that is appropriate for the α plane. The peak occurs just above the frequency of oscillation of Whinnery's compensating current, and the dip occurs somewhat below this fre-

quency. It might be stated that at the lower frequencies there is almost complete compensation of the initial disturbing pulse, and at the higher frequencies there is insufficient compensation. It seems that there occurs one frequency where the compensation is nearly complete. We should state that the effect of transverse velocities has not been evaluated and might well camouflage this effect.

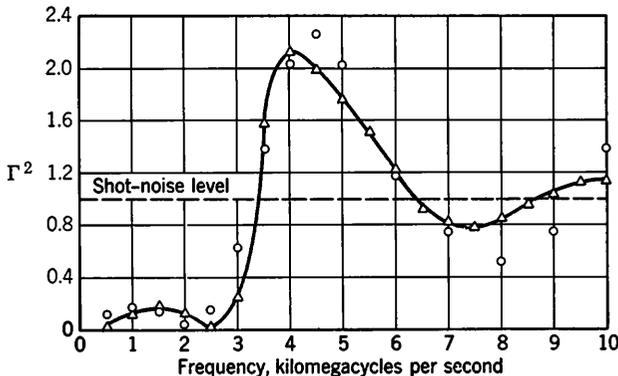


Fig. 1.8. Computed space-charge reduction factor Γ^2 as a function of frequency.

We also need to know the noise velocity at the α plane. The results of the computation can be summarized by stating that the computed noise velocity corresponds very closely with the expression of Eq. 1.74. Furthermore, within the limits of the accuracy of the computation, there is apparently no correlation between velocity and current.

This treatment is for an a-c short-circuited diode. At the frequency corresponding to the "dip" of the Γ^2 curve the transit angle is slightly less than 2π . Since physical diodes, or electron guns, are more nearly open-circuited, there is some concern as to whether the "dip" in noise current can be realized. Siegman and Bloom²¹ have discussed† some linear models that enlarge upon the work of Whinnery and Watkins for the open-circuited diode and find no evidence of the minimum in noise current. On either side of this frequency region the agreement is fairly good.

Another limitation of this analysis must be kept in mind. At the location of the $x = 1.2x_m$ plane there is still a large spread in electron velocities. Thus in the region immediately beyond the plane, the single-velocity description may not be adequate. Work by Siegman, Watkins, and Hsieh²² indicates that the multiveLOCITY character of the

† See Chapter 5, Section 2, page 229.

beam is important for some distance beyond $1.2x_m$. They use a linearized theory to predict that correlation is produced between the current and velocity fluctuations as the beam passes from the potential minimum to the α plane.

In summarizing this discussion we see that the noise from a thermionic cathode is fairly well understood at low frequencies. In a high-frequency diode, however, the description of noise requires a knowledge of noise current, noise velocity, and their correlation at the α plane. The α plane is defined as that point beyond the potential minimum where the spread in velocities between electrons is small. Beyond the α plane one can use, with these noise parameters as input conditions, the single-velocity theory, and this will be fully treated in later sections. According to the work of Tien and Moshman at the plane $x = 1.2x_m$, the noise current is given by Fig. 1.8, the velocity has the value given in Eq. 1.74, and their correlation is zero. The later work of Siegman, Watkins, and Hsieh predicts that some correlation is introduced in the multiveLOCITY beam as it drifts between the $1.2x_m$ plane and the α plane. It will be evident from later chapters how these parameters enter into the over-all noise figure for a high-frequency amplifier. It is sufficient here to point out that some degree of correlation between the noise velocity and current may lead to a noise figure for an amplifier which is less than that from an uncorrelated beam.†

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† *Editor's Note:* Recent measurements by Prof. S. Saito are in essential agreement with the predictions of Siegman, Watkins, and Hsieh. The measurements indicated a finite correlation between the velocity and current fluctuations under space-charge-limited operation. Under temperature-limited condition, the correlation disappeared.²³

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