CHAPTER I

A GENERAL VIEW OF MATHEMATICS

An adequate presentation of any science cannot consist of detailed information alone, however extensive. It must also provide a proper view of the essential nature of the science as a whole. The purpose of the present chapter is to give a general picture of the essential nature of mathematics. For this purpose there is no great need to introduce any of the details of recent mathematical theories, since elementary mathematics and the history of the science already provide a sufficient foundation for general conclusions.

§1. The Characteristic Features of Mathematics

1. Abstractions, proofs, applications. With even a superficial knowledge of mathematics, it is easy to recognize certain characteristic features: its abstractness, its precision, its logical rigor, the indisputable character of its conclusions, and finally, the exceptionally broad range of its applications.

The abstractness of mathematics is easy to see. We operate with abstract numbers without worrying about how to relate them in each case to concrete objects. In school we study the abstract multiplication table, that is, a table for multiplying one abstract number by another, not a number of boys by a number of apples, or a number of apples by the price of an apple.

Similarly in geometry we consider, for example, straight lines and not stretched threads, the concept of a geometric line being obtained by abstraction from all other properties, excepting only extension in one direction. More generally, the concept of a geometric figure is the result of abstraction from all the properties of actual objects except their spatial form and dimensions.

Abstractions of this sort are characteristic for the whole of mathematics. The concept of a whole number and of a geometric figure are only two of the earliest and most elementary of its concepts. They have been followed by a mass of others, too numerous to describe, extending to such abstractions as complex numbers, functions, integrals, differentials, functionals, n-dimensional, and even infinite-dimensional spaces, and so forth. These abstractions, piled up as it were on one another, have reached such a degree of generalization that they apparently lose all connection with daily life and the "ordinary mortal" understands nothing about them beyond the mere fact that "all this is incomprehensible."

In reality, of course, the case is not so at all. Although the concept of n-dimensional space is no doubt extremely abstract, yet it does have a completely real content, which is not very difficult to understand. In the present book it will be our task to emphasize and clarify the concrete content of such abstract concepts as those mentioned earlier, so that the reader may convince himself that they are all connected with actual life, both in their origin and in their applications.

But abstraction is not the exclusive property of mathematics; it is characteristic of every science, even of all mental activity in general. Consequently, the abstractness of mathematical concepts does not in itself give a complete description of the peculiar character of mathematics.

The abstractions of mathematics are distinguished by three features. In the first place, they deal above all else with quantitative relations and spatial forms, abstracting them from all other properties of objects. Second, they occur in a sequence of increasing degrees of abstraction, going very much further in this direction than the abstractions of other sciences. We will illustrate these two features in detail later, using as examples the fundamental notions of number and figure. Finally, and this is obvious, mathematics as such moves almost wholly in the field of abstract concepts and their interrelations. While the natural scientist turns constantly to experiment for proof of his assertions, the mathematician employs only argument and computation.

It is true that mathematicians also make constant use, to assist them in the discovery of their theorems and methods, of models and physical analogues, and they have recourse to various completely concrete examples. These examples serve as the actual source of the theory and as a means of discovering its theorems, but no theorem definitely belongs to mathematics until it has been rigorously proved by a logical argument. If a geometer, reporting a newly discovered theorem, were to demonstrate it by means of models and to confine himself to such a demonstration, no mathematician would admit that the theorem had been proved. The demand for a proof of a theorem is well known in high school geometry, but it pervades the whole of mathematics. We could measure the angles at the base of a thousand isosceles triangles with extreme accuracy, but such a procedure would never provide us with a mathematical proof of the theorem that the angles at the base of an isosceles triangle are equal. Mathematics demands that this result be deduced from the fundamental concepts of geometry, which at the present time, in view of the fact that geometry is nowadays developed on a rigorous basis, are precisely formulated in the axioms. And so it is in every case. To prove a theorem means for the mathematician to deduce it by a logical argument from the fundamental properties of the concepts occurring in that theorem. In this way, not only the concepts but also the methods of mathematics are abstract and theoretical.

The results of mathematics are distinguished by a high degree of logical rigor, and a mathematical argument is conducted with such scrupulousness as to make it incontestable and completely convincing to anyone who understands it. The scrupulousness and cogency of mathematical proofs are already well known in a high school course. Mathematical truths are in fact the prototype of the completely incontestable. Not for nothing do people say "as clear as two and two are four." Here the relation "two and two are four" is introduced as the very image of the irrefutable and incontestable.

But the rigor of mathematics is not absolute; it is in a process of continual development; the principles of mathematics have not congealed once and for all but have a life of their own and may even be the subject of scientific quarrels.

In the final analysis the vitality of mathematics arises from the fact that its concepts and results, for all their abstractness, originate, as we shall see, in the actual world and find widely varied application in the other sciences, in engineering, and in all the practical affairs of daily life; to realize this is the most important prerequisite for understanding mathematics.

The exceptional breadth of its applications is another characteristic feature of mathematics.

In the first place we make constant use, almost every hour, in industry and in private and social life, of the most varied concepts and results of mathematics, without thinking about them at all; for example, we use arithmetic to compute our expenses or geometry to calculate the floor area of an apartment. Of course, the rules here are very simple, but we should remember that in some period of antiquity they represented the most advanced mathematical achievements of the age.

Second, modern technology would be impossible without mathematics. There is probably not a single technical process which can be carried through without more or less complicated calculations; and mathematics plays a very important role in the development of new branches of technology.

Finally, it is true that every science, to a greater or lesser degree, makes essential use of mathematics. The "exact sciences," mechanics, astronomy, physics, and to a great extent chemistry, express their laws, as every schoolboy knows, by means of formulas and make extensive use of mathematical apparatus in developing their theories. The progress of these sciences would have been completely impossible without mathematics. For this reason the requirements of mechanics, astronomy, and physics have always exercised a direct and decisive influence on the development of mathematics.

In other sciences mathematics plays a smaller role, but here too it finds important applications. Of course, in the study of such complicated phenomena as occur in biology and sociology, the mathematical method cannot play the same role as, let us say, in physics. In all cases, but especially where the phenomena are most complicated, we must bear in mind, if we are not to lose our way in meaningless play with formulas, that the application of mathematics is significant only if the concrete phenomena have already been made the subject of a profound theory. In one way or another, mathematics is applied in almost every science, from mechanics to political economy.

Let us recall some particularly brilliant applications of mathematics in the exact sciences and in technology.

The planet Neptune, one of the most distant in the Solar System, was discovered in the year 1846 on the basis of mathematical calculations. By analyzing certain irregularities in the motion of Uranus, the astronomers Adams and Leverrier came to the conclusion that these irregularities were caused by the gravitational attraction of another planet. Leverrier calculated on the basis of the laws of mechanics exactly where this planet must be, and an observer to whom he communicated his results caught sight of it in his telescope in the exact position indicated by Leverrier. This discovery was a triumph not only for mechanics and astronomy, and in particular for the system of Copernicus, but also for the powers of mathematical calculation.

Another example, no less impressive, was the discovery of electromagnetic waves. The English physicist Maxwell, by generalizing the laws of electromagnetic phenomena as established by experiment, was able to express these laws in the form of equations. From these equations he deduced, by purely mathematical methods, that electromagnetic waves could exist and that they must be propagated with the speed of light. On the basis of this result, he proposed the electromagnetic theory of light, which was later developed and deepened in every direction. Moreover, Maxwell's results led to the search for electromagnetic waves of purely electrical origin, arising for example from an oscillating charge. These waves were actually discovered by Hertz. Shortly afterwards, A. S. Popov, by discovering means for exciting, transmitting, and receiving electromagnetic oscillations made them available for a wide range of applications and thereby laid the foundations for the whole technology of radio. In the discovery of radio, now the common possession of everyone, an important role was played by the results of a purely mathematical deduction.

So from observation, as for example of the deflection of a magnetic needle by an electric current, science proceeds to generalization, to a theory of the phenomena, and to formulation of laws and to mathematical expression of them. From these laws come new deductions, and finally, the theory is embodied in practice, which in its turn provides powerful new impulses for the development of the theory.

It is particularly remarkable that even the most abstract constructions of mathematics, arising within that science itself, without any immediate motivation from the natural sciences or from technology, nevertheless have fruitful applications. For example, imaginary numbers first came to light in algebra, and for a long time their significance in the actual world remained uncomprehended, a circumstance indicated by their very name. But when about 1800 a geometrical interpretation (see Chapter IV, §2) was given to them, imaginary numbers became firmly established in mathematics, giving rise to the extensive theory of functions of a complex variable, i.e., of a variable of the form $x + y\sqrt{-1}$. This theory of "imaginary" functions of an "imaginary" variable proved itself to be far from imaginary, but rather a very practical means of solving technological problems. Thus, the fundamental results of N. E. Jukovski concerning the lift on the wing of an airplane are proved by means of this theory. The same theory is useful, for example, in the solution of problems concerning the oozing of water under a dam, problems whose importance is obvious during the present period of construction of huge hydroelectric stations.

Another example, equally impressive, is provided by non-Euclidean geometry,* which arose from the efforts, extending for 2000 years from the time of Euclid, to prove the parallel axiom, a problem of purely

* Here we merely point out this example without further explanation, for which the reader may turn to Chapter XVII.

mathematical interest. N. I. Lobačevskiĭ himself, the founder of the new geometry, was careful to label his geometry "imaginary," since he could not see any meaning for it in the actual world, although he was confident that such a meaning would eventually be found. The results of his geometry appeared to the majority of mathematicians to be not only "imaginary" but even unimaginable and absurd. Nevertheless, his ideas laid the foundation for a new development of geometry, namely the creation of theories of various non-Euclidean spaces; and these ideas subsequently became the basis of the general theory of relativity, in which the mathematical apparatus consists of a form of non-Euclidean geometry of four-dimensional space. Thus the abstract constructions of mathematics, which at the very least seemed incomprehensible, proved themselves a powerful instrument for the development of one of the most important theories of physics. Similarly, in the present-day theory of atomic phenomena, in the so-called quantum mechanics, essential use is made of many extremely abstract mathematical concepts and theories, as for example the concept of infinite-dimensional space.

There is no need to give any further examples, since we have already shown with sufficient emphasis that mathematics finds widespread application in everyday life and in technology and science; in the exact sciences and in the great problems of technology, applications are found even for those theories which arise within mathematics itself. This is one of the characteristic peculiarities of mathematics, along with its abstractness and the rigor and conclusiveness of its results.

2. The essential nature of mathematics. In discussing these special features of mathematics we have been far from explaining its essence; rather we have merely pointed out its external marks. Our task now is to explain the essential nature of these characteristic features. For this purpose it will be necessary to answer, at the very least, the following questions:

What do these abstract mathematical concepts reflect? In other words, what is the actual subject matter of mathematics?

Why do the abstract results of mathematics appear so convincing, and its initial concepts so obvious? In other words, on what foundation do the methods of mathematics rest?

Why, in spite of all its abstractness, does mathematics find such wide application and does not turn out to be merely idle play with abstractions? In other words, how is the significance of mathematics to be explained?

Finally, what forces lead to the further development of mathematics, allowing it to unite abstractness with breadth of application? What is the basis for its continuing growth?

In answering these questions we will form a general picture of the content of mathematics, of its methods, and of its significance and its development; that is, we will understand its essence.

Idealists and metaphysicists not only fall into confusion in their attempts to answer these basic questions but they go so far as to distort mathematics completely, turning it literally inside out. Thus, seeing the extreme abstractness and cogency of mathematical results, the idealist imagines that mathematics issues from pure thought.

In reality, mathematics offers not the slightest support for idealism or metaphysics. We will convince ourselves of this as we attempt, in general outline, to answer the listed questions about the essence of mathematics. For a preliminary clarification of these questions, it is sufficient to examine the foundations of arithmetic and elementary geometry, to which we now turn.

§2. Arithmetic

1. The concept of a whole number. The concept of number (for the time being, we speak only of whole positive numbers), though it is so familiar to us today, was worked out very slowly. This can be seen from the way in which counting has been done by various races who until recent times have remained at a relatively primitive level of social life. Among some of them, there were no names for numbers higher than two or three; among others, counting went further but ended after a few numbers, after which they simply said "many" or "countless." A stock of clearly distinguished names for numbers was only gradually accumulated among the various peoples.

At first these peoples had no concept of what a number is, although they could in their own fashion make judgments about the size of one or another collection of objects met with in their daily life. We must conclude that a number was directly perceived by them as an inseparable property of a collection of objects, a property which they did not, however, clearly distinguish. We are so accustomed to counting that we can hardly imagine this state of affairs, but it is possible to understand it.*

At the next higher level a number already appears as a property of a

* In fact, every collection of objects, whether it be a flock of sheep or a pile of firewood, exists and is immediately perceived in all its concreteness and complexity. The distinguishing in it of separate properties and relationships is the result of conscious analysis. Primitive thought does not yet make this analysis, but considers the object only as a whole. Similarly, a man who has not studied music perceives a musical composition without distinguishing in it the details of melody, tonality, and so forth, while at the same time a musician easily analyzes even a complicated symphony.

collection of objects, but it is not yet distinguished from the collection as an "abstract number," as a number in general, not connected with concrete objects. This is obvious from the names of numbers among certain peoples, as "hand" for five and "wholeman" for twenty. Here five is to be understood not abstractly but simply in the sense of "as many as the fingers on a hand," twenty is "as many as the fingers and toes on a man" and so forth. In a completely analogous way, certain peoples had no concept of "black," "hard," or "circular." In order to say that an object is black, they compared it with a crow for example, and to say that there were five objects, they directly compared these objects with a hand. In this way it also came about that various names for numbers were used for various kinds of objects; some numbers for counting people, others for counting boats, and so forth, up to as many as ten different kinds of numbers. Here we do not have abstract numbers, but merely a sort of "appellation," referring only to a definite kind of objects. Among other peoples there were in general no separate names for numbers, as for example, no word for "three," although they could say "three men" or "in three places," and so forth.

Similarly among ourselves, we quite readily say that this or that object is black but much more rarely speak about "blackness" in itself, which is a more abstract concept.*

The number of objects in a given collection is a property of the collection, but the number itself, as such, the "abstract number," is a property abstracted from the concrete collection and considered simply in itself, like "blackness" or "hardness." Just as blackness is the property common to all objects of the color of coal, so the number "five" is the common property of all collections containing as many objects as there are fingers on a hand. In this case the equality of the two numbers is established by simple comparison: We take an object from the collection, bend one finger over, and count in this way up to the end of the collection. More generally, by pairing off the objects of two collections, it is possible, without making any use of numbers at all, to establish whether or not the collections contain the same number of objects. For example, if guests are taking their places at the table they can easily, without any counting, make it clear to the hostess that she has forgotten one setting, since one guest will be without a setting.

* In the formation of concepts about properties of objects, such as color or the numerosity of a collection, it is possible to distinguish three steps, which we must not, of course, try to separate too sharply from one another. At the first step the property is defined by direct comparison of objects: like a crow, as many as on a hand. At the second, an adjective appears: a black stone or (the numerical adjective being quite analogous) five trees. At the third step the property is abstracted from the objects and may appear "as such"; for example "blackness," or the abstract number "five."

In this way it is possible to give the following definition of a number: Each separate number like "two," "five," and so forth, is that property of collections of objects which is common to all collections whose objects can be put into one-to-one correspondence with one another and which is different for those collections for which such a correspondence is impossible. In order to discover this property and to distinguish it clearly, that is, in order to form the concept of a definite number and to give it a name "six," "ten," and so forth, it was necessary to compare many collections of objects. For countless generations people repeated the same operation millions of times and in that way discovered numbers and the relations among them.

2. Relations among the whole numbers. Operations with numbers arose in their turn as a reflection of relations among concrete objects. This is observable even in the names of numbers. For example, among certain American Indians the number "twenty-six" is pronounced as "above two tens I place a six," which is clearly a reflection of a concrete method of counting objects. Addition of numbers corresponds to placing together or uniting two or more collections, and it is equally easy to see the concrete meaning of subtraction, multiplication, and division. Multiplication in particular arose to a great extent, it seems clear, from the habit of counting off equal collections: that is, by twos, by threes, and so forth.

In the process of counting, men not only discovered and assimilated the relations among the separate numbers, as for example that two and three are five, but also they gradually established certain general laws. By practical experience, it was discovered that a sum does not depend on the order of the summands and that the result of counting a given set of objects does not depend on the order in which the counting takes place, a fact which is reflected in the essential identity of the "ordinal" and "cardinal" numbers: first, second, third, and one, two, three. In this way the numbers appeared not as separate and independent, but as interrelated with one another.

Some numbers are expressed in terms of others in their very names and in the way they are written. Thus, "twenty" denotes "two (times) ten"; in French, eighty is "four-twenties" (quatre-vingt), ninety is "four-twenties-ten"; and the Roman numerals VIII, IX denote that $8 = 5 + 3$, $9 = 10 - 1$.

In general, there arose not just the separate numbers but a system of numbers with mutual relations and rules.

The subject matter of arithmetic is exactly this, the system of numbers with its mutual relations and rules.* The separate abstract number by itself does not have tangible properties, and in general there is very little to be said about it. If we ask ourselves, for example, about the properties of the number six, we note that $6 = 5 + 1, 6 = 3 \cdot 2$, that 6 is a factor of 30 and so forth. But here the number 6 is always connected with other numbers; in fact, the properties of a given number consist precisely of its relations with other numbers.† Consequently, it is clear that every arithmetical operation determines a connection or relation among numbers. Thus the subject matter of arithmetic is relations among numbers. But these relations are the abstract images of actual quantitative relations among collections of objects; so we may say that arithmetic is the science of actual quantitative relations considered abstractly, that is, purely as relations. Arithmetic, as we see, did not arise from pure thought, as the idealists represent, but is the reflection of definite properties of real things; it arose from the long practical experience of many generations.

3. Symbols for the numbers. As social life became more extensive and complicated, it posed broader problems. Not only was it necessary to take note of the number of objects in a set and to tell others about it, a necessity which had already led to formulation of the concept of number and to names for the numbers, but it became essential to learn to count increasingly larger collections, of animals in a herd, of objects for exchange, of days before a fixed date, and so forth, and to communicate the results of the count to others. This situation absolutely demanded improvement in the names and also in the symbols for numbers.

The introduction of symbols for the numbers, which apparently occured as soon as writing began, played a great role in the development of arithmetic. Moreover, it was the first step toward mathematical signs and formulas in general. The second step, consisting of the introduction of signs for arithmetical operations and of a literal designation for the unknown (x), was taken considerably later.

The concept of number, like every other abstract concept, has no immediate image; it cannot be exhibited but can only be conceived in the

* The word "arithmetic," meaning the "art of calculation," is derived from the Greek adjective "arithmetic" formed from the noun "arithmos," meaning "number." The adjective modifies a noun "techne" (art, technique), which is here understood.

† This is understandable from the most general considerations. An arbitrary abstraction, removed from its concrete basis (just as a number is abstracted from a concrete collection of objects), has no sense "in itself"; it exists only in its relations with other concepts. These relations are already implicit in any statement about the abstraction, in the most incomplete definition of it. Without them the abstraction lacks content and significance, i.e., it simply does not exist. The content of the concept of an abstract number lies in the rules, in the mutual relations of the system of numbers.

mind. But thought is formulated in language, so that without a name there can be no concept. The symbol is also a name, except that it is not oral but written and presents itself to the mind in the form of a visible image. For example, if I say "seven," what do you picture to yourself? Probably not a set of seven objects of one kind or another, but rather the symbol "7," which forms a sort of tangible framework for the abstract number "seven." Moreover, a number 18273 is considerably harder to pronounce than to write and cannot be pictured with any accuracy in the form of a set of objects. In this way it came about, though only after some lapse of time, that the symbols gave rise to the conception of numbers so large that they could never have been discovered by direct observation or by enumeration. With the appearance of government, it was necessary to collect taxes, to assemble and outfit an army, and so forth, all of which required operations with very large numbers.

Thus the importance of symbols for the numbers consists, in the first place, in their providing a simple embodiment of the concept of an abstract number.* This is the role of mathematical designations in general: They provide an embodiment of abstract mathematical concepts. Thus $+$ denotes addition, x denotes an unknown number, a an arbitrary given number, and so forth. In the second place the symbols for numbers provide a particularly simple means of carrying out operations on them. Everyone knows how much easier it is "to calculate on paper" than "in one's head." Mathematical signs and formulas have this advantage in general: They allow us to replace a part of our arguments with calculations, with something that is almost mechanical. Moreover, if a calculation is written down, it already possesses a definite authenticity; everything is visible, everything can be checked, and everything is defined by exact rules. As examples one might mention addition by individual columns or any algebraic transformation such as "taking over to the other side of the equation with change of sign." From what has been said, it is clear that without suitable symbols for the numbers arithmetic could not have made much progress. Even more is it true that contemporary mathematics would be impossible without its special signs and formulas.

It is obvious that the extremely convenient method of writing numbers that is in use today could not have been worked out all at once. From ancient times there appeared among various peoples, from the very

* It is worth remarking that the concept of number, which was worked out with such difficulty in a long period of time, is mastered nowadays by a child with relative ease. Why? The first reason is, of course, that the child hears and sees adults constantly making use of numbers, and they even teach him to do the same. But a second reason, and this is the one to which we wish to draw special attention, is that the child already has at hand words and symbols for the numbers. He first learns these external symbols for number and only later masters the meaning of them.

Table

SYMBOLS FOR THE NUMBERS

	Slavic		Chinese			
	Cyrillic	Glagolitic	Ancient	Commercial	Scientific	Greek
0				〇	○	
1	а҃	Ⰰ	一	〡	𝍠	α̅
2	в҃	Ⰱ	二	〢	𝍡	β̅
3	г҃	Ⰲ	三	〣	𝍢	γ̅
4	д҃	Ⰳ	四	〤	𝍣	δ̅
5	є҃	Ⰴ	五	〥	𝍤	ε̅
6	ѕ҃	Ⰵ	六	〦	𝍥	ϛ̅
7	з҃	Ⰶ	七	〧	𝍦	ζ̅
8	и҃	Ⰷ	八	〨	𝍧	η̅
9	ѳ҃	Ⰸ	九	〩	𝍨	θ̅
10	і҃	Ⰹ	十	〸	𝍠○	ι̅
20	к҃	Ⰻ	二十	〹	𝍡○	κ̅
30	л҃	Ⰼ	三十	〺	𝍢○	λ̅
100	р҃	Ⱃ	百	百	𝍠○○	ρ̅
1000	҂а	Ⱍ	千	千	𝍠○○○	͵α

1

Among Various Peoples

	Arabic	Georgian	Egyptian Hieroglyphic	Egyptian Hieratic	Roman	Mayan
0						𝋠
1	ا	ა	𓏺	[illegible]	I	𝋡
2	ب	ბ	𓏻	[illegible]	II	𝋢
3	ج	გ	𓏼	[illegible]	III	𝋣
4	د	დ	𓏽	[illegible]	IV	𝋤
5	ه	ე	𓏾	[illegible]	V	𝋥
6	و	ვ	𓏿	[illegible]	VI	𝋦
7	ز	ზ	𓐀	[illegible]	VII	𝋧
8	ح	ჱ	𓐁	[illegible]	VIII	𝋨
9	ط	თ	𓐂	[illegible]	IX	𝋩
10	ي	ი	𓎆	[illegible]	X	𝋪
20	ك	კ	𓎇	[illegible]	XX	
30	ل	ლ	𓎈	[illegible]	XXX	
100	ق	რ	𓍢	[illegible]	C	
1000	غ	ჭ	𓆼	[illegible]	M	

beginnings of their culture, various symbols for the numbers, which were very unlike our contemporary ones not only in their general appearance but also in the principles on which they were chosen. For example, the decimal system was not used everywhere, and among the ancient Babylonians there was a system that was partly decimal and partly sexagesimal. Table 1 gives some of the symbols for numbers among various peoples. In particular, we see that the ancient Greeks, and later also the Russians, made use of letters to designate numbers. Our contemporary "Arabic" symbols and, more generally, our method of forming the numbers, were brought from India to Europe by the Arabs in the 10th century and became firmly rooted there in the course of the next few centuries.

The first peculiarity of our system is that it is a decimal system. But this is not a matter of great importance, since it would have been quite possible to use, for example, a duodecimal system by introducing special symbols for ten and eleven. The most important peculiarity of our system of designating numbers is that it is "positional"; that is, that one and the same number has a different significance depending upon its position. For example, in 372 the number 3 denotes the number of hundreds and 7 the numbers of tens. This method of writing is not only concise and simple but makes calculations very easy. The Roman numerals were in this respect much less convenient, the same number 372 being written in the form CCCLXXII; it is a very laborious task to multiply together two large numbers written in Roman numerals.

Positional writing of numbers demands that in one way or another we take note of any category of numbers that has been omitted, since if we do not do this, we will confuse, for example, thirty-one with three-hundred-and-one. In the position of the omitted category we must place a zero, thereby distinguishing 301 and 31. In a rudimentary form, zero already appears in the late Babylonian cuneiform writings, but its systematic introduction was an achievement of the Indians:* It allowed them to proceed to a completely positional system of writing just as we have it today.

But in this way zero also became a number and entered into the system of numbers. By itself zero is nothing; in the Sanskrit language of ancient India, it is called exactly that: "empty" (çūnga); but in connection with other numbers, zero acquires content, and well-known properties; for example, an arbitrary number plus zero is the same number, or when an arbitrary number is multiplied by zero it becomes zero.

* The first Indian manuscript in which zero appears comes from the end of the 9th century; in it the number 270 is written exactly as we would write it today. But it is probable that zero was introduced in India still earlier, in the 6th century.

4. The theory of numbers as a branch of pure mathematics. Let us return to the arithmetic of the ancients. The oldest texts that have been preserved from Babylon and Egypt go back to the second millennium B.C. These and later texts contain various arithmetical problems with their solutions, among them certain ones that today belong to algebra, such as the solution of quadratic and even cubic equations or progressions; all this being presented, of course, in the form of concrete problems and numerical examples. Among the Babylonians we also find certain tables of squares, cubes, and reciprocals. It is to be supposed that they were already beginning to form mathematical interests which were not immediately connected with practical problems.

In any case arithmetic was well developed in ancient Babylon and Egypt. However, it was not yet a mathematical theory of numbers but rather a collection of solutions for various problems and of rules of calculation. It is exactly in this way that arithmetic is taught up to the present time in our elementary schools and is understood by everyone who is not especially interested in mathematics. This is perfectly legitimate, but arithmetic in this form is still not a mathematical theory. There are no general theorems about numbers.

The transition to theoretical arithmetic proceeded gradually.

As was pointed out, the existence of symbols allows us to operate with numbers so large that it is impossible to visualize them as collections of objects or to arrive at them by the process of counting in succession from the number one. Among primitive tribes special numbers were worked out up to 3, 10, 100 and so forth, but after these came the indefinite "many." In contrast to this situation the use of symbols for numbers enabled the Chinese, the Babylonians, and the Egyptians to proceed to tens of thousands and even to millions. It was at this stage that the possibility was noticed of indefinitely extending the series of numbers, although we do not know how soon this possibility was clearly perceived. Even Archimedes (287–212 B.C.) in his remarkable essay "The Sand Reckoner" took the trouble to describe a method for naming a number greater than the number of grains of sand sufficient to fill up the "sphere of the fixed stars." So the possibility of naming and writing such a number still required at his time a detailed explanation.

By the 3rd century B.C., the Greeks had clearly recognized two important ideas: first, that the sequence of numbers could be indefinitely extended and second, that it was not only possible to operate with arbitrarily given numbers but to discuss numbers in general, to formulate and prove general theorems about them. This idea represents the generalization of an immense amount of earlier experience with concrete numbers, from which arose the rules and methods for *general* reasoning about numbers. A transition took place to a higher level of abstraction: from separate given (though abstract) numbers to number in general, to any possible number.

From the simple process of counting objects one by one, we pass to the unbounded process of formation of numbers by adding one to the number already formed. The sequence of numbers is regarded as being indefinitely continuable, and with it there enters into mathematics the notion of infinity. Of course, we cannot in fact, by the process of adding one, proceed arbitrarily far along the sequence of numbers: Who could reach as far as a million-million, which is almost forty times the number of seconds in a thousand years? But that is not the point; the process of adding ones, the process of forming arbitrary large collections of objects is in principle unlimited, so that the possibility exists of continuing the sequence of numbers beyond all limits. The fact that in actual practice counting is limited is not relevant; an abstraction is made from it. It is with this indefinitely prolonged sequence that general theorems about numbers have to deal.

General theorems about any property of an arbitrary number already contain in implicit form infinitely many assertions about the properties of separate numbers and are therefore qualitatively richer than any particular assertions that could be verified for specific numbers. It is for this reason that general theorems must be proved by general arguments proceeding from the fundamental rule for the formation of the sequence of numbers. Here we perceive a profound peculiarity of mathematics: Mathematics takes as its subject not only given quantitative relationships but all possible quantitative relationships and therefore infinity.

In the famous "Elements" of Euclid, written in the 3rd century B.C., we already find general theorems about whole numbers, in particular, the theorem that there exist arbitrarily large prime numbers.*

Thus arithmetic is transformed into the theory of numbers. It is already removed from particular concrete problems to the region of abstract concepts and arguments. It has become a part of "pure" mathematics. More precisely, this was the moment of the birth of pure mathematics itself with the characteristic features discussed in our first section. We must, of course, take note of the fact that pure mathematics was born simultaneously from arithmetic and geometry and that there were already to be found in the general rules of arithmetic some of the rudiments of algebra, a subject which was separated from arithmetic at a later stage. But we will discuss this later.

It remains now to summarize our conclusions up to this point, since we

* We recall that a prime number is defined as a positive integer greater than unity which is divisible without remainder only by the number itself and by unity.

have now traced out, though in very hurried fashion, the process whereby theoretical arithmetic arose from the concept of number.

5. The essential nature of arithmetic. Since the birth of theoretical arithmetic is part of the birth of mathematics, we may reasonably expect that our conclusions about arithmetic will throw light on our earlier questions concerning mathematics in general. Let us recall these questions, particularly in their application to arithmetic.

1. How did the abstract concepts of arithmetic arise and what do they reflect in the actual world?

This question is answered by the earlier remarks about the birth of arithmetic. Its concepts correspond to the quantitative relations of collections of objects. These concepts arose by way of abstraction, as a result of the analysis and generalization of an immense amount of practical experience. They arose gradually; first came numbers connected with concrete objects, then abstract numbers, and finally the concept of number in general, of any possible number. Each of these concepts was made possible by a combination of practical experience and preceding abstract concepts. This, by the way, is one of the fundamental laws of formation of mathematical concepts: They are brought into being by a series of successive abstractions and generalizations, each resting on a combination of experience with preceding abstract concepts. The history of the concepts of arithmetic shows how mistaken is the idealistic view that they arose from "pure thought," from "innate intuition," from "contemplation of a priori forms," or the like.

2. Why are the conclusions of arithmetic so convincing and unalterable?

History answers this question too for us. We see that the conclusions of arithmetic have been worked out slowly and gradually; they reflect experience accumulated in the course of unimaginably many generations and have in this way fixed themselves firmly in the mind of man. They have also fixed themselves in language: in the names for the numbers, in their symbols, in the constant repetition of the same operations with numbers, in their constant application to daily life. It is in this way that they have gained clarity and certainty. The methods of logical reasoning also have the same source. What is essential here is not only the fact that they can be repeated at will but their soundness and perspicuity, which they possess in common with the relations among things in the actual world, relations which are reflected in the concepts of arithmetic and in the rules for logical deduction.

This is the reason why the results of arithmetic are so convincing; its conclusions flow logically from its basic concepts, and both of them, the methods of logic and the concepts of arithmetic, were worked out and firmly fixed in our consciousness by three thousand years of practical experience, on the basis of objective uniformities in the world around us.

3. Why does arithmetic have such wide application in spite of the abstractness of its concepts?

The answer is simple. The concepts and conclusions of arithmetic, which generalize an enormous amount of experience, reflect in abstract form those relationships in the actual world that are met with constantly and everywhere. It is possible to count the objects in a room, the stars, people, atoms, and so forth. Arithmetic considers certain of their general properties, in abstraction from everything particular and concrete, and it is precisely because it considers only these general properties that its conclusions are applicable to so many cases. The possibility of wide application is guaranteed by the very abstractness of arithmetic, although it is important here that this abstraction is not an empty one but is derived from long practical experience. The same is true for all mathematics, and for any abstract concept or theory. The possibilities for application of a theory depend on the breadth of the original material which it generalizes.

At the same time every abstract concept, in particular the concept of number, is limited in its significance as a result of its very abstractness. In the first place, when applied to any concrete object it reflects only one aspect of the object and therefore gives only an incomplete picture of it. How often it happens, for example, that the mere numerical facts say very little about the essence of the matter. In the second place, abstract concepts cannot be applied everywhere without certain limiting conditions; it is impossible to apply arithmetic to concrete problems without first convincing ourselves that their application makes some sense in the particular case. If we speak of addition, for example, and merely unite the objects in thought, then naturally no progress has been made with the objects themselves. But if we apply addition to the actual uniting of the objects, if we in fact put the objects together, for example by throwing them into a pile or setting them on a table, in this case there takes place not merely abstract addition but also an actual process. This process does not consist merely of the arithmetical addition, and in general it may even be impossible to carry it out. For example, the object thrown into a pile may break; wild animals, if placed together, may tear one another apart; the materials put together may enter into a chemical reaction: a liter of water and a liter of alcohol when poured together produced not 2, but 1.9 liters of the mixture as a result of partial solution of the liquids; and so forth.

If other examples are needed they are easy to produce.

To put it briefly, truth is concrete; and it is particularly important to remember this fact with respect to mathematics, exactly because of its abstractness.

4. Finally, the last question we raised had to do with the forces that led to the development of mathematics.

For arithmetic the answer to this question also is clear from its history. We saw how people in the actual world learned to count and to work out the concept of number, and how practical life, by posing more difficult problems, necessitated symbols for the numbers. In a word, the forces that led to the development of arithmetic were the practical needs of social life. These practical needs and the abstract thought arising from them exercise on each other a constant interaction. The abstract concepts provide in themselves a valuable tool for practical life and are constantly improved by their very application. Abstraction from all nonessentials uncovers the kernel of the matter and guarantees success in those cases where a decisive role is played by the properties and relations picked out and preserved by the abstraction; namely, in the case of arithmetic, by the quantitative relations.

Moreover, abstract reflection often goes farther than the immediate demands of a practical problem. Thus the concept of such large numbers as a million or a billion arose on the basis of practical calculations but arose earlier than the practical need to make use of them. There are many such examples in the history of science; it is enough to recall the imaginary numbers mentioned earlier. This is just a particular case of a phenomenon known to everyone, namely the interaction of experience and abstract thought, of practice and theory.

§3. Geometry

1. The concept of a geometric figure. The history of the origin of geometry is essentially similar to that of arithmetic. The earliest geometric concepts and information also go back to prehistoric times and also result from practical activity.

Early man took over geometric forms from nature. The circle and the crescent of the moon, the smooth surface of a lake, the straightness of a ray of light or of a well-proportioned tree existed long before man himself and presented themselves constantly to his observation. But in nature itself our eyes seldom meet with really straight lines, with precise triangles or squares, and it is clear that the chief reason why men gradually worked out a conception of these figures is that their observation of nature was an active one, in the sense that, to meet their practical needs, they manufactured objects more and more regular in shape. They built dwellings, cut stones, enclosed plots of land, stretched bowstrings in their bows, modeled their clay pottery, brought it to perfection and correspondingly formed the notion that a pot is *curved,* but a stretched bowstring is *straight.* In short, they first gave form to their material and only then recognized form as that which is impressed on material and can therefore be considered in itself, as an abstraction from material. By recognizing the form of bodies, man was able to improve his handiwork and thereby to work out still more precisely the abstract notion of form. Thus practical activity served as a basis for the abstract concepts of geometry. It was necessary to manufacture thousands of objects with straight edges, to stretch thousands of threads, to draw upon the ground a large number of straight lines, before men could form a clear notion of the straight line in general, as that quality which is common to all these particular cases. Nowadays we learn early in life to draw a straight line, since we are surrounded by objects with straight edges that are the result of manufacture, and it is only for this reason that in our childhood we already form a clear notion of the straight line. In exactly the same way the notion of geometric magnitudes, of length, area, and volume, arose from practical activity. People measured lengths, determined distances, estimated by eye the area of surfaces and the volumes of bodies, all for their practical purposes. It was in this way that the simplest general laws were discovered, the first geometric relations: for example, that the area of a rectangle is equal to the product of the lengths of its sides. It is useful for a farmer to be aware of such a relation, in order that he may estimate the area he has sowed and consequently the harvest he may expect.

So we see that geometry took its rise from practical activity and from the problems of daily life. On this question the ancient Greek scholar, Eudemus of Rhodes, wrote as follows: "Geometry was discovered by the Egyptians as a result of their measurement of land. This measurement was necessary for them because of the inundations of the Nile, which constantly washed away their boundaries.* There is nothing remarkable in the fact that this science, like the others, arose from the practical needs of men. All knowledge that arises from imperfect circumstances tends to perfect itself. It arises from sense impressions but gradually becomes an object of our contemplation and finally enters the realm of the intellect."

Of course, the measurement of land was not the only problem that led the ancients toward geometry. From the fragmentary texts that have survived, it is possible to form some idea of various problems of the ancient Egyptians and Babylonians and of their methods for solving them. One of the oldest Egyptian texts goes back to 1700 B.C. This is a manual

* What is meant here is the boundaries between shares of land. Let us note, parenthetically, that *geometry* means land-measurement (in ancient Greek "ge" is land, and "metron" is measure).

of instruction for "secretaries" (royal officers), written by a certain Ahmes. It contains a collection of problems on calculating the capacity of containers and warehouses, the area of shares of land, the dimensions of earthworks, and so forth..

The Egyptians and Babylonians were able to determine the simplest areas and volumes, they knew with considerable exactness the ratio of the circumference to the diameter of a circle, and perhaps they were even able to calculate the surface area of a sphere; in a word, they already possessed a considerable store of geometrical knowledge. But so far as we can tell, they were still not in possession of geometry as a theoretical science with theorems and proofs. Like the arithmetic of the time, geometry was basically a collection of rules deduced from experience. Moreover, geometry was in general not distinguished from arithmetic. Geometric problems were at the same time problems for calculation in arithmetic.

In the 7th century B.C., geometry passed from Egypt to Greece, where it was further developed by the great materialist philosophers, Thales, Democritus, and others. A considerable contribution to geometry was also made by the successors of Pythagoras, the founders of an idealistic religiophilosophical school.

The development of geometry took the direction of compiling new facts and clarifying their relations with one another. These relations were gradually transformed into logical deductions of certain propositions of geometry from certain others. This had two results: first, the concept of a geometrical theorem and its proof; and second, the clarification of those fundamental propositions from which the others may be deduced, namely, the axioms.

In this way geometry gradually developed into a mathematical theory.

It is well known that systematic expositions of geometry appeared in Greece as far back as the 5th century B.C., but they have not been preserved, for the obvious reason that they were all supplanted by the "Elements" of Euclid (3rd century B.C.). In this work, geometry was presented as such a well-formed system that nothing essential was added to its foundations until the time of N. I. Lobačevskiĭ, more than two thousand years later. The well-known school text of Kiselev, like school books over the whole world, represented in its older editions, nothing but a popular reworking of Euclid. Very few other books in the world have had such a long life as the "Elements" of Euclid, this perfect creation of Greek genius. Of course, mathematics continued to advance, and our understanding of the foundations of geometry has been considerably deepened; nevertheless the "Elements" of Euclid became, and to a great extent remains, the model of a book on pure mathematics. Bringing together the accomplishments of his predecessors, Euclid presented the mathematics of his time as an independent theoretical science; that is, he presented it essentially as it is understood today.

2. The essential nature of geometry. The history of geometry leads to the same conclusions as that of arithmetic. We see that geometry arose from practical life and that its transformation to a mathematical theory required an immense period of time.

Geometry operates with "geometric bodies" and figures; it studies their mutual relations from the point of view of magnitude and position. But a geometric body is nothing other than an actual body considered solely from the point of view of its spatial form,* in abstraction from all its other properties such as density, color, or weight. A geometric figure is a still more general concept, since in this case it is possible to abstract from spatial extension also; thus a surface has only two dimensions, a line, only one dimension, and a point, none at all. A point is the abstract concept of the end of a line, of a position defined to the limit of precision so that it no longer has any parts. It is in this way that all these concepts are defined by Euclid.

Thus geometry has as its object the spatial forms and relations of actual bodies, removed from their other properties and considered from the purely abstract point of view. It is just this high level of abstraction that distinguishes geometry from the other sciences that also investigate the spatial forms and relations of bodies. In astronomy for example, the mutual positions of bodies are studied, but they are the actual bodies of the sky; in geodesy it is the form of the earth that is studied, in crystallography, the form of crystals, and so forth. In all these other sciences, the form and the position of concrete bodies are studied in their dependence on other properties of the bodies.

This abstraction necessarily leads to the purely theoretical method of geometry; it is no longer possible to set up experiments with breadthless straight lines, with "pure forms." The only possibility is to make use of logical argument, deriving some conclusions from others. A geometrical theorem must be proved by reasoning, otherwise it does not belong to geometry; it does not deal with "pure forms."

The self-evidence of the basic concepts of geometry, the methods of reasoning and the certainty of their conclusions, all have the same source as in arithmetic. The properties of geometric concepts, like the concepts themselves, have been abstracted from the world around us. It was necessary for people to draw innumerable straight lines before they could take it as an axiom that through every two points it is possible to draw a

* By form we mean also dimensions.

straight line; they had to move various bodies about and apply them to one another on countless occasions before they could generalize their experience to the notion of superposition of geometric figures and make use of this notion for the proof of theorems, as is done in the well-known theorems about congruence of triangles.

Finally, we must emphasize the generality of geometry. The volume of a sphere is equal to $4/3\pi R^3$ quite independently of whether we are speaking of a spherical vessel, of a steel sphere, of a star, or of a drop of water. Geometry can abstract what is common to all bodies, because every actual body does have more or less definite form, dimensions, and position with respect to other bodies. So it is no cause for wonder that geometry finds application almost as widely as arithmetic. Workmen measuring the dimensions of a building or reading a blueprint, an artillery man determining the distance to his target, a farmer measuring the area of his field, an engineer estimating the volume of earthworks, all these people make use of the elements of geometry. The pilot, the astronomer, the surveyor, the engineer, the physicist, all have need of the precise conclusions of geometry.

A clear example of the abstract-geometrical solution of an important problem in physics is provided by the investigations of the well-known crystallographer and geometer, E. S. Fedorov. The problem he set himself of finding all the possible forms of symmetry for crystals is one of the most fundamental in theoretical crystallography. To solve this problem, Fedorov made an abstraction from all the physical properties of a crystal, considering it only as a regular system of geometric bodies "in place of a system of concrete atoms." Thus the problem became one of finding all the forms of symmetry which could possibly exist in a system of geometric bodies. This purely geometrical problem was completely solved by Fedorov, who found all the possible forms of symmetry, 230 in number. His solution proved to be an important contribution to geometry and was the source of many geometric investigations.

In this example, as in the whole history of geometry, we detect the prime moving force in the development of geometry. It is the mutual influence of practical life and abstract thought. The problem of discovering possible symmetries originated in physical observation of crystals but was transformed into an abstract problem and so gave rise to a new mathematical theory, the theory of regular systems, or of the so-called Fedorov groups.* Subsequently this theory not only found brilliant confirmation in the practical observation of crystals but also served as a general guide in the development of crystallography, giving rise to new investigations, both in experimental physics and in pure mathematics.

* Compare Chapter XX.

§4. Arithmetic and Geometry

1. The origin of fractions in the interrelation of arithmetic and geometry. Up to now we have considered arithmetic and geometry apart from each other. Their mutual relation, and consequently the more general interrelation of all mathematical theories, has so far escaped our attention. Nevertheless this relation has exceptionally great significance. The interaction of mathematical theories leads to advances in mathematics itself and also uncovers a rich treasure of mutual relations in the actual world reflected by the these theories.

Arithmetic and geometry are not only applied to each other but they also serve thereby as sources for further general ideas, methods, and theories. In the final analysis, arithmetic and geometry are the two roots from which has grown the whole of mathematics. Their mutual influence goes back to the time when both of them had just come into being. Even the simple measurement of a line represents a union of geometry and arithmetic. To measure the length of an object we *apply* to it a certain unit of length and *calculate* how many times it is possible to do this; the first operation (application) is geometric, the second (calculation) is arithmetical. Everyone who counts off his steps along a road is already uniting these two operations.

In general, the measurement of any magnitude combines calculation with some specific operation which is characteristic of this sort of magnitude. It is sufficient to mention measurement of a liquid in a graduated container or measurement of an interval of time by counting the number of strokes of a pendulum.

But in the process of measurement it turns out, generally speaking, that the chosen unit is not contained in the measured magnitude an integral number of times, so that a simple calculation of the number of units is not sufficient. It becomes necessary to divide up the unit of measurement in order to express the magnitude more accurately by parts of the unit; that is, no longer by whole numbers but by fractions. It was in this way that fractions actually arose, as is shown by an analysis of historical and other data. They arose from the division and comparison of continuous magnitudes; in other words, from measurement. The first magnitudes to be measured were geometric, namely lengths, areas of fields, and volumes of liquids or friable materials, so that in the earliest appearance of fractions we see the mutual action of arithmetic and geometry. This interaction leads to the appearance of an important new concept, namely of fractions, as an extension of the concept of number from whole numbers to fractional numbers (or as the mathematicians say, to rational numbers, expressing the ratio of whole numbers). Fractions did not arise, and could not arise, from the division of whole numbers, since only whole objects are counted by whole numbers. Three men, three arrows, and so forth, all these make sense, but two-thirds of a man and even two-thirds of an arrow are senseless concepts; even three separate thirds of an arrow will not kill a deer, for this it is necessary to have a *whole* arrow.

2. Incommensurable magnitudes. In the development of the concept of number, arising from the mutual action of arithmetic and geometry, the appearance of fractions was only the first step. The next was the discovery of incommensurable intervals. Let us recall that intervals are called incommensurable if no interval exists which can be applied to each of them a whole number of times or, in other words, if their ratio cannot be expressed by an ordinary fraction; that is, by a ratio of whole numbers.

At first people simply did not think about the question whether every interval can be expressed by a fraction. If in dividing up or measuring an interval they came upon very small parts, they merely discarded them; in practice, it made no sense to speak of infinite precision of measurement. Democritus even advanced the notion that geometrical figures consist of atoms of a particular kind. This notion, which to our view seems quite strange, proved very fruitful in the determination of areas and volumes. An area was calculated as the sum of rows consisting of atoms, and a volume as the sum of atomic layers. It was in this way, for example, that Democritus found the volume of a cone. A reader who understands the integral calculus will note that this method already forms the prototype of the determination of areas and volumes by the methods of the integral calculus. Moreover, in returning in thought to the times of Democritus, one must attempt to free oneself of the customary notions of today, which have become firmly fixed in our minds by the development of mathematics. At the time of Democritus, geometrical figures were not yet separated from actual ones to the same extent as is now the case. Since Democritus considered actual bodies as consisting of atoms, he naturally also regarded geometrical figures in the same light.

But the notion that intervals consist of atoms comes into contradiction with the theorem of Pythagoras, since it follows from this theorem that incommensurable intervals exist. For example, the diagonal of a square is incommensurable with its side; in other words, the ratio of the two cannot be expressed as the ratio of whole numbers.

We shall prove that the side and the diagonal of a square are in fact incommensurable. If a is the side and b is the diagonal of a square, then according to the theorem of Pythagoras $b^2 = a^2 + a^2 = 2a^2$ and therefore

$$\left(\frac{b}{a}\right)^2 = 2.$$

But there is no fraction such that its square is equal to 2. In fact, if we suppose that there is, let p and q be whole numbers for which

$$\left(\frac{p}{q}\right)^2 = 2,$$

where we may assume that p and q have no common factor, since otherwise we could simplify the fraction. But if $(p/q)^2 = 2$, then $p^2 = 2q^2$, and therefore p^2 is divisible by 2. In this case p^2 is also divisible by 4, since it is the square of an even number. So $p^2 = 4q_1$; that is, $2q^2 = 4q_1$, and $q^2 = 2q_1$. From this it follows that q must also be divisible by 2. But this contradicts the supposition that p and q have no common factor. This contradiction proves that the ratio b/a cannot be expressed by a rational number. The diagonal and the side of a square are incommensurable.

This discovery made a great impression on the Greek scientists. Nowadays, when we are accustomed to irrational numbers and calculate freely with square roots, the existence of incommensurable intervals does not disturb us. But in the 5th century B.C., the discovery of such intervals had a completely different aspect for the Greeks. Since they did not have the concept of an irrational number and never wrote a symbol like $\sqrt{2}$, the previous result indicated that the ratio of the diagonal and the side of the square was not represented by any number at all.

In the existence of incommensurable intervals the Greeks discovered a profound paradox inherent in the concept of continuity, one of the expressions of the dialectical contradiction comprised in continuity and motion. Many important Greek philosophers considered this contradiction; particularly well-known among them, because of his paradoxes, is Zeno the Eleatic.

The Greeks founded a theory of ratios of intervals, or of magnitudes in general, which takes into consideration the existence of incommensurable intervals;* it is expounded in the "Elements" of Euclid, and in simplified form is explained today in high school courses in geometry. But to recognize that the ratio of one interval to another (if the second interval is taken as the unit of length, this ratio is simply the length of the first interval) may also be considered as a number, whereby the very concept of number is generalized, to this idea the Greeks were not able to rise: The concept of an irrational number simply did not originate among them.† This step was taken at a later period by the mathematicians

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* This theory is ascribed to the Greek scientist Eudoxus, who lived in the 4th century B.C.

† As a result of the fact that the theory of the measurement of magnitudes did not become part of arithmetic but passed over into geometry, mathematics among the

of the East; and in general, a mathematically rigorous definition of a real number, not depending immediately on geometry, was given only recently: in the seventies of the last century.* The passage of such an immense period of time after the founding of the theory of ratios shows how difficult it is to discover abstract concepts and give them exact formulation.

3. The real number. In describing the concept of a real number, Newton in his "General Arithmetic" wrote: "by number we mean not so much a collection of units as an abstract ratio of a certain quantity to another quantity taken as the unit." This number (ratio) may be integral, rational, or if the given magnitude is incommensurable with the unit, irrational.

A real number in its original sense is therefore nothing but the ratio of one magnitude to another taken as a unit; in particular cases this is a ratio of intervals, but it may also be a ratio of areas, weights, and so forth.

Consequently, a real number is a ratio of magnitudes in general, considered in abstraction from their concrete nature.

Just as *abstract* whole numbers are of mathematical interest only in their relations with one another, so *abstract* real numbers have content and become an object of mathematical attention only in relation with one another in the system of real numbers.

In the theory of real numbers, just as in arithmetic, it is first necessary to define operations on numbers: addition, subtraction, multiplication, division, and also the relations expressed by such words as "greater than" or "less than." These operations and relations reflect actual connections among the various magnitudes; for example, addition reflects the placing together of intervals. A beginning on operations with abstract real numbers was made in the Middle Ages by the mathematicians of the East. Later came the gradual discovery of the most important property of the system

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Greeks was engulfed by geometry. Such questions, for example, as the solution of quadratic equations, which today we treat in an algebraic way, they stated and solved geometrically. The "Elements" of Euclid contain a considerable number of such questions, which obviously represented for contemporary mathematicians a summary of the foundations not only of geometry in our sense but of mathematics in general. This domination by geometry continued up to the time when Descartes, on the contrary, subjected geometry to algebra. Traces of the long domination by geometry are preserved, for example, in such names as "square" and "cube" for the second and third powers: "a cubed" is a cube with side a.

* We are speaking here not of a descriptive definition, but of a definition which serves as the immediate basis for proofs of theorems about the properties of real numbers. It is natural that such definitions should arise at a later period, when the development of mathematics, and in particular of the infinitesimal analysis, required a suitable definition of the real number represented by "the variable x." This definition was given in various forms in the seventies of the last century by the German mathematicians Weierstrass, Dedekind, and Cantor.

of real numbers, its continuity. The system of real numbers is the abstract image of all the possible values of a continuously varying magnitude.

In this way, as in the similar case of whole numbers, the arithmetic of real numbers deals with the actual quantitative relations of continuous magnitudes, which it studies in their general form, in complete abstraction from all concrete properties. It is precisely because real numbers deal with what is common to all continuous magnitudes that they have such wide application: The values of various magnitudes, a length, a weight, the strength of an electric current, energy and so forth, are expressed by numbers, and the interdependence or relations among these entities are mirrored as relations among their numerical values.

To show how the general concept of real numbers can serve as the basis of a mathematical theory, we must give their mathematical definition in a formal way. This may be done by various methods, but perhaps the most natural is to proceed from the very process of measurement of magnitudes which actually did lead in practical life to this generalization of the concept of number. We will speak about the length of intervals, but the reader will readily perceive that we could argue in exactly the same way about any other magnitudes which permit indefinite subdivision.

Let us suppose that we wish to measure the interval AB by means of the interval CD taken as a unit (figure 1).

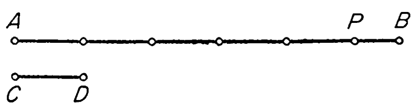

FIG. 1.

We apply the interval CD to AB, beginning for example with the point A, as long as CD goes into AB. Suppose this is n_0 times. If there still remains from the interval AB a remainder PB, then we divide the interval CD into ten parts and measure the remainder with these tenths. Suppose that n_1 of the tenths go into the remainder. If after this there is still a remainder, we divide our measure into ten parts again; that is, we divide CD into a hundred parts, and repeat the same operation, and so forth. Either the process of measurement comes to an end, or it continues. In either case we reach the result that in the interval AB the whole interval CD is contained n_0 times, the tenths are contained n_1 times, the hundredths n_2 times and so forth. In a word, we derive the ratio of AB to CD with increasing accuracy: up to tenths, to hundredths, and so forth. So the ratio itself is represented by a decimal fraction with n_0 units, n_1 tenths and so forth

$$\frac{AB}{CD} = n_0 \cdot n_1 n_2 n_3 \cdots$$

This decimal fraction may be infinite, corresponding to the possibility of indefinite increase in the precision of measurement.

Thus the ratio of two intervals, or of two magnitudes in general, is always representable by a decimal fraction, finite or infinite. But in the decimal fraction there is no longer any trace of the concrete magnitude itself; it represents exactly the abstract ratio, the real number. Thus a real number may be formally defined if we wish, as a finite or infinite decimal fraction.*

Our definition will be complete if we say what we mean by the operations of addition and so forth for decimal fractions. This is done in such a way that the operations defined on decimal fractions correspond to the operations on the magnitudes themselves. Thus, when intervals are put together their lengths are added; that is, the length of the interval $AB + BC$ is equal to the sum of the length AB and BC. In defining the operations on real numbers, there is a difficulty that these numbers are represented in general by *infinite* decimal fractions, while the well-known rules for these operations refer to finite decimal fractions. A rigorous definition of the operations for infinite decimals may be made in the following way. Suppose, for example, that we must add the two numbers a and b. We take the corresponding decimal fractions up to a given decimal place, say the millionth, and add them. We thus obtain the sum $a + b$ with corresponding accuracy, up to two millionths, since the errors in a and b may be added together. So we are able to define the sum of two numbers *with an arbitrary degree of accuracy*, and in that sense their sum is completely defined, although at each stage of the calculation it is known only with a a certain accuracy. But this corresponds to the essential nature of the case, since each of the magnitudes a and b is also measured only with a certain accuracy, and the exact value of each of the corresponding infinite fractions is obtained as the result of an indefinitely extended increase in accuracy. The relations "greater than" and "less than" may then be defined by means of addition: a is greater than b if there exists a magnitude c such that $a = b + c$, where we are speaking, of course, of positive numbers.

The continuity of the sequence of real numbers finds expression in the fact that if the numbers $a_1, a_2, \cdots$ increase and $b_1, b_2, \cdots$ diminish but

* Fractions with the periodic digit nine are not considered here, they are identical with the corresponding fraction without nines according to the well-known rule, which is clear from the example: $0.139999 \cdots = 0.140000 \cdots$.

always remain greater than the a_i, then between the one series of numbers and the other there is always a number c. This may be visualized on a straight line if its points are put into correspondence with the numbers (figure 2) according to the well-known rule.

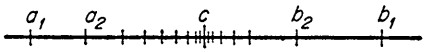

FIG. 2.

Here it is clearly seen that the presence of the number c and of the point corresponding to it signify the absence of a break in the series of numbers, which is what is meant by their continuity.

4. The conflict of opposites: concrete and abstract. Already in the example of the interaction of arithmetic and geometry we can see that the development of mathematics is a process of conflict among the many contrasting elements: the concrete and the abstract, the particular and the general, the formal and the material, the finite and the infinite, the discrete and the continuous, and so forth. Let us try, for example, to trace the contrast between concrete and abstract in the formation of the concept of a real number. As we have seen, the real number reflects an infinitely improvable process of measurement or, in slightly different terms, an absolutely accurate determination of a magnitude. This corresponds to the fact that in geometry we consider ideally precise forms and dimensions of bodies, abstracting altogether from the mobility of concrete objects and from a certain indefiniteness in their actual forms and dimensions; for example, the interval measured (figure 1) was a completely ideal one.

But ideally precise geometric forms and absolutely precise values for magnitudes represent abstractions. No concrete object has absolutely precise form nor can any concrete magnitude be measured with absolute accuracy, since it does not even *have* an absolutely accurate value. The length of a line segment, for example, has no sense if one tries to make it precise beyond the limits of atomic dimensions. In every case when one passes beyond well-known limits of quantitative accuracy, there appears a qualitative change in the magnitude, and in general it loses its original meaning. For example, the pressure of a gas cannot be made precise beyond the limits of the impact of a single molecule; electric charge ceases to be continuous when one tries to make it precise beyond the charge on an electron and so forth. In view of the absence in nature of objects of ideally precise form, the assertion that the ratio of the diagonal of a square to the side is equal to the $\sqrt{2}$ not only cannot be deduced with absolute accuracy from immediate measurement but does not even have any absolutely accurate meaning for an actual concrete square.

The conclusion that the diagonal and the side of a square are incommensurable comes, as we have seen, from the theorem of Pythagoras. This is a *theoretical* conclusion based on a development of the data of experience; it is a result of the application of logic to the original premises of geometry, which are taken from experience.

In this way the concept of incommensurable intervals, and all the more of real numbers, is not a simple immediate reflection of the facts of experience but goes beyond them. This is quite understandable. The real number does not reflect any given concrete magnitude but rather magnitude in general, in abstraction from all concreteness; in other words, it reflects what is *common* to particular concrete magnitudes. What is common to all of them consists in particular in this, that the value of the magnitude can be determined more and more precisely; and if we abstract from concrete magnitudes, then the limit of this possible increase in precision, which depends on the concrete nature of the magnitude, becomes indefinite and disappears.

In this way a *mathematical* theory of magnitudes, since it considers magnitudes in abstraction from their individual nature, must inevitably consider the possibility of unlimited accuracy for the value of the magnitude and *must* thereby lead to the concept of a real number. At the same time, since it reflects only what is common to various magnitudes, mathematics takes no account of the peculiarity of each individual magnitude.

Since mathematics selects only general properties for consideration, it operates with its clearly defined abstractions quite independently of the actual limits of their applicability, as must happen precisely because these limits are different in different particular cases. These limits depend on the concrete properties of the phenomena under consideration and on the qualitative changes that take place in them. So in making an application of mathematics, it is necessary to verify the actual applicability of the theory in question. To consider matter as continuous and to describe its properties by continuous magnitudes is permissible only if we may abstract from its atomic structure, and this is possible only under well-known conditions.

Nevertheless, the real numbers represent a trustworthy and powerful instrument for the mathematical investigation of actual continuous magnitudes and processes. Their theory is based on practice, on an immense field of applications in physics, technology, and chemistry. Consequently, practice shows that the concept of the real number correctly reflects the general properties of magnitudes. But this correctness is not without limits; it is not possible to consider the theory of real numbers as something absolute, allowing an unlimited abstract development in complete separation from reality. The very concept of the real number is continuing to develop and is in fact still far from being complete.

5. The conflict of opposites: discrete and continuous. The role of another of the mentioned contrasts, the contrast between the discrete and the continuous, may also be illustrated by the development of the concept of number. We have already seen that fractions arose from the division of continuous magnitudes.

On this theme of division there is a humorous question which is extraordinarily instructive. Grandmother has bought three potatoes and must divide them equally between two grandsons. How is she to do it? The answer is: make mashed potatoes.

The joke reveals the very essence of the matter. Separate objects are indivisible in the sense that, when divided, the object almost always ceases to be what it was before, as is clear from the example of "thirds of a man" or "thirds of an arrow." On the other hand, continuous and homogenous magnitudes or objects may easily be divided and put together again without losing their essential character. Mashed potatoes offer an excellent example of a homogeneous object, which in itself is not separated into parts but may nevertheless be divided in practice into as small parts as desired. Lengths, areas, and volumes have the same property. Although they are continuous in their very essence and are not actually divided into parts, nevertheless they offer the possibility of being divided without limit.

Here we encounter two contrasting kinds of objects: on the one hand, the indivisible, separate, discrete objects; and on the other, the objects which are completely divisible and yet are not divided into parts but are continuous. Of course, these contrasting characteristics are always united, since there are no absolutely indivisible and no completely continuous objects. Yet these aspects of the objects have an actual existence, and it often happens that one aspect is decisive in one case and the other in another.

In abstracting forms from their content, mathematics by this very act sharply divides these forms into two classes, the discrete and the continuous.

The mathematical model of a separate object is the unit, and the mathematical model of a collection of discrete objects is a sum of units, which is, so to speak, the image of pure discreteness, purified of all other qualities. On the other hand, the fundamental, original mathematical model of continuity is the geometric figure; in the simplest case, the straight line.

We have before us therefore two contrasts, discreteness and continuity, and their abstract mathematical images: the whole number and the geometric extension. Measurement consists of the union of these contrasts:

The continuous is measured by separate units. But the inseparable units are not enough; we must introduce fractional parts of the original unit. In this way the fractional numbers arise and the concept of numbers develops precisely as a result of the union of the mentioned contrasts.

Then, on a more abstract level, appeared the concept of incommensurable intervals, and, as a result, the real number as an abstract image of unlimited increase in accuracy in the determination of a magnitude. This concept was not formed immediately, and the long path of its development led through many a conflict between these same two contrasting elements, the discrete and the continuous.

In the first place, Democritus represented figures as consisting of atoms and in this way reduced the continuous to the discrete. But the discovery of incommensurable intervals led to the abandonment of such a representation. After this discovery continuous magnitudes were no longer thought of as consisting of separate elements, atoms or points, and they were not represented by numbers, since numbers other than the whole numbers and the fractions were not known at that time.

The contrast between the continuous and the discrete appeared in mathematics again with renewed force in the 17th century, when the foundations of the differential and integral calculus were being laid. Here it was the infinitesimal that was under discussion. In some accounts the infinitesimal was thought of as a real, "actually" infinitesimal, "indivisible" particle of the continuous magnitude, like the atoms of Democritus, except that now the number of these particles was considered to be infinitely great. Calculation of areas and volumes, or in other words integration, was thought of as summation of an infinite number of these infinitely small particles. An area, for example, was understood as "the sum of the lines from which it is formed" (figure 3). Consequently, the continuous was again reduced to the discrete, but now in a more complicated way, on a higher level. But this point of view also proved unsatisfactory, and, as a counterweight to it, there appeared, on the basis of Newton's work, the notion of *continuous* variables, of the infinitesimal as a *continuous* variable decreasing without limit. This conception finally carried the day at the beginning of the 19th century, when the rigorous theory of limits was founded. An interval was now thought of as consisting not of points or "indivisibles," but as an extension, as a continuous medium, where it was only possible to fix separate points, separate values of a continuous magnitude. Mathe-

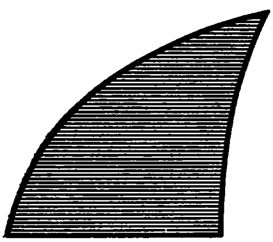

FIG. 3.

maticians then spoke of "extension." In the union of the discrete and the continuous, it was again the continuous that dominated.

But the development of analysis demanded further precision in the theory of variable magnitudes and above all in the general definition of a real number as an arbitrary possible value of a variable magnitude. In the seventies of the last century there arose a theory of real numbers which represents an interval as a set of points, and correspondingly the range of variation of a variable as a set of real numbers. The continuous again consisted of separate discrete points and the properties of continuity were again expressed in the structure of the set of points that formed it. This conception led to immense progress in mathematics and became dominant. But again profound difficulties were discovered in it, and these led to attempts to return on a new level to the notion of pure continuity. Other attempts were made to change the concept of an interval as a set of points. New points of view appeared for the concepts of number, variable, and function. The development of the theory is continuing, and we must await its further progress.

6. Further results of the interaction of arithmetic and geometry. The interaction of geometry and arithmetic played a role elsewhere than in the formation of the concept of a real number. The same interaction of geometry with arithmetic, or more accurately with algebra, also showed itself in the formation of negative and complex numbers, that is of numbers of the form $a + b\sqrt{-1}$. Negative numbers are represented by points of the straight line to the left of the point representing zero. It was exactly this geometric representation which gave imaginary numbers a firm place in mathematics; up to that time they had not been understood. New concepts of magnitude appeared: for example, vectors, which are represented by directed line segments; and tensors, which are still more general magnitudes; in these again algebra is united with geometry.

The union of various mathematical theories has always played a great and sometimes decisive role in the development of mathematics. We shall see this further on in the rise of analytic geometry, differential and integral calculus, the theory of functions of a complex variable, the recent so-called functional analysis, and other theories. Even in the theory of numbers itself, that is in the study of whole numbers, methods are applied with great success which depend on continuity (namely on the infinitesimal analysis) and on geometry. These methods have given rise to extensive chapters in the theory of numbers, the "analytic theory of numbers," and the "geometry of numbers."

From a certain well-known point of view, it is possible to regard the foundations of mathematics as the union of concepts arising from geometry and arithmetic; that is to say, of the general concepts of continuity and of algebraic operations (as generalizations of arithmetic operations). But we will not be able to speak here of these difficult theories. The aim of the present chapter has been to give an impression of the general interaction of concepts, of the union and the conflict between contrasting ideas in mathematics, as illustrated by the interaction of arithmetic and geometry in the development of the concept of number.

§5. The Age of Elementary Mathematics

1. The four periods of mathematics. The development of mathematics cannot be reduced to the simple accumulation of new theorems but includes essential qualitative changes. These qualitative changes take place, however, not in a process of destruction or abolition of already existing theories but in their being deepened and generalized, so as to form more general theories, for which the way has been prepared by preceding developments.

From the most general point of view, we may distinguish in the history of mathematics four fundamental, qualitatively distinct periods. Of course, it is not possible to draw exact boundary lines between these stages, since the essential traits of each period appeared more or less gradually, but the distinctions among the stages and the passages from one to another are completely clear.

The first stage (or period) is the period of the rise of mathematics as an independent and purely theoretical science. It begins in the most ancient times and extends to the 5th century B.C., or perhaps earlier, when the Greeks laid the foundations of "pure" mathematics with its logical connection between theorems and proofs (in that century there appeared, in particular, systematic expositions of geometry like the "Elements" of Hippocrates of Chios). This first stage was the period of the formation of arithmetic and geometry, in the form considered earlier. At this time mathematics consisted of a collection of separate rules deduced from experience and immediately connected with practical life. These rules did not yet form a logically unified system, since the theoretical character of mathematics with its logical proof of theorems was formed very slowly, as material for it was accumulated. Arithmetic and geometry were not separated but were closely interwoven with each other.

The second period may be characterized as the period of elementary mathematics, of the mathematics of constant magnitudes; its simple fundamental results now form the content of a high school course. This period extended for almost 2000 years and ended in the 17th century with the rise of "higher" mathematics. It is with this period that we will be concerned in greater detail in the present section. The following sections will be devoted to the third and fourth periods, namely to the founding and development of analysis and to the period of contemporary mathematics.

2. Mathematics in Greece. The period of elementary mathematics may in its turn be divided into two parts, distinguished by their basic content: the period of the development of geometry (up to the 2nd century A.D.) and the period of the predominance of algebra (from the 2nd to the 17th century). With respect to historical conditions it is divided into three parts, which may be called "Greek," "Eastern," and "European Renaissance." The Greek period coincides in time with the general flowering of Greek culture, beginning with the 7th century B.C., reaching its culmination in the 3rd century B.C. at the time of the great geometers of antiquity, Euclid, Archimedes, and Apollonius, and ending in the 6th century A.D. Mathematics, and especially geometry, enjoyed a wonderful development in Greece. We know the names and the results of numerous Greek mathematicians, although only a few genuine works have come down to us. It is to be remarked that Rome gave nothing to mathematics though it reached its zenith in the 1st century A.D. at a time when the science of Greece, which had been conquered by Rome, was still flourishing.

The Greeks not only developed and systematized elementary geometry to the extent to which it is given in the "Elements" of Euclid and is now taught in our secondary schools, but achieved considerably higher results. They studied the conic sections: ellipse, hyperbola, parabola; they proved certain theorems relating to the elements of what is called projective geometry; guided by the needs of astronomy, they worked out spherical geometry (in the 1st century A.D.) and also the elements of trigonometry, and calculated the first tables of sines (Hipparchus, 2nd century B.C. and Claudius Ptolemy, 2nd century A.D.);* they determined the areas and volumes of a number

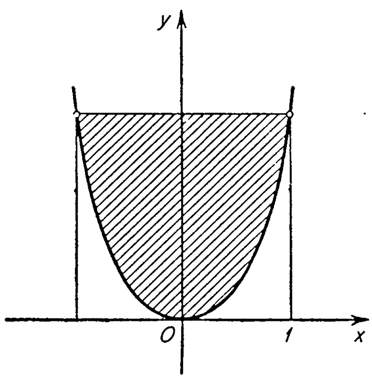

FIG. 4.

* Ptolemy is widely known as the author of a system in which the Earth is considered as the center of the universe and the motion of the heavenly bodies is described as proceeding around it. This system was supplanted by the Copernician system.

of complicated figures; for example, Archimedes found the area of the segment of a parabola by proving that it is 2/3 of the area of the rectangle containing it (figure 4). The Greeks were also acquainted with the theorem that of all bodies with a given surface area the sphere has the greatest volume, but their proof has not been preserved and was probably not complete. Such a proof is quite difficult and was first discovered in the 19th century, by means of the integral calculus.

In arithmetic and in the elements of algebra, the Greeks also made considerable progress. As was mentioned earlier, they laid the foundation for the theory of numbers. Here belong, for example, their investigations on prime numbers (the theorem of Euclid on the existence of an infinite number of prime numbers and the "sieve" of Eratosthenes for finding prime numbers) and the solution of equations in whole numbers (Diophantus about 246–330 A.D.).

We have already said that the Greeks discovered irrational magnitudes but considered them geometrically, as line segments. So the problems that today we deal with algebraically were treated geometrically by the Greeks. It was in this way that they solved quadratic equations and transformed irrational expressions. For example, the equation that we today write in the form $x^2 + ax = b^2$, they stated as follows: Find a segment x such that if to the square constructed on it we add a rectangle constructed on the same segment and on the given segment a, we obtain a rectangle equal in area to a given square. This dominance of geometry lasted a long time after the Greeks. They were also acquainted with (geometric) methods for extracting square roots and cube roots and with the properties of arithmetic and geometric progressions.

In this way the Greeks were already in possession of much of the material of contemporary elementary algebra but not, however, of the following essential elements: negative numbers and zero, irrational numbers abstracted entirely from geometry, and finally a well-developed system of literal symbols. It is true that Diophantus made use of literal symbols for the unknown quantity and its powers and also of special symbols for addition, subtraction, and equality, but his algebraic equations were still written with concrete numerical coefficients.

In geometry the Greeks attained what we now call "higher" mathematics. Archimedes made use of integral calculus for the calculation of areas and volumes and Apollonius used analytic geometry in his investigations on conic sections. Apollonius actually gives the equations of these curves*

* He gives the "equations" of conic sections referred to a vertex. For example "the equation" of the parabola $y^2 = 2px$ is formulated thus: The square on the side y is equal in area to the rectangle with sides $2p$ and x. Of course, in place of the symbols p, x, y he uses the corresponding line segments.

but expresses them in geometric language. In these equations there does not yet appear the general notion of an arbitrary constant or of a variable magnitude; and the necessary means of expressing such concepts, namely the literal symbols of algebra, appear only at a later age; they alone could convert such investigations into a source of new theories, which would be truly a part of higher mathematics. The founders of these new theories were guided, a thousand years later, by the legacy of the Greek scientists; in fact, the "Geometry" of Descartes (1637), which laid the foundation for analytic geometry, begins with a selection of problems left by the Greeks.

Such is the general rule. The old theories, by giving rise to new and profound problems, outgrow themselves, as it were, and demand for further progress new forms and new ideas. But these forms and ideas may demand new historical conditions for their birth. In ancient society the conditions necessary for the passage to higher mathematics did not and could not exist; they came on the scene with the development of the natural sciences in modern times, a development which in its turn was conditioned in the 16th and 17th centuries by the new demands of technology and of manufacturing and was connected in this way with the birth and development of capitalism.

The Greeks practically exhausted the possibilities of elementary mathematics, which is the explanation of the fact that the brilliant progress of geometry dried up at the beginning of our era and was replaced by trigonometry and algebra in the works of Ptolemy, Diophantus, and others. In fact, one may consider the works of Diophantus as the beginning of the period in which algebra played the leading role. But the society of the ancients, already verging to its decline, was no longer able to advance science in this new direction.

It should be noted that, a few centuries earlier, arithmetic had already reached a high level in China. The Chinese scientists of the 2nd and 1st centuries B.C. described the rules for arithmetical solution of a system of three equations of the first degree. It is here for the first time in history that negative coefficients are made use of and the rules for operating with negative quantities are formulated. But the solutions themselves were sought only in the form of positive numbers, just as later in the works of Diophantus. These Chinese books also include a method for the extraction of square roots and cube roots.

3. The Middle East. With the end of Greek science a period of scientific stagnation began in Europe, the center of mathematical development being shifted to India, Central Asia, and the Arabic countries.*

* To give some orientation in the dates we list here the times of some of the out-

For a period of about a thousand years, from the 5th to the 15th century, mathematics developed chiefly in connection with the demands of computation, particularly in astronomy, since the mathematicians of the East were for the most part also astronomers. It is true that they added nothing of importance to Greek geometry; in this field they only preserved for later times the results of the Greeks. But the Indian, Arabic, and Central Asian mathematicians achieved immense successes in the fields of arithmetic and algebra.*

As has been mentioned in §2, the Indians invented our present system of numeration. They also introduced negative numbers, comparing the contrast between positive and negative numbers with the contrast between property and debt or between the two directions on a straight line. Finally, they began to operate with irrational magnitudes exactly as with rational, without representing them geometrically, in contrast to the Greeks. They also had special symbols for the algebraic operations, including extraction of roots. For the very reason that the Indian and Central Asian scholars were no longer embarrassed by the difference between the irrational and rational magnitudes, they were able to overcome the "dominance" of geometry, which was characteristic of Greek mathematics, and to open up paths for the development of contemporary algebra, free of the heavy geometric framework into which it had been forced by the Greeks.

The great poet and mathematician, Omar Khayyam (about 1048–1122), and also the Azerbaijanian mathematician, Nasireddin Tusi (1201–1274), clearly showed that every ratio of magnitudes, whether commensurable or incommensurable, may be called a number; in their works we find the same general definition of number, both rational and irrational, as was introduced above in Newton's formulation, in §4. The magnitude of these achievements becomes particularly clear when we recall that complete recognition of negative and irrational numbers was attained by European mathematicians only very slowly, even after the beginning of the Renaissance of mathematics in Europe. For example, the celebrated French mathematician Viète (1540–1603), to whom algebra owes a great deal, avoided negative numbers, and in England protests against them lasted even into the 18th century. These numbers were considered absurd, since they were less than zero, that is "less than nothing at all." Nowadays they

standing mathematicians of the East. From India: Aryabhata, born about 476 A.D.; Brahmagupta, about 598–660; Bhaskara, 12th century; from Kharizm: Al-Kharizmi, 9th century; Al-Biruni, 973–1048; from Azerbaijan: Nasireddin Tusi, 1201–74; from Samarkand: Gyaseddin Jamschid, 15th century.

* One should keep in mind that it is wrong to associate the development of mathematics in this period chiefly with the Arabs. The term "Arabic" mathematics came into use chiefly because most of the scholars of the East wrote in the Arabic language, which had been spread abroad by the Arab conquests.

have become familiar, if only in the form of negative temperature; everyone reads the newspapers and understands what is mean by "the temperature in Moscow is $-8°$."

The word "algebra" comes from the name of a treatise of the mathematician and astronomer Mahommed ibn Musa al-Kharizmi (Mahommed, son of Musa, native of Kharizm), who lived in the 9th century. His treatise on algebra was called Al-jebr w'al-muqabala, which means "transposition and removal." By transposition (al-jebr) is understood the transfer of negative terms to the other side of an equation, and by removal (al-muqabala), cancellation of equal terms on both sides.

The Arabic word "al-jebr" became in Latin transcription "algebra" and the word al-muqabala was discarded, which accounts for the modern term "algebra."*

The origin of this term corresponds very well to the actual content of the science itself. Algebra is basically the doctrine of arithmetical operations considered formally from a general point of view, with abstraction from the concrete numbers. Its problems bring to the fore the formal rules for transformation of expressions and solution of equations. Al-Kharizmi placed on the title page of his book the actual names of two most general formal rules, expressing in this way the true spirit of algebra.

Subsequently, Omar Khayyam defined algebra as the science of solving equations. This definition retained its significance up to the end of the 19th century, when algebra, along with the theory of equations, struck out in new directions, essentially changing its character but not changing its spirit of generality as the science of formal operations.

The mathematicians of Central Asia found methods for calculation, both exact and approximate, of the roots of a number of equations; they discovered the general formula for the "binomial of Newton," although they expressed it in words; they greatly advanced and systematized the science of trigonometry, and calculated very accurate tables of sines. These tables were computed, for astronomical purposes, by the mathematician Gyaseddin (about 1427) who was working with the famous Uzbek astronomer Ulug Begh; Gyaseddin also invented decimal fractions 150 years before they were reinvented in Europe.

To sum up, in the course of the Middle Ages in India and in Central Asia the present decimal system of numeration (including fractions) was almost completely built up, as were also elementary algebra and trigonometry. During the same period the achievements of Chinese science began to make their way into the neighboring countries; about the 6th

* It is to be noted also that the mathematical term "algorithm," denoting a method or set of rules for computation, comes from the name of the same al-Kharizmi.

century B.C. the Chinese already had methods for the solution of the simplest indeterminate equations, for approximate calculations in geometry, and for the first steps in approximate solution of equations of the third degree. Essentially the only parts of our present high school course in algebra that were not known before the 16th century were logarithms and imaginary numbers. However, there did not yet exist a system of literal symbols: The content of algebra had outdistanced its form. Yet the form was indispensable: The abstraction from concrete numbers and the formulation of general rules demanded a corresponding method of expression; it was essential to have some way of denoting *arbitrary* numbers and operations on them. The algebraic symbolism is the necessary form corresponding to the content of algebra. Just as in remote antiquity it had been necessary, in order to operate with whole numbers, that symbols should be invented for them, so now, to operate with arbitrary numbers and to give general rules for their use, it was necessary to work out corresponding symbols. This task, begun at the time of the Greeks, was not brought to completion until the 17th century, when the present system of symbols was finally set up in the works of Descartes and others.

4. Renaissance Europe. At the time of the Renaissance the Europeans became acquainted with Greek mathematics by way of the Arabic translations. The books of Euclid, Ptolemy, and Al-Kharizmi were translated in the 12th century from Arabic into Latin, the common scientific language of Western Europe, and at the same time, the earlier system of calculation, as derived from the Greeks and Romans, was gradually replaced by the present-day Indian method, which was borrowed by the Europeans from the Arabs.

It was only in the 16th century that European science finally surpassed the achievements of its predecessors. Thus the Italians, Tartaglia and Ferrari, solved the general cubic equation, and later, the general equation of the fourth degree (see Chapter IV). Let us note that although these results are not taught in school, they belong, with respect to the methods employed in them, to elementary algebra. To higher algebra we must however refer the general theory of equations.

During the same period imaginary numbers began for the first time to be used; at first this was done in a purely formal manner, without logical foundation, which came considerably later at the beginning of the 19th century. Our present-day algebraic symbols were also worked out; in particular, literal symbols were used by Viète in 1591 not only for unknown quantities but also for given ones.

Many mathematicians took a share in this development of algebra. At the same time decimal fractions appeared in Europe; they were invented by the Dutch scholar Stevin, who wrote about them in 1585.

Finally, Napier in Great Britain invented logarithms as an aid in astronomical calculations and wrote about them in 1614; Briggs calculated the first decimal tables of logarithms, which were published in 1624.*

At the same time there appeared in Europe the "theory of combinations" and the general formula for the "binomial of Newton";† the progressions being already known, and in this way the structure of elementary algebra was completed. Therewith came to an end, at the beginning of the 17th century, the whole period of the mathematics of constant magnitudes, of elementary mathematics as it is now taught, with a few additions, in our schools. Arithmetic, elementary geometry, trigonometry, and elementary algebra were now essentially complete. There followed a transition to higher mathematics, to the mathematics of variable magnitudes.

It is not to be thought, however, that the development of elementary mathematics ceased at this time; for example, new results were discovered and are being constantly discovered today in elementary geometry. Furthermore, it is precisely because of the subsequent development of higher mathematics that we now understand more clearly the essential nature of elementary mathematics itself. But the leading role in mathematics was now taken over by the concepts of variable magnitude, function, and limit. The problems, that led from elementary mathematics to higher mathematics are nowadays clarified and solved by the concepts and methods of higher mathematics (occasionally they are not solvable at all by elementary methods), and there are other problems which may be stated in terms of elementary mathematics but which serve even today as a source of more general results and even of entire theories. Examples are provided by the earlier mentioned theory of regular systems of figures or by problems of the theory of numbers which are elementary in their formulation but far from elementary in the methods by which they are solved. For further details the reader may consult Chapter X.

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* It is interesting to note that Napier did not define logarithms as they are defined nowadays, when we say that in the formula $x = a^y$ the number y is the logarithm of x to the base a. This definition of logarithms appeared later. Napier's definition was related to the concepts of a variable magnitude and an infinitesimal and amounted to saying that the logarithm of x is a function $y = f(x)$ whose rate of growth is inversely proportional to x; that is, $y' = c/x$ (see Chapter II). In this way the basis of the definition was essentially a differential equation, defining the logarithm, although differentials had not yet been invented.

† The formula bears the name of Newton not because he was the first to discover it but because he generalized it from integral exponents to arbitrary fractional and irrational exponents.

§6. Mathematics of Variable Magnitudes

1. Variable and function. In the 16th century the investigation of motion was the central problem of physics. The physical sciences were led to this problem, and to the study of various others involving interdependence of variable magnitudes, by the demands of practical life and by the whole development of science itself.

As a reflection of the general properties of change, there arose in mathematics the concepts of a variable magnitude and a function, and it was this cardinal extension of the subject matter of mathematics that determined the transition to a new stage, to the mathematics of variable magnitudes.

The law of motion of a body in a given trajectory, for example along a straight line, is defined by the manner in which the distance covered by the body increases with time.

Thus Galileo (1564–1642) discovered the law of falling bodies by establishing that the distance fallen increases proportionally to the square of the time. This fact is expressed in the well-known formula

$$s = \frac{gt^2}{2}, \tag{1}$$

where g is approximately equal to 9.81 m/sec².

In general, the law of motion expresses the distance covered in the time t. Here the time t and the distance s are respectively the "independent" and the "dependent" variable, and the fact that to each time t there corresponds a definite distance s is what is meant by saying that the distance s is a function of the time t.

The mathematical concepts of variable and function are the abstract generalization of concrete variables (such as time, distance, velocity, angle of rotation, and area of surface traced out) and of the interdependences among them (the distance depends on the time and so forth). Just as the concept of a real number is the abstract image of the actual value of an arbitrary magnitude, so a "variable" is the abstract image of a varying magnitude, which assumes various values during the process under consideration. A mathematical variable x is "something" or, more accurately, "anything" that may take on various numerical values. This is the meaning of a variable in general; in particular, we may understand by it the time, the distance, or any other variable magnitude.

In exactly the same way, a function is the abstract image of the dependence of one magnitude on another. The assertion that y is a function of x means in mathematics only that to each possible value of x there corresponds a definite value of y. This correspondence between the values of y and the values of x is called a function. For example, according to the law of falling bodies, the distance covered corresponds to the time of fall by formula (1). The distance is a function of the time. Let us look at some other examples.

The energy of a falling body is expressed by its mass and its velocity according to the formula

$$E = \frac{mv^2}{2}. \tag{2}$$

For a given body the energy is a function of the velocity v.

By a familiar law the quantity of heat generated in a conductor in unit time by the passage of an electric current is expressed by the formula

$$Q = \frac{RI^2}{2}, \tag{3}$$

where I is the magnitude of the current and R is the resistance of the conductor. For a given resistance there corresponds to every current I a definite amount of heat Q, generated in unit time. That is, Q is a function of I.

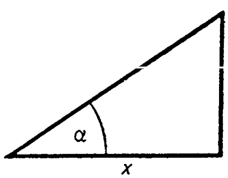

FIG. 5

The area of a right-angled triangle S with a given acute angle α and corresponding side x (see figure 5) is expressed by the formula

$$S = \tfrac{1}{2} x^2 \tan \alpha. \tag{4}$$

For a given angle α the area is a function of the side x.

All these formulas (1)–(4) may be united in the one

$$y = \tfrac{1}{2} ax^2. \tag{5}$$

This general formula represents a transition from the concrete variable magnitudes t, s, E, Q, v and so forth to the general variables x and y, and from the concrete dependences (1), (2), (3), (4) to their general form (5). Mechanics and the theory of electricity have to do with concrete formulas (1), (2), (3), interrelating concrete magnitudes, but the mathematical theory of functions deals with the general formula (5), without associating this formula with any concrete magnitudes.

The next degree of abstraction from the concrete consists in our examining not a given dependence of y on x, like $y = \frac{1}{2}ax^2$, $y = \sin x$, $y = \log x$ and so forth, but the general dependence of y on x expressed in the abstract formula

$$y = f(x).$$

This formula states that the magnitude y is in general some function of x; that is, to each value assumed by x there corresponds, in some fashion or another, a definite value y. The subject matter of mathematics thus consists not only of certain given functions ($y = \frac{1}{2} ax^2$, $y = \sin x$, and so forth), but of *arbitrary* (more accurately, more or less arbitrary) functions. These degrees of abstraction, first from concrete magnitudes and then from concrete functions, are analogous to the degrees of abstraction observed in the formation of the concept of a whole number: First, abstraction from concrete collections of objects led to the concept of whole numbers (1, 3, 12, and so forth), and then a further abstraction led to the concept of an arbitrary whole number in general. This generalization is the result of a profound interraction between analysis and synthesis: analysis of separate interrelations and synthesis, in the form of new concepts, of their common features.

The branch of mathematics devoted to the study of functions is called analysis, or often, infinitesimal analysis, since one of the most important elements in the study of functions is the concept of the infinitesimal (the meaning of this concept and its significance are explained in Chapter II).

Since a function is the abstract image of a dependence of one magnitude on another, we may say that analysis takes as its subject matter dependences between variable magnitudes, not between one concrete magnitude and another but between variables in general, in abstraction from their content. An abstraction of this sort guarantees great breadth of application, since one formula or one theorem contains an infinite number of possible concrete cases. An example of this is given already by our simple formulas (1)–(5). So the complete analogy of analysis with arithmetic and algebra becomes evident. They all originate in definite practical problems and give a general abstract expression to concrete relationships in the actual world.

2. Analytic geometry and analysis. Thus the new period of mathematics, beginning in the 17th century, may be defined as the period of the birth and development of analysis. (This is the third of the three important periods mentioned earlier.) It is to be understood, of course, that no theory arises as a result of the mere formation of new concepts, that analysis could not result from the mere existence of the concepts of variable and function. For the founding of a theory, and all the more of a complete branch of science like mathematical analysis, it is necessary that the new concepts become active, so to speak, that among them there be discovered new relationships, and that they permit the solution of new problems.

But more than that, new concepts can originate and develop, and become more general and precise, only on the basis of the very problems they enable us to solve, only through those theorems of which they form a part. The concepts of variable and function did not arise in complete form in the mind of Galileo, Descartes, Newton, or anybody else. They occurred to many mathematicians (for example Napier in connection with logarithms) and gradually assumed a more or less clear, but still by no means final, form with Newton and Leibnitz, being made still more precise and general in the subsequent development of analysis. Their present-day definition was laid down only in the 19th century, but even it is not *absolutely* rigorous or *altogether* final. The development of the concept of a function is continuing even at the present time.

Mathematical analysis was based on material furnished by the new science of mechanics, and on problems of geometry and algebra. The first definite step toward the mathematics of variable magnitudes was the appearance in 1637 of the "geometry" of Descartes, where the foundations were laid for the so-called analytic geometry. The basic ideas of Descartes are as follows.

Suppose we are given, for example, the equation

$$x^2 + y^2 = a^2. \tag{6}$$

In algebra x and y were understood as unknowns, and since the given equation does not allow us to determine them, it did not present any essential interest for algebra. But Descartes did not consider x and y as unknowns, to be found from the equation, but as *variables*; so that the given equation expresses the interdependence of two variables. Such an equation may be written in general form, by taking all its terms to the left-hand side, thus:

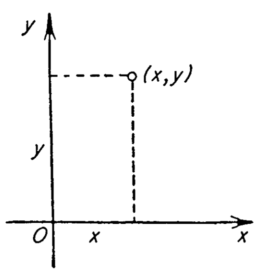

FIG. 6.

$$F(x, y) = 0.$$

Further, Descartes introduced into the plane the coordinates x, y which are now called Cartesian (figure 6). In this way, to each pair of values x and y there corresponds a point, and conversely to each point there corresponds a pair of coordinates x, y. Consequently, the equation $F(x, y) = 0$ determines the geometric locus of those points on the plane whose coordinates satisfy the equation. In general, this will be a curve. For example, equation (6) determines the circumference of a circle of radius a with center at the origin. In fact, as is obvious from figure 7, by the theorem of Pythagoras, $x^2 + y^2$ is the square of the distance from the origin O to the point M with coordinates x and y. So equation (6) represents the geometric locus of those points whose distance from the origin is equal to a, which is the circumference of a circle.

Conversely, a geometric locus of points, given by a geometric condition, may also be given by an equation expressing the same condition in the language of algebra by means of coordinates. For example, the geometric condition defining the circumference of a circle, namely that it is a geometric locus of points equidistant from a given point, may be expressed in algebraic language by equation (6).

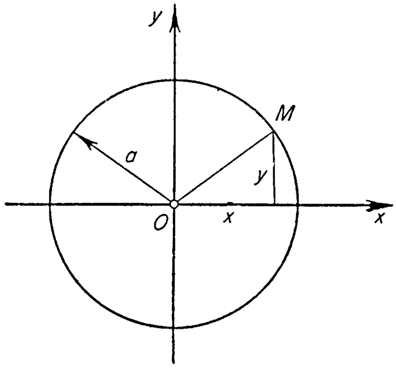

FIG. 7.

Thus the general problem and the general method of analytic geometry are as follows: We represent a given equation in two variables by a curve on the plane, and from the algebraic properties of the equation we investigate the geometric properties of the corresponding curve; and conversely, from the geometric properties of the curve we find the equation, and then from the algebraic properties of the equation we investigate the geometric properties of the curve. In this way geometric problems may be reduced to algebraic, and so finally to computation.

The content of analytic geometry will be discussed in detail in Chapter III. We now wish to direct attention to the fact that, as is evident from our short explanation, it originated in a union of geometry, algebra, and the general idea of a variable magnitude. The main geometric content of the early beginnings of analytic geometry was the theory of conic sections, ellipse, hyperbola, and parabola. This theory, as we have pointed out, was developed by the ancient Greeks; the results of Apollonius already contained in geometric form the equations of the conic sections. The union of this geometric content with algebraic form, developed after the time of the Greeks, and with the general idea of a variable magnitude, arising from the study of motion, produced analytic geometry.

Among the Greeks the conic sections were a subject of purely mathematical interest, but by the time of Descartes they were of practical importance for astronomy, mechanics, and technology. Kepler (1571–1630) discovered that the planets move around the sun in ellipses, and Galileo established the fact that a body thrown in the air, whether it is a stone or a cannonball, moves along a parabola (to the first approximation, if we may neglect air resistance). As a result, the calculation of various magnitudes referring to the conic sections became an urgent necessity, and it was the method of Descartes that solved this problem. So the way was prepared for his method by the preceding development of mathematics, and the method itself was brought into existence by the insistent demands of science and technology.

3. Differential and integral calculus. The next decisive step in the mathematics of variable magnitudes was taken by Newton and Leibnitz during the second half of the 17th century, in the founding of the differential and integral calculus. This was the actual beginning of analysis, since the subject matter of this calculus is the properties of functions themselves, as distinct from the subject matter of analytic geometry, which is geometric figures. In fact Newton and Leibnitz only brought to completion an immense amount of preparatory work, shared by many mathematicians and going back to the methods for determining areas and volumes worked out by the ancient Greeks.

Here we shall not explain the fundamental concepts of differential and integral calculus and of the theories of analysis that followed them, since this will be done in the special chapters devoted to these theories. We wish only to draw attention to the sources of the calculus, which were mainly the new problems of mechanics and the old problems of geometry, the latter consisting of drawing a tangent to a given curve and of determining areas and volumes. These geometric problems had already been studied by the ancients (it is sufficient to mention Archimedes), and also by Kepler, Cavalieri, and others at the beginning of the 17th century. But the decisive event was the discovery of the remarkable relation between these two types of problems and the formulation of a general method for solving them; this was the achievement of Newton and Leibnitz.

This relation, allowing us to connect the problems of mechanics with those of geometry, was discovered because of the possibility, arising from the method of coordinates, of making a graphical representation of the dependence of one variable on another, or in other words of a function. With the help of this graphical representation, it is easy for us to formulate the earlier mentioned relation, between the problems of mechanics and geometry, which was the source of the differential and integral calculus, and consequently to describe the general content of these two types of calculus.

The differential calculus is basically a method for finding the velocity of motion when we know the distance covered at any given time. This problem is solved by "differentiation." It turns out that the problem is completely equivalent to that of drawing a tangent to the curve representing the dependence of distance on time. The velocity at the moment t

is equal to the slope of the tangent to the curve at the point corresponding to t (figure 8).

The integral calculus is basically a method of finding the distance covered when the velocity is known, or more generally of finding the total result of the action of a variable magnitude. This problem is obviously

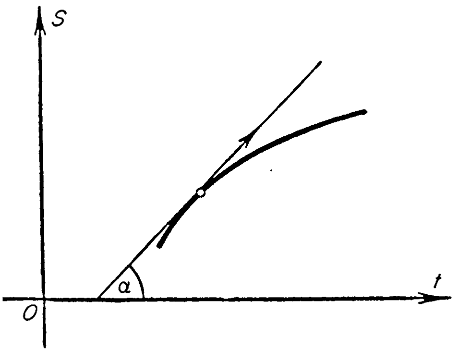

FIG. 8.

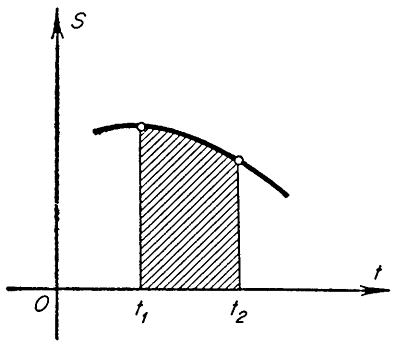

FIG. 9.

the converse of the problem of the differential calculus (the problem of finding the velocity); it is solved by "integration." It turns out that the problem of integration is completely equivalent to that of finding the area under the curve representing the dependence of the velocity on time. The distance covered in the interval of time from the moment t_1 to the moment t_2 is equal to the area under the curve between the straight lines corresponding on the graph to the values t_1 and t_2 (figure 9).

By abstracting from the mechanical formulation of the problems of the calculus and by dealing with functions rather than with dependence of distance or velocity on time, we obtain the general concept of the problems of differential and integral calculus in abstract form.

Fundamental to the calculus, as to the whole subsequent development of analysis, is the concept of a limit, which was formulated somewhat later than the other fundamental concepts of variable and function. In the early days of analysis the role later played by the limit was taken by the somewhat nebulous concept of an infinitesimal. The methods for actual calculation of velocity, given the distance covered (namely, differentiation), and of distance, given the velocity (integration), were founded on a union of algebra with the concept of limit. Analysis originated in the application of these concepts and methods to the aforementioned problems of mechanics and geometry (and also to certain other problems; for example, problems of maxima and minima). The science of analysis was in turn absolutely necessary for the development of mechanics, in the formulation of whose laws its concepts had already appeared in latent form. For example, the second law of Newton, as formulated by Newton himself, states that "the change in momentum is proportional to the acting force" (more precisely: The rate of change of momentum is proportional to the force). Consequently, if we wish to make any use of this law, we must be able to define the rate of change of a variable, that is, to differentiate. (If we state the law in the form that the acceleration is proportional to the force, the problem remains the same, because acceleration is proportional to rate of change of momentum.) Also, it is perfectly clear that in order to state the law governing a motion when the force is variable (in other words, the motion proceeds with a variable acceleration), we must be able to solve the inverse problem of finding a magnitude given its rate of change; in other words, we must be able to integrate. So one might say that Newton was simply *compelled* to invent differentiation and integration in order to develop the science of mechanics.

4. Other branches of analysis. Along with the differential and integral calculus, other branches of analysis arose: The theory of series (see Chapter II, §14), the theory of differential equations (Chapters V and VI), and the application of analysis to geometry, which later became a special branch of geometry, called differential geometry and dealing with the general theory of curves and surfaces (Chapter VII). All these theories were brought to life by the problems of mechanics, physics, and technology.

The theory of differential equations, the most important branch of analysis, has to do with equations in which the unknown is no longer a magnitude but a function, or in other words a law governing the dependence of one magnitude on another or on several others. It is easy to understand how such equations arose. In mechanics we seek to determine the whole law of motion of a body under given conditions and not just one value of the velocity or of the distance covered. In the mechanics of fluids it is necessary to find the distribution of velocity over the whole mass of fluid in motion, or in other words to find the dependence of the velocity on all three space coordinates and on time. Analogously, in the theory of electricity and magnetism we must find the tension in the field throughout all space; that is, the dependence of this tension on the same three space coordinates, and similarly in other cases.

Problems of this sort arose continually in the various branches of mechanics, including hydrodynamics and the theory of elasticity, in acoustics, in the theory of electricity and magnetism, and in the theory of heat. From the very moment of its birth, analysis remained in close contact with mechanics and with physics in general, its most important achievements being invariably connected with the solution of problems posed by the exact sciences. Beginning with Newton, the greatest analysts, D. Bernoulli (1700–1782), L. Euler (1707–1783), J. Lagrange (1736–1813), H. Poincaré (1854–1912), M. V. Ostrogradskiĭ (1801–1861) and A. M. Lyapunov (1857–1918), as well as many others who laid new foundations in analysis, started as a rule from the urgent problems of contemporary physics.

In this way new theories arose: In direct connection with mechanics, Euler and Lagrange founded a new branch of analysis, called the calculus of variations (see Chapter VIII), and at the end of the 19th century Poincaré and Lyapunov, starting again from the problems of mechanics, founded the so-called qualitative theory of differential equations (see Chapter V, §7).

In the 19th century analysis was enriched by an important new branch, the theory of functions of a complex variable (see Chapter IX). The rudiments of it are to be found in the works of Euler and certain other mathematicians, but its transformation into a well-formed theory took place in the middle of the 19th century and was carried out to a great extent by the French mathematician Cauchy (1789–1857). This theory rapidly underwent an imposing development with numerous significant results that allowed mathematicians to penetrate more deeply into many of the laws of analysis and found important applications in problems of mathematics itself, and of physics and technology.

Analysis developed rapidly; not only did it form the center and the most important part of mathematics but it also penetrated into the older regions: algebra, geometry, and even the theory of numbers. Algebra began to be thought of as basically the doctrine of functions expressed in the form of polynomials of one or several variables.* Analytic and differential geometry began to dominate the field of geometry. As far back as Euler, methods of analysis were introduced into the theory of numbers and formed in this way the beginning of the so-called analytic theory of numbers, which contains some of the most profound achievements of the science of whole numbers.

Through the influence of analysis, with its concepts of variable, function, and limit, the whole of mathematics was penetrated by the idea of motion and change, and therefore of dialectic. In exactly the same way, basically through analysis, mathematics was affected by the exact sciences and

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* Polynomials are functions of the form $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$. The fundamental problem of the algebra of the period, namely the solution of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, simply means the search for values of x for which the function $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is equal to zero. The very existence of a solution, of a root of the equation, which is called the fundamental theorem of algebra, is proved by means of analysis (see Chapter IV, §3).

technology and in turn played a role in their development, since it was the means of giving exact expression to their laws and of solving their problems. Just as among the Greeks mathematics was basically geometry, one may say that after Newton it was basically analysis. Of course, analysis did not completely absorb the whole of mathematics; in geometry, in the theory of numbers, and in algebra the problems and methods characteristic of these sciences were everywhere continued. Thus in the 17th century there arose, along with analytic geometry, another branch of geometry, namely projective geometry, in which purely geometric methods played a dominant role. It originated chiefly in problems of the representation of objects on a plane (projection), and as a result it is particularly useful in descriptive geometry.

At the same time there was developed an important new branch of mathematics, the theory of probability, which takes as its subject matter the uniformities observable in large masses of phenomena, such as a long series of rifle shots or tosses of a coin. In the succeeding period it acquired a special importance in physics and technology and its development was conditioned by the problems which came to it from those branches of science. The characteristic feature of this theory is that it deals with the laws of "random events," providing mathematical methods for investigation of the irregularities that necessarily appear in random events. The basic features of the theory of probability will be explained in Chapter XI.

5. Applications of analysis. Analysis in all its branches provided physics and technology with powerful methods for the solution of problems of many different kinds. We have already mentioned the earliest of these: to find the rate of change of a magnitude when we know how the magnitude itself depends on time; to find the area of curvilinear figures and the volumes of solids; and to find the total result of some process or another or the total action of a variable magnitude. Thus, the integral calculus allows us to determine the work done by an expanding gas as the pressure changes according to a well-known law; the same integral calculus allows us to compute, for example, the tension of an electric field with an arbitrarily given system of charges, basing our work on the law of Coulomb which determines the tension of a field resulting from a point charge, and so forth.

Further, analysis provided a method for finding the maximum and the minimum values of a magnitude under given conditions. Thus, with the help of analysis it is easy to determine the shape of a cylindrical cistern which for a given volume will have the smallest surface and consequently will require the smallest outlay of material. It turns out that the cistern will have this property if its height is equal to the diameter of its base (figure 10). Analysis allows us to determine the shape of the curve along which a body must roll in order to fall in the shortest time from one given point to another (this curve is the so-called cycloid; figure 11).

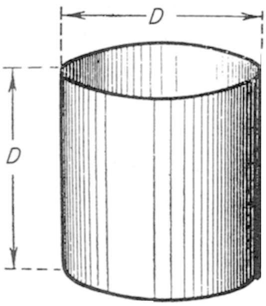

FIG. 10.

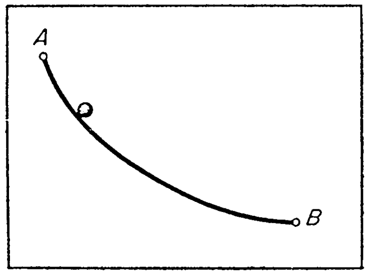

FIG. 11.

For the solution of these and other problems the reader may turn to Chapters II and VIII.

Analysis, or more precisely the theory of differential equations, allows us not merely to find separate values for variable magnitudes but also to determine unknown functions; that is, to find laws of dependence of certain variables on others. Thus we have the possibility, on the basis of the general laws of electricity, of computing how the current varies with time in a circuit with arbitrary resistance, capacitance, and self-induction. We can determine laws for the distribution of velocities throughout the whole mass of a fluid under given conditions. We can deduce general laws for the vibration of strings and membranes, and for the propagation of vibrations in various media; here we are referring to sound waves, electromagnetic waves, or elastic vibrations propagated through the Earth by earthquakes or explosions. Parenthetically, we may remark that new methods are thereby provided for searching for useful minerals and for carrying out investigations far below the surface of the Earth. Individual problems of this sort will be found in Chapters V and VI.

Finally, analysis not only provides us with methods for solving special problems; it also gives us general methods for mathematical formulation of the quantitative laws of the exact sciences. As was mentioned, earlier, the general laws of mechanics could not be formulated mathematically without recourse to the concepts of analysis, and without such a formulation we would not be able to solve the problems of mechanics. In exactly the same way the general laws for heat conduction, diffusion through porous materials, propagation of vibrations, the course of chemical reactions, the basic laws of electromagnetism, and many other laws simply could not be given a mathematical formulation without the concepts of analysis. It is only as the result of such a formulation that these laws can be applied to the most varied concrete cases, providing a basis for exact mathematical conclusions in the special problems of heat conduction, vibrations, chemical solution, electromagnetic fields, and other problems of mechanics, astronomy, and all the numerous branches of physics, chemistry, heat engineering, power, machine construction, electrical engineering, and so forth.

6. Critical examination of the foundations of analysis. Just as in the history of geometry among Greeks the rigorous and systematic presentation given by Euclid brought to completion a long previous development, so in the development of analysis there arose the necessity of placing it upon a firmer basis than had been provided by the first creators of its powerful methods: Newton, Euler, Lagrange, and others. As the analysis founded by them grew more extensive, it began, on the one hand, to deal with more profound and difficult problems, and on the other, to require from its very extent a more systematic and carefully reasoned basis. The growth of the theory necessitated a systematization and critical analysis of its foundations. To put a theory on a firm foundation requires examination of its entire development and should by no means be considered as a starting point for the theory itself, since without the theory we would simply have no idea of what it is that we need to provide with a foundation. By the way, certain contemporary formalists forget this fact when they consider it advisable to found and develop a theory starting from axioms that have not been selected on the basis of any analysis of the actual material which they are supposed to summarize. But the axioms themselves require a justification of their content; they only sum up other material and provide a foundation for the logical construction of a theory.*

The necessary period of criticism, systematization, and laying of foundations occurred in analysis at the beginning of the last century. Through the efforts of a number of eminent scientists this important and difficult work was brought to a successful completion. In particular, precise definitions were given for the basic concepts of real number, variable, function, limit, and continuity.

However, as we have already had occasion to mention, none of these definitions may be considered as absolutely rigorous or final. The development of these concepts is continuing. Euclid and all the mathematicians in the course of 2,000 years after him no doubt considered his "Elements"

* This double role of the axioms is sometimes lost from view even in works of a methodological character, which thereby attribute to the construction of axioms a significance which does not at all belong to it, namely that of the total construction of a theory.

as the practical limit of logical rigor. But to a contemporary view the Euclidean foundations of geometry seem quite superficial. This historical example shows that we ought not to flatter ourselves with any idea of "absolute" or "final" rigor in contemporary mathematics. In a science that is not yet dead and mummified, there is not and cannot be anything perfect. But we can say with confidence that the foundations of analysis as they exist at present correspond in a quite satisfactory way to the contemporary problems of science and the contemporary conception of logical precision; and second, that the continued deepening of these concepts and the discussions that are now taking place about them give us no cause, and will not give us cause, simply to reject them; these discussions will lead us to a new, more precise, and more profound understanding, the results of which it is still difficult to estimate.

Although the establishment of the basic principles of a theory forms a summary of its development, it does not represent the end of the theory; on the contrary, it is conducive to further development. This is exactly what happened in analysis. In connection with the deepening of its foundations there arose a new mathematical theory, created by the German mathematician Cantor in the seventies of the last century, namely the general theory of infinite sets of arbitrary abstract objects, whether numbers, points, functions or any other "elements". On the basis of these ideas there grew up a new chapter in analysis, the so-called theory of functions of a real variable, whose concepts, along with those of the foundations of analysis and the theory of sets, are explained in Chapter XV. At the same time the general ideas of the theory of sets penetrated every branch of mathematics. But this "set-theoretical point of view" is inseparably connected with a new stage in the development of mathematics, which we will now consider briefly.

§7. Contemporary Mathematics

1. The more advanced character of present-day mathematics. To the four stages of the develoment of mathematics mentioned in §5 there naturally correspond stages in our mathematical education, the material learned at each stage of our study consisting, to a fair degree of approximation, of the basic content of the corresponding period in the history of mathematics.

The basic results of arithmetic and geometry, obtained in the first period of the development of mathematics, form the subject of primary education and are known to us all. For example, when we determine the quantity of material necessary to carry out a certain job, let us say to cover a floor, we are already making use of these first results of mathematics. The most important achievements of the second period, the period of elementary mathematics, are taught in the high schools. The basic results of the third period, the foundations of analysis, the theory of differential equations, higher algebra, and so forth, form the mathematical instruction of an engineer; they are studied in all the schools of higher education, except those devoted purely to the humanities. In this way the basic ideas and results of the mathematics of that period are widely known, use being made of them to some extent by almost every engineer and scientist.

On the other hand, the ideas and results of the present-day period of mathematics are studied almost exclusively in graduate departments of mathematics and physics. Beside mathematical specialists, they are used by researchers in the fields of mechanics and physics, and in a number of the newer branches of technology. Of course, this does not at all mean that they have no practical application, but since they represent the most recent results of science, they are naturally more complicated. Consequently, as we now pass to a general description of the latest stage in the development of mathematics we can no longer consider that everything which we mention briefly will be altogether clear. We will try to present in a few lines the most general character of the new branches of mathematics; their content will be explained in greater detail in the corresponding chapters of the book.

If the present section seems overly difficult it may be passed over at first reading and taken up again after study of the special chapters.

2. Geometry. The beginning of the present-day development of mathematics is characterized by profound changes in all its basic fields: algebra, geometry, and analysis. This change may perhaps be followed most clearly in the field of geometry. In the year 1826 Lobačevskiĭ, and almost simultaneously with him the Hungarian mathematician Janos Bolyai, developed the new non-Euclidean geometry. The ideas of Lobačevskiĭ were far from being immediately understood by all mathematicians. They were too bold and unexpected. But from this moment there began a fundamental new development of geometry; the very conception of what is meant by geometry was changed. Its subject matter and the range of its applications were rapidly extended. The most important step, after Lobačesvskiĭ, in this direction was taken in 1854 by the celebrated German mathematician, Riemann. He clearly formulated the general idea that an unlimited number of "spaces" could be investigated by geometry, and at the same time he indicated their possible significance in the real world. In the new development of geometry two features were characteristic.

In the first place the earlier geometry studied only the *spatial* forms and relations of the material world, and then only to the extent in which they appear in the framework of Euclidean geometry, but now the subject matter of geometry began to include also many *other* forms and relations of the actual world, provided only they were similar to the spatial ones and therefore allowed the use of geometric methods. The term "space" thereby took on in mathematics a new meaning, broader and at the same time more special. Simultaneously, the methods of geometry became much richer and more varied. In their turn they provide us with more complete means for learning about the physical world around us, the world from which geometry in its original form was abstracted.

In the second place, even in Euclidean geometry important progress was made: In it were studied the properties of incomparably more complicated figures, even including arbitrary sets of points. Also a fundamentally new attitude appeared toward the properties of the figures under investigation. Separate groups of properties were distinguished, which could be investigated in abstraction from others, and this very abstraction *within* geometry gave rise to many characteristic branches of the subject, which essentially became independent "geometries." The development of geometry in all these directions is being continued and more and more new "spaces" and their "geometries" are being studied: the space of Lobačesvskiĭ, projective space, Euclidean and other spaces of various dimensions, in particular four-dimensional space, Riemann spaces, Finsler spaces, topological spaces, and so forth. These theories find important application in mathematics itself, outside of geometry, and also in physics and mechanics; particularly noteworthy are their applications in the theory of relativity of contemporary physics, which is a theory of space, time, and gravitation. From what has been said it is clear that we are dealing here with a qualitative change in geometry.

The ideas of contemporary geometry and some of the elements of the theory of various spaces investigated in it will be explained in Chapters XVII and XVIII.

3. Algebra. Algebra too underwent a qualitative change. In the first half of the 19th century new theories arose, which led to changes in its character, and to an extension of its subject matter and its range of application.

In its original form, as pointed out in §5, algebra dealt with mathematical operations on numbers considered from a formal point of view, in abstraction from given concrete numbers. This abstraction found expression in the fact that in algebra magnitudes are denoted by letters, on which calculations are carried out according to well-known formal rules.

Contemporary algebra retains this basis but widens it in a very extensive way. It now considers "magnitudes" of a much more general nature than numbers, and studies operations on these "magnitudes" which are to some extent analogous in their formal properties to the ordinary operations of arithmetic: addition, subtraction, multiplication, and divison. A very simple example is offered by vector magnitudes, which may be added by the well-known parallelogram rule. But the generalization carried out in contemporary algebra is such that even the very term "magnitude" often loses its meaning and one speaks more generally of "elements" on which it is possible to perform operations similar to the usual algebraic ones. For example, two motions carried out one after the other are evidently equivalent to a certain single motion, which is the sum of the two; two algebraic transformations of a formula may be equivalent to a single transformation that produces the same result, and so forth; and so it is possible to speak of a characteristic "addition" of motions or transformations. All this and much else is studied in a general abstract form in contemporary algebra.

The new algebraic theories in this direction arose in the first half of the 19th century in the investigations of a number of mathematicians, among whom we should particularly mention the French mathematician Galois (1811–1832). The concepts, methods, and results of contemporary algebra find important applications in analysis, geometry, physics, and crystallography. In particular, the theory mentioned at the end of §3 concerning the symmetry of crystals, which was developed by E. S. Fedorov, is based on a union of geometry with one of the new algebraic theories, the so-called theory of groups.

As we see, we are dealing here with a fundamental, qualitative generalization of the subject matter of algebra with a change in the very concept of what algebra is. The ideas of contemporary algebra and the basic elements of some of its theories will be explained in Chapter XX and XVI.

4. Analysis. Analysis in all its branches also made profound progress. In the first place, as was already mentioned in the preceding section, its foundations were made more precise; in particular, its basic concepts were given exact and general definitions: such concepts as function, limit, integral and finally, the basic concept of a variable magnitude (a rigorous definition was given for the real number). A beginning of the process of putting analysis on a more precise foundation was made by the Czech mathematician Bolzano (1781–1848), the French mathematician Cauchy (1789–1857), and a number of others. This greater precision was gained at the same time as the new developments in algebra and geometry were being made; it was brought to completion in its present well-known form in the eighties of the 19th century by the German mathematicians Weierstrass, Dedekind, and Cantor. As was mentioned at the end of §6, Cantor also laid the foundation for the theory of transfinite sets, which plays such a large role in the development of the newer ideas in mathematics.

The increase in precision in the concepts of variable and function in connection with the theory of sets laid the foundation for a further development of analysis. A transition was made to the study of more general functions; and in the same direction the apparatus of analysis, namely the integral and differential calculus, was also generalized. Thus, on the threshold of the present century, there arose the new branch of analysis already mentioned in §6, the so-called theory of functions of a real variable. The development of this theory is chiefly connected with the French mathematicians, Borel and Lebesgue and others, and with N. N. Luzin (1883–1950) and his school. In general, the newer branches of analysis are called modern analysis in contradistinction to the earlier so-called classical analysis.

Other new theories arose in analysis. Thus a special branch was formed by the theory of approximation of functions, which studies questions of the best approximate representation of general functions by various "simple" functions, above all by polynomials, that is by functions of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n .$$

The theory of approximation of functions has great importance, if only for the reason that it lays down general foundations for the practical calculation of functions, for the approximate replacement of complicated functions by simpler ones. The rudiments of this theory go back to the very beginnings of analysis. Its modern direction was given to it by the great Russian mathematician P. L. Čebyšev (1821–1894). This direction was later developed into the so-called constructive theory of functions, chiefly in the works of Soviet mathematicians, particularly S. N. Bernšteĭn (born 1880), to whom belong the most important results in this field. Chapter XII deals with approximation of functions.

We spoke earlier about the development of the theory of functions of a complex variable. We must still mention the so-called qualitative theory of differential equations, originating in the works of Poincaré (1854–1912) and A. M. Lyapunov (1857–1918), about which some ideas will be given in Chapter V, and also the theory of integral equations. These theories have great practical importance in mechanics, physics, and technology. Thus, the qualitative theory of differential equations provides solutions of problems concerning stability of motion, and the action of mechanisms or of vibrating electric systems and the like. Stability of a process means in the most general sense that if small changes are made in the initial data or in the conditions of the motion, then the motion itself during the whole of its course will change only slightly. The technical significance of questions of this sort hardly needs to be emphasized.

5. Functional analysis. On the ground prepared by the development of analysis and mathematical physics, along with the new ideas of geometry and algebra, there has grown up an extensive new division of mathematics, the so-called functional analysis, which plays an exceptionally important role in modern mathematics. Many mathematicians shared in creating it; let us mention, for example, the greatest German mathematician of recent times, Hilbert (1862–1943), the Hungarian mathematician Riesz (1880–1956) and the Polish mathematician Banach (1892–1945). The separate Chapter XIX is devoted to functional analysis.

The essence of this new branch of mathematics consists briefly in the following. In classical analysis the variable is a magnitude, or "number," but in functional analysis the function itself is regarded as the variable. The properties of the given function are determined here not in themselves but in relation to other functions. What is under study is not a separate function but a whole collection of functions characterized by one property or another; for example, the collection of all continuous functions. Such a collection of functions forms the so-called functional space. This procedure corresponds, for example, to the fact that we may consider the collection of all curves on a surface or of all possible motions of a given mechanical system, thereby defining the properties of the separate curves or motions in their relation to other curves or motions.

The transition from the investigation of separate functions to a *variable* function is similar to the transition from unknown numbers x, y to variables x, y; that is, it is similar to the idea of Descartes mentioned in a preceding paragraph. On the basis of this idea Descartes produced his well-known union of algebra and geometry, of an equation and a curve, which is one of the most important elements in the rise of analysis. Similarly, the union of the concept of a variable function with the ideas of contemporary algebra and geometry produced the new functional analysis. Just as analysis was necessary for the development of the mechanics of the time, so functional analysis provided new methods for the solution of present-day problems of mathematical physics and produced the mathematical apparatus for the new quantum mechanics of the atom. History repeats itself as usual, but in a new way, on a higher plane. As we have said, functional analysis unites the basic ideas and methods of analysis, of modern algebra, and of geometry and in its turn exercises an influence on the development of these branches of mathematics. The problems arising in classical analysis now find new, more general solutions, often almost at a single step, by means of functional analysis. Here, as at a focus, are gathered together, in a very productive way, the most general and abstract ideas of modern mathematics.

From this short sketch, from this mere enumeration of the new directions of analysis (the theory of functions of a real variable, theory of approximation of functions, qualitative theory of differential equations, theory of integral equations, and functional analysis) it may be seen that we are dealing here in fact with an essentially new stage in the development of analysis.

6. Computational mathematics and mathematical logic. At all periods the technical level of the means of computation has had an essential influence on mathematical methods. But the equipment for carrying out calculations which has been at our disposal up until most recent times has been very limited. The simplest devices, such as the abacus, tables of logarithms and the logarithmic sliderule, the calculating machine, and finally more complicated calculators and the automatic calculating machine, these were the basic implements for computation existing up to the forties of the 20th century. These implements made it possible to carry out more or less quickly the separate operations of addition, multiplication, and so forth. But to carry through to final numerical result the practical problems that arise nowadays requires a colossal number of such operations, following one another in a complicated program that sometimes depends on results obtained during the course of the calculation. The solution of such problems proved to be practically impossible or completely valueless on account of the length of the process of solution. But in the last ten years a radical change has taken place in the whole science of computation. Modern calculating machines, constructed on new principles, allow us to make computations with exceptionally great speed and at the same time to carry out complicated chains of calculations automatically, according to extremely flexible programs arranged in advance. Some of the questions connected with the construction and significance of modern calculating machines will be discussed in Chapter XIV.

The new techniques not only enable us to carry out investigations that were formerly quite impracticable but also lead us to change our estimate of the value of many well-known mathematical results. For example, they have given a special stimulus to the development of approximative methods; that is, methods which allow us, by a chain of elementary operations, to reach a desired numerical result with sufficiently great accuracy. The mathematical methods themselves must now be estimated from the point of view of their suitability for corresponding machines.

In close connection with the development of calculating techniques is the subject of mathematical logic. It was developed primarily as a result of intrinsic difficulties arising in mathematics itself, its subject matter being the analysis of mathematical proof. It is itself a branch of mathematics, and includes those branches of general logic that can be objectively formulated and developed by the mathematical method.

Although on the one hand mathematical logic thus goes back to the very sources and foundations of mathematics, it is closely connected, on the other hand, with the most modern questions of computational technique. Naturally, for example, a proof that leads to the setting up of a definite preassigned process, permitting us to approach a desired result with an arbitrary degree of accuracy, is essentially different from more abstract proofs on the existence of the given result.

There also arises here a characteristic range of questions concerning the degree of generality possible in problems that can be dealt with by a method which is completely defined in advance at every step. Profound results have been reached along these lines in mathematical logic, results that are extremely important from a general epistemological point of view.

It would not be an exaggeration to say that with the development of the new computational techniques and the achievements of mathematical logic a new period has begun in modern mathematics, characterized by the fact that its subject matter is not only the study of one object or another but also all the ways and means by which such an object can be defined; not only certain problems, but also all possible methods of solving them.

To what has been said it is only necessary to add that also in the older branches of mathematics, the theory of numbers, Euclidean geometry, classical algebra and analysis, and the theory of probability, rapid development has continued throughout the whole period of modern mathematics so that these fields have been enriched by many new fundamental ideas and results; let us mention, for example, the results attained in the theory of numbers and in the geometry of everyday space by the Russian and Soviet mathematicians P. L. Čebyšev, E. S. Fedorov, I. M. Vinogradov, and others. The development on a wide front of the theory of probability has been connected with the extraordinarily important regularities observable in statistical physics and in contemporary problems of technology.

7. Characteristic features of modern mathematics. What are the most general characteristics of modern mathematics as a whole, distinguishing it from the earlier development of geometry, algebra, and analysis?

First of all is the immense extension of the subject matter of mathematics and of its applications. Such an extension of subject matter and range of application represents an enormous quantitative and qualitative growth, brought about by the appearance of powerful new theories and methods that allow us to solve problems completely inaccessible up to now. This extension of the subject matter of mathematics is characterized by the fact that contemporary mathematics conscientiously sets itself the task of studying all possible types of quantitative relationships and spatial forms.

A second characteristic feature of modern mathematics is the formation of general concepts on a new and higher level of abstraction. It is precisely this feature that guarantees preservation of the unity of mathematics, in spite of its immense growth in widely differing branches. Even in parts of mathematics that are extremely far from one another similarities of structure are brought to light by the general concepts and theories of the present day. They guarantee that contemporary mathematical methods will have great generality and breath of application; in particular, they produce a profound interpenetration of the fundamental branches of mathematics: geometry, algebra, and analysis.

As one of the characteristic features of modern mathematics, we must also mention the obvious dominance of the set-theoretical point of view. Of course, this point of view owes its significance to the fact that it summarizes in a certain sense the rich content of all the preceding developments of mathematics. Finally, one of the most characteristic features of modern mathematics is the profound analysis of its foundations, of the mutual influence of its concepts, of the structure of its separate theories, and of the methods of mathematical proof. Without such an analysis of foundations it would not be possible to improve or develop any further the principles and theories that have led to the present generalizations.

The characteristic feature of modern mathematics may be said to be that its subject matter consists not only of given quantitative relations and forms but of all possible ones. In geometry, we speak not only of spatial relations and forms but of all possible forms similar to spatial ones. In algebra, we speak of various abstract systems of objects with all possible laws of operation on them. In analysis, not only magnitudes are considered as variables but the very functions themselves. In a functional space all the functions of a given type (all the possible interdependences among the variables) are brought together. Summing up, it is possible to say that while elementary mathematics deals with constant magnitudes, and the next period with variable magnitudes, *contemporary mathematics is the mathematics of all possible* (*in general*, *variable*) *quantitative relations and interdependences among magnitudes*. This definition is, of course, incomplete, but it does emphasize the characteristic feature of modern mathematics which distinguishes it from the mathematics of preceding ages.*

Suggested Reading

Preliminary remark. The original Russian text of *Mathematics: its content, methods, and meaning* contains a list of recommended books at the end of each of its twenty chapters. In the present translation these books have been retained only if they have been translated into English. In compensation, the lists given here contain many other, readily available, works in the English language.

Books dealing with mathematics in general

E. T. Bell, *The development of mathematics*, 2d ed., McGraw-Hill, New York, 1945.

R. Courant and H. Robbins, *What is mathematics?* Oxford University Press, New York, 1941.

H. Eves and C. V. Newsom, *An introduction to the foundations and fundamental concepts of mathematics*, Rinehart, New York, 1958.

G. H. Hardy, *A mathematician's apology*, Macmillan, New York, 1940.

R. L. Wilder, *Introduction to the foundations of mathematics*, Wiley, New York, 1952.

Books of a historical character

R. C. Archibald, *Outline of the history of mathematics*, 5th ed., Mathematical Association of America, Oberlin, Ohio, 1941.

F. Cajori, *History of mathematics*, 2d ed., Macmillan, New York, 1919.

Euclid, The thirteen books of Euclid's *Elements* translated with an introduction and commentary by T. L. Heath, 2d ed., 3 vols., Dover, New York, 1956.

O. E. Neugebauer, *The exact sciences in antiquity*, Princeton University Press, Princeton, N. J., 1952.

D. E. Smith, *History of mathematics*, Vol. I. *General survey of the history of elementary mathematics*, Vol. II. *Special topics of elementary mathematics*, Dover, New York, 1958.

D. J. Struik, *A concise history of mathematics*, Dover, New York, 1948.

B. L. van der Waerden, *Science awakening*, P. Noordhoff, Groningen, 1954.

* This section is followed in the original Russian text by two sections entitled "The essential nature of mathematics" and "The laws of the development of mathematics." These sections are omitted in the present translation in view of the fact that they discuss in more detail, and in the more general philosophical setting of dialectical materialism, points of view already stated with great clarity in the preceding sections.