CHAPTER I

Determinants

1. Definitions and useful properties

A discussion of the theory of determinants may be approached in a variety of ways. For the reader who already has an acquaintance with this subject and can, therefore, dispense with introductory remarks, the following procedure* is particularly effective since it strikes directly at those ideas which make the determinant a useful tool.

A determinant is commonly written in the form

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad [1]$$

in which the vertical lines enclosing the array of elements a_{ik} are intended to take the place of parentheses as an indication that these elements are the variables of the function A, just as $f(x)$ is written as a symbol for a function of x.

The determinant is said to be of the nth *order* when it involves n rows and n columns, the total number of elements then being n^2. The italic capital letter A is used as an abbreviation for the function whose elements are denoted by the lower case letter a. Thus, B may represent another determinant with the elements b_{ik}, etc. The first index on an element indicates the row, the second index the column in which that element is situated.

The determinant may be defined uniquely in terms of the following *three* fundamental properties:

I. *The value of the function is unchanged if the elements of any row (column) are replaced by the sums of the elements of that row (column) and the corresponding ones of another row (column); for example, if* a_{11}, a_{12}, $\cdots$ a_{1n} *are replaced by* $(a_{11} + a_{31})$, $(a_{12} + a_{32})$, $\cdots$ $(a_{1n} + a_{3n})$.

II. *The value of the function is multiplied by the constant* k *if all the elements of any row or column are multiplied by* k.

III. *The value of the function is unity if all the elements on the principal diagonal, that is,* a_{11}, a_{22}, $\cdots$ a_{nn}, *are unity and all others are zero.*

*C. Carathéodory, *Vorlesungen über reelle Funktionen* (Leipzig, 1918), Ch. VI.

To these three fundamental properties may be added the following derived ones:

IV. *The first fundamental property may be amplified to the effect that an arbitrary factor times the elements of any row or column may be added to (or subtracted from) the corresponding elements of another row or column.*

V. *The algebraic sign of the function is reversed when any two rows or columns are interchanged.*

VI. *The value of the function is zero if all the elements of a row or column are zero, or if the corresponding elements of any two rows or columns are identical or have a common ratio.*

Rule IV may be seen to follow from I and II. As shown in the numerical example below, the elements of the third column are first multiplied by k; the resulting k-multiplied elements are then added to the respective ones of the first column, after which column three is multiplied by k^{-1}, thus restoring to its elements their original values.

$$A = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 2 & 6 \\ 3 & 1 & 7 \end{vmatrix} \qquad kA = \begin{vmatrix} 1 & 3 & 2k \\ 4 & 2 & 6k \\ 3 & 1 & 7k \end{vmatrix} = \begin{vmatrix} (1+2k) & 3 & 2k \\ (4+6k) & 2 & 6k \\ (3+7k) & 1 & 7k \end{vmatrix}$$

$$A = \begin{vmatrix} (1+2k) & 3 & 2 \\ (4+6k) & 2 & 6 \\ (3+7k) & 1 & 7 \end{vmatrix} \qquad [2]$$

Rule V is a consequence of I and the extended form IV of II. Thus, suppose column 1 is first added to column 3, next the resultant column 3 is subtracted from column 1, and, finally, this resulting first column is added to the resultant column 3. The net effect is to interchange columns 1 and 3 and prefix all the elements of the first column with minus signs, as illustrated below:

$$A = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 2 & 6 \\ 3 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & (2+1) \\ 4 & 2 & (6+4) \\ 3 & 1 & (7+3) \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 3 & (2+1) \\ -6 & 2 & (6+4) \\ -7 & 1 & (7+3) \end{vmatrix} = \begin{vmatrix} -2 & 3 & 1 \\ -6 & 2 & 4 \\ -7 & 1 & 3 \end{vmatrix} \qquad [3]$$

The first part of rule VI follows from the property II for $k = 0$, and the second part is seen to be true on account of IV because a row or column of zeros is obtained when, for a suitably chosen factor, the k-multiplied elements of one of the proportional rows or columns are subtracted from the respective elements of the other row or column.

2. Evaluation of numerical determinants

The properties discussed above may be applied to the numerical evaluation of determinants, as is best illustrated by the following numerical example. Let

$$A = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 2 & 6 \\ 3 & 1 & 7 \end{vmatrix} \tag{4}$$

Step 1. Subtract from the second row the 4-multiplied elements of the first row:

$$A = \begin{vmatrix} 1 & 3 & 2 \\ 0 & -10 & -2 \\ 3 & 1 & 7 \end{vmatrix} \tag{5}$$

Step 2. Subtract from the third row the 3-multiplied elements of the first row:

$$A = \begin{vmatrix} 1 & 3 & 2 \\ 0 & -10 & -2 \\ 0 & -8 & 1 \end{vmatrix} \tag{6}$$

Step 3. Subtract from the third row the $\frac{8}{10}$-multiplied elements of the second row:

$$A = \begin{vmatrix} 1 & 3 & 2 \\ 0 & -10 & -2 \\ 0 & 0 & \frac{13}{5} \end{vmatrix} \tag{7}$$

Step 4. Subtract from the second column the 3-multiplied elements of the first column:

$$A = \begin{vmatrix} 1 & 0 & 2 \\ 0 & -10 & -2 \\ 0 & 0 & \frac{13}{5} \end{vmatrix} \tag{8}$$

Step 5. Subtract from the third column the 2-multiplied elements of the first column:

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -10 & -2 \\ 0 & 0 & \frac{13}{5} \end{vmatrix} \tag{9}$$

Step 6. Subtract from the third column the $\frac{2}{10}$-multiplied elements of the second column:

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & \frac{13}{5} \end{vmatrix} \tag{10}$$

Application of the fundamental properties II and III then gives

$$A = (1)(-10)(\tfrac{13}{5})\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(-10)(\tfrac{13}{5}) = -26 \qquad [11]$$

It is useful to note that the modifications involved in the last three steps of this process do not influence the values of the elements on the principal diagonal, the product of which is equal to the value of the determinant. This fact may be stated in the form of an additional derived property:

VII. *The value of the special determinant in triangular form:*

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} \qquad [12]$$

is given by the product $(a_{11}a_{22}\cdots a_{nn})$ *of the elements on its principal diagonal.*

With the help of this rule the value of the determinant in the above example may be set down after the completion of the third step.

If in the determinant A the rows and columns are interchanged, the values of the elements on the principal diagonal are not affected; and if the above operations with respect to rows are then replaced by the same operations with respect to corresponding columns and vice versa, the same final value is evidently arrived at. This fact demonstrates the equivalence of rows and columns as far as the value of a determinant is concerned. For convenience in reference this is stated as the property:

VIII. *The value of a determinant is unchanged if its rows are written as corresponding columns or vice versa.*

In numerical work, the method of evaluation illustrated in the above example is short and convenient to apply. When an analytic result is desired, however, other methods are usually preferable. They are given in Arts. 4 and 5, to which the discussion immediately following serves as an introduction.

3. Minors and cofactors

If in the determinant A of Eq. 1, one or more rows and a corresponding number of columns are deleted, the remaining square array of elements is again a determinant. It is referred to as the $(n - p)$-rowed *minor* (or *minor determinant*) of A, where the integer p denotes the number of rows or the corresponding number of columns which have been deleted. Thus the n-rowed minor is the determinant itself. An $(n-1)$-rowed minor is also spoken of as a *first* minor, an $(n-2)$-rowed one as a *second* minor, etc.*

The minor is customarily denoted by a symbol whose indexes refer to the canceled rows and columns. Thus the minor M_{ik} is formed by canceling the ith row and the kth column in A. It is quite common to speak of M_{ik} as the minor of a_{ik}, or as the minor corresponding to the element a_{ik}, although (according to the immediately following discussion) it should more properly be referred to as the complement of a_{ik}.

A minor of second order, denoted by $M_{\substack{ik\\rs}}$, is formed by canceling the ith and rth rows and the kth and sth columns. The extension of this notation to the designation of minors of higher order is readily recognized, but when the number of canceled rows and columns is large (cases of this sort are infrequent in engineering applications), such notation becomes awkward and is usually replaced by some other expedient which seems more effective at the moment.

The elements which lie at the intersections of the canceled rows and columns, arranged in a square array in the same order (from left to right and from top to bottom) as they appear in the original determinant, form another minor determinant N which is called the *complement of* M. The complement of a first minor is a single element; that of a second minor is a two-rowed determinant, etc.

In particular, the minors formed by canceling the *same* rows as columns (these intersect on the principal diagonal) are called *principal* minors, and their complements are again principal minors.

An alternative view may be taken with regard to the formation of minors. Instead of obtaining the minors by canceling rows and columns in the original determinant, they may be formed by first selecting from the determinant certain rows, and subsequently selecting from this rectangular (nonsquare) array any like number of columns. Or, from a given set of columns in the determinant A, minors may be formed by the selection of corresponding numbers of rows. The minors thus formed are evidently the complements of those obtained by the process of canceling the same combinations of rows and columns.

If M is a minor of the determinant A and N is its complement, and if the rows and columns *contained* in M are formed from the $i, j, k, \cdots$ th rows and the $q, r, s, \cdots$ th columns of A, then

$$N \times (-1)^{(i+j+k+\cdots+q+r+s+\cdots)} \qquad [13]$$

*The terms "minor of first order," "second order," etc., are also used.

is referred to as the *algebraic* complement of M. This differs from the ordinary complement only in its algebraic sign. If the sum of the indexes referring to the (first, second, etc.) rows and columns of A contained in M is an even integer, the algebraic sign, by Eq. 13, is $+1$; if this sum is an odd integer, the sign is -1.

The relation between a minor and its complement is evidently a mutual one in the sense that the two designations may be interchanged. Whereas M may be called a minor and N its complement, N may be looked upon as the minor and M as its complement. Thus, a single element may be thought of as a one-rowed minor. If the element is a_{ik}, its complement is the minor M_{ik}.

The algebraic complement of the single element a_{ik} is sufficiently important to deserve a special name and symbol. It is called the *cofactor* of a_{ik} and is quite commonly denoted by the corresponding upper-case letter with like subscripts (although various other notations are also encountered in the literature). Thus, the cofactor of a_{ik} is given by

$$A_{ik} = (-1)^{i+k} M_{ik} \tag{14}$$

It differs from the minor (which is the complement of a_{ik}) only in algebraic sign; hence the cofactor is sometimes referred to as the *signed* minor.

The indexes i and k, whose integer values determine the sign-controlling factor $(-1)^{i+k}$, refer respectively to the row and column intersecting at the point where the element a_{ik} is located. If the cofactor is formed for an element in the original determinant, its indexes and those appearing in the sign-controlling factor obviously agree with the indexes appearing on the element in question, because the indexes on an element of the original determinant indicate respectively its row and its column positions. This correspondence is, however, no longer consistently true in a minor of the original determinant.

For example, the minor M_{23} of A, Eq. 1, reads

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} & a_{15} & \cdots & a_{1n} \\ a_{31} & a_{32} & a_{34} & a_{35} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{44} & a_{45} & \cdots & a_{4n} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n4} & a_{n5} & \cdots & a_{nn} \end{vmatrix} \tag{15}$$

Here the element a_{32}, for instance, is located at the intersection of the second row and the second column. This is referred to as the (2,2) position in the minor determinant of Eq. 15. In general, the term (r,s) *position* is used to indicate the location at which the rth row and sth column of a given rectangular array intersect. The object of the present argument is to point out as a typical case that in forming the cofactor for the element

a_{32} for the minor determinant of Eq. 15, the sign-controlling factor is $(-1)^{2+2}$ and not $(-1)^{3+2}$.

If only the algebraic signs of the cofactors are set down at the positions of the corresponding elements in a rectangular array, the following checkerboard of $+$ and $-$ signs is obtained:

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad [16]$$

This pictorial statement for the signs of the cofactors is sometimes referred to as the "checkerboard rule."

4. Laplace's development

In the manipulation of determinants, and sometimes to facilitate their numerical evaluation, a process of development formulated by Laplace is frequently useful. It may be stated in the following form:

If all the minors are formed from a selected set of rows or columns of a determinant and the products of these minors with their respective algebraic complements are added, the resulting sum is equal to the determinant.

If a single row is selected, this development reads

$$A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \qquad (i = 1, \; 2, \; \cdots n) \qquad [17]$$

For a single column, the result is written

$$A = a_{1k}A_{1k} + a_{2k}A_{2k} + \cdots + a_{nk}A_{nk} \qquad (k = 1, \; 2, \; \cdots n) \qquad [18]$$

In Eq. 17 the determinant is represented by the sum of the products of the elements of any row with their respective cofactors. In Eq. 18 a corresponding summation is carried out with respect to the elements and cofactors of any column. This simplest form for the Laplace development, which is also called an expansion of the determinant along one of its rows or columns, is the one most frequently used.

It may be of interest, however, to illustrate a more complicated example of this type of development. Let the following fourth-order determinant

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \qquad [19]$$

be developed through the selection of the first two columns for the formation of minors. All possible two-rowed minors are systematically formed as the rows: 1, 2; 1, 3; 1, 4; 2, 3; 2, 4; 3, 4 are selected from these columns. The sign-controlling factors of the corresponding algebraic complements are respectively:

$$\begin{aligned}
(-1)^{1+2+1+2} &= +1\\
(-1)^{1+2+1+3} &= -1\\
(-1)^{1+2+1+4} &= +1\\
(-1)^{1+2+2+3} &= +1\\
(-1)^{1+2+2+4} &= -1\\
(-1)^{1+2+3+4} &= +1
\end{aligned}$$

With the terms written down in this order, the development reads

$$\begin{aligned}
A = &\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \times \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \times \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix}\\
&+ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} \times \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \times \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix}\\
&- \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} \times \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \times \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}
\end{aligned} \qquad [20]$$

By means of the Laplace development a determinant may evidently be evaluated in a variety of ways. One possible method of evaluation consists in repeatedly applying the simplest form of development given by Eqs. 17 and 18. In the first step of this process, the development is given by the sum of n terms, each of which is the product of an element and an $(n-1)$-rowed cofactor. In the second step, each of these cofactors is similarly developed, thus yielding for the determinant A a sum of $n(n-1)$ terms, each of which consists of the product of two elements and an $(n-2)$-rowed cofactor. As the process is continued one recognizes that the final evaluation of A is given by the sum of $n!$ terms, each of which consists of the product of n elements.

The determinant is, therefore, a rational integral function, homogeneous, and of the nth degree in its elements. In any term of the final evaluated form, the appearance of the product of an element with another element of the *same* row or column is not possible. This fact is readily appreciated by noting in the term $a_{12}A_{12}$, for example, that the cofactor A_{12} contains no elements of the first row or second column. Hence none of these elements can subsequently appear in a term containing a_{12}. The determinant is, therefore, a *linear* function in the elements of any one row or column.

5. Other methods of evaluation in numerical or functional form

The evaluation of a determinant by means of the Laplace development, although useful for numerous analytic investigations, is a long and tedious process. The solution of simultaneous linear equations by means of determinants, as discussed in Art. 8, is usually found in numerical problems to involve a larger number of component operations than a systematic process of elimination. This situation is true even when the determinant and cofactors are evaluated by the method given in Art. 2, although this method parallels the elimination process in its essential steps.

Alternative abbreviated methods of solving such equations are given in Arts. 7 and 11 of Ch. II. From a broader standpoint it is well to be familiar with numerous processes of evaluating determinants, so that the particular conditions of a specific problem may be met most expeditiously. In this regard the following remarks may also prove useful.

Evaluations of two of the simplest cases by means of the Laplace development method are written down so that their resultant forms may be examined.

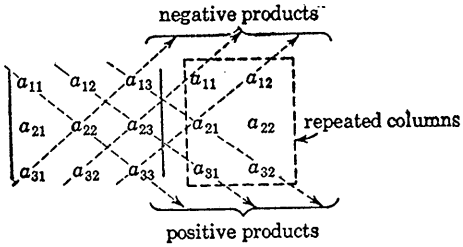
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \qquad [21]$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{matrix} \qquad [22]$$

By inspection of Eq. 21 it may be said that the value of a two-rowed determinant is given by the product of the elements on the principal diagonal less the product of the elements on the conjugate diagonal (lower left to upper right) as indicated in the following by arrows:

$$\begin{array}{ccc} \text{principal diagonal} \searrow & & \swarrow \text{negative product} \\ & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \\ \text{conjugate diagonal} \nearrow & & \nwarrow \text{positive product} \end{array} \qquad [23]$$

This is called the *diagonal product rule*. It is applicable in extended form to the evaluation of a three-rowed determinant. Here there are three positive and three negative products, the positive ones being formed by elements on the principal and adjacent parallel diagonals and the negative ones by elements on the conjugate and adjacent parallel diagonals in a manner which is more easily understood if the first two columns are repeated so that the arrows may continue straight, thus:



[24]

The result is seen to check with Eq. 22.

An extension of this rule does not yield the value of fourth and higher order determinants, as may readily be appreciated from the fact that the number of terms in the final evaluation must be $n!$, whereas the diagonal product rule yields only $2n$ terms. If $n = 4$, there remain $4! - 2 \times 4 = 16$ terms unaccounted for after all diagonal products have been formed.

From a more comprehensive study of determinants, it is seen that all the terms in the final evaluation may be found by writing down the group of elements on the principal diagonal

$$a_{11} \quad a_{22} \quad a_{33} \cdots a_{nn}$$

and carrying out all permutations of the first subscripts, keeping the second subscripts fixed, or vice versa. In either case there are as many different products as there are permutations of n things taken n at a time, which is $n!$

In this process, the algebraic signs of the various terms are controlled by the rule that all permutations formed by an *even* number of inversions of the permuted subscripts represent *positive* terms, all others being negative. Thus for $n = 4$ the possible permutations are

Even number of inversions				*Odd number of inversions*			
1	2	3	4	2	1	3	4
2	3	1	4	2	3	4	1
1	3	4	2	1	3	2	4
3	1	2	4	3	1	4	2
3	4	1	2	3	2	1	4
3	2	4	1	3	4	2	1
2	1	4	3	1	2	4	3
2	4	3	1	2	4	1	3
1	4	2	3	1	4	3	2
4	1	3	2	4	1	2	3
4	2	1	3	4	3	1	2
4	3	2	1	4	2	3	1

The twenty-four terms written with these as the first or second set of indexes, the fixed set being 1 2 3 4, and prefixed with algebraic signs according to the stated rule, represent the evaluation of the fourth-order determinant.

6. Bordered determinants

A determinant of given order may be transformed into a determinant of higher order without changing its value, as is readily seen by applying the ideas of the Laplace development to the following example:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ 0 & 1 & \beta_1 & \beta_2 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{vmatrix} \quad [25]$$

The elements α_0, α_1, α_2, β_1, β_2 may have any values. The process can evidently be varied by placing the zeros to the right or above or below the rectangle containing the a_{ik}'s. The resulting form is referred to as a *bordered* determinant.

7. Products of determinants

The product of two determinants of like order can be expressed as a single determinant of the same order. If the two determinants are initially not of the same order, one of them can be bordered. In the present discussion the determinants can, therefore, be assumed to have the same order.

The procedure for obtaining the elements of the product determinant is best illustrated by means of a simple example. By the Laplace development the transformation of the following product is justified:

$$A \times B = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \times \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{vmatrix} \quad [26]$$

According to rule IV, Art. 1, the resultant fourth-order determinant may be modified in the following manner without changing its value. First the b_{11}-multiplied elements of the first row and the b_{12}-multiplied elements of the second row are added to the corresponding elements of the third row, giving

$$A \times B = \begin{vmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \\ c_{11} & c_{21} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} \end{vmatrix} \quad [27]$$

in which

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{21}b_{12} \\ c_{21} &= a_{12}b_{11} + a_{22}b_{12} \end{aligned} \quad [28]$$

The object of this transformation is to produce zeros in place of the elements b_{11} and b_{12} in the fourth-order determinant of Eq. 26. Now both the b_{21}-multiplied elements of the first row and the b_{22}-multiplied elements of the second row are added to the corresponding elements of the fourth row, giving

$$A \times B = \begin{vmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \\ c_{11} & c_{21} & 0 & 0 \\ c_{12} & c_{22} & 0 & 0 \end{vmatrix} \qquad [29]$$

where

$$\begin{aligned} c_{12} &= a_{11}b_{21} + a_{21}b_{22} \\ c_{22} &= a_{12}b_{21} + a_{22}b_{22} \end{aligned} \qquad [30]$$

By the method of Laplace's development the determinant of Eq. 29 is simply

$$A \times B = \begin{vmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \qquad [31]$$

the second form being obtained by an interchange of rows and columns. Examining the expressions for the elements c_{ik}, as given by Eqs. 28 and 30, it is observed that they are formed by multiplying successive elements in the columns of the determinant A by successive elements in the rows of the determinant B and adding the results, the specific columns and rows involved being indicated by the first and second subscripts respectively on c_{ik}. Thus c_{11} is formed from the elements of the first column of A and those of the first row of B; c_{12} is formed from the elements of the first column of A and those of the second row of B, etc. More briefly, the c's are said to be formed by multiplying the columns of A by the rows of B.

Since the individual values of the determinants A and B are unchanged by writing their rows as columns, it is clear that the value of the product determinant is unaltered if its elements are formed by multiplying either the rows *or* columns of A by either the rows *or* columns of B. The elements of this resulting determinant may, therefore, be formed in any of *four* different ways. Although the individual elements thus obtained are different, the value of the resultant determinant remains the same.

A straightforward extension of the method used in the above example shows that the process of forming the elements of a determinant representing the product of two given determinants A and B follows the same general rules when A and B have any order. This result is summarized in the statement

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix} = \begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots \\ c_{n1} & & c_{nn} \end{vmatrix} \qquad [32]$$

with any one of the following four relations:

$$c_{ik} = a_{i1}b_{k1} + a_{i2}b_{k2} + \cdots + a_{in}b_{kn} \tag{33}$$

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} \tag{34}$$

$$c_{ik} = a_{1i}b_{k1} + a_{2i}b_{k2} + \cdots + a_{ni}b_{kn} \tag{35}$$

$$c_{ik} = a_{1i}b_{1k} + a_{2i}b_{2k} + \cdots + a_{ni}b_{nk} \tag{36}$$

On the right-hand sides of the last four expressions, the indexes i and k have fixed values for any one of the c's. Since the first index refers to a row and the second to a column, it is recognized that the four representations for c_{ik} above correspond respectively to multiplying rows of A by rows of B, rows of A by columns of B, columns of A by rows of B, and columns of A by columns of B.

8. Linear Equations

One of the most important uses for determinants is in the solution of linear simultaneous equations. A set of n such equations involving n unknowns reads

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{array}\right\} \tag{37}$$

The object is to determine the values $x_1 \cdots x_n$ in terms of the values $y_1 \cdots y_n$ and the coefficients a_{ik} which are the elements of the determinant of this system of equations. This determinant is given by Eq. 1.

Let the equations in the set (Eq. 37) be multiplied successively by the cofactors $A_{11}, A_{21}, \cdots + A_{n1}$. Subsequent addition yields the equation

$$\begin{array}{l} (a_{11}A_{11}+a_{21}A_{21}+\cdots+a_{n1}A_{n1})x_1 \\ +(a_{12}A_{11}+a_{22}A_{21}+\cdots+a_{n2}A_{n1})x_2 \\ +(a_{13}A_{11}+a_{23}A_{21}+\cdots+a_{n3}A_{n1})x_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +(a_{1n}A_{11}+a_{2n}A_{21}+\cdots+a_{nn}A_{n1})x_n = A_{11}y_1+A_{21}y_2+\cdots+A_{n1}y_n \end{array} \tag{38}$$

Here the coefficient of x_1 is recognized as the Laplace development of the determinant A with respect to the elements of the first column, as given by Eq. 18 for $k = 1$. Similarly, the coefficient of x_2 is seen to be the Laplace development, with respect to the elements of the first column, of a determinant which is formed from A by replacing the elements of the first column by those of the second column. This coefficient, therefore, is the determinant

$$\begin{vmatrix} a_{12} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ & \cdot & & \cdots & \cdot \\ & \cdot & \cdot & \cdots & \cdot \\ a_{n2} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{39}$$

which by rule VI, Art. 1, has the value zero.

Similarly, the coefficient of x_3 is equal to the determinant A with its first column replaced by the third column. This is likewise zero, as are the coefficients of all the remaining x's. Equation 38 is, therefore, equivalent to

$$Ax_1 = A_{11}y_1 + A_{21}y_2 + \cdots + A_{n1}y_n \tag{40}$$

whence

$$x_1 = \frac{A_{11}y_1 + A_{21}y_2 + \cdots + A_{n1}y_n}{A} \tag{41}$$

In like manner the solution for x_2 may be obtained by multiplying the equations in the set 37 by the cofactors $A_{12}, A_{22}, \cdots A_{n2}$, respectively and adding the results. The coefficient of x_2 then equals A, and the remaining ones are zero. Hence there results

$$x_2 = \frac{A_{12}y_1 + A_{22}y_2 + \cdots + A_{n2}y_n}{A} \tag{42}$$

This result may be stated in general terms by assuming Eqs. 37 to be multiplied respectively by the cofactors $A_{1k}, A_{2k}, \cdots A_{nk}$ and adding the results. The coefficient of x_k then equals A, and the remaining ones are zero, so that

$$x_k = \frac{A_{1k}y_1 + A_{2k}y_2 + \cdots + A_{nk}y_n}{A} \tag{43}$$

For $k = 1, 2, \cdots n$, this represents the desired solutions.

The numerator of Eq. 43 is recognized as the Laplace development of a determinant which is formed from A by replacing its kth column by the column of y's appearing on the right-hand sides of Eqs. 37. Thus the result, Eq. 43, may be written

$$x_k = \frac{\begin{vmatrix} a_{11} & \vdots & a_{1,k-1} & y_1 & a_{1,k+1} & \vdots & a_{1n} \\ a_{21} & \vdots & a_{2,k-1} & y_2 & a_{2,k+1} & \vdots & a_{2n} \\ \cdot & \vdots & \cdot & \cdot & \cdot & \vdots & \cdot \\ \cdot & \vdots & \cdot & \cdot & \cdot & \vdots & \cdot \\ \cdot & \vdots & \cdot & \cdot & \cdot & \vdots & \cdot \\ a_{n1} & \vdots & a_{n,k-1} & y_n & a_{n,k+1} & \vdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}} \qquad [44]$$

A statement describing this form of the solution is known as *Cramer's rule.*

A significant feature in the derivation of these solutions is a recognition of the validity of the relation

$$a_{1i}A_{1k} + a_{2i}A_{2k} + \cdots + a_{ni}A_{nk} = \begin{cases} A & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \qquad [45]$$

which justifies the step from Eq. 38 to Eq. 40. The companion relation, which is established in an analogous fashion, reads

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = \begin{cases} A & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \qquad [46]$$

For $i \neq k$ this represents the Laplace development of a determinant whose ith and kth *rows* are identical. Equations 45 and 46 may he looked upon as an extension of the relations expressed by Eqs. 18 and 17 respectively.

The solutions to the set of Eqs. 37 may be written in the form

$$\left.\begin{aligned} b_{11}y_1 + b_{12}y_2 + \cdots + b_{1n}y_n &= x_1 \\ b_{21}y_1 + b_{22}y_2 + \cdots + b_{2n}y_n &= x_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_{n1}y_1 + b_{n2}y_2 + \cdots + b_{nn}y_n &= x_n \end{aligned}\right\} \qquad [47]$$

in which, according to Eq. 43, the coefficients are given by

$$b_{rs} = \frac{A_{sr}}{A} \qquad [48]$$

In this result it is significant to note the reversal of the subscripts on A_{sr} as compared with those on b_{rs}.

In case the elements of the determinant A fulfill the condition

$$a_{ik} = a_{ki} \qquad [49]$$

the determinant is said to be *symmetrical* about its principal diagonal. It is clear from rule VIII, Art. 2, that the minors and cofactors of A then also have this property, that is,

$$A_{ik} = A_{ki} \tag{50}$$

and it then follows from Eq. 48 that the determinant of the system of Eqs. 47 is likewise symmetrical. In that case the subscript order in Eq. 48 is, of course, unimportant.

Equations 37 and 47 are mutually inverse systems. The one set represents the solution of the other. Consequently by analogy to Eq. 48 the coefficients of Eqs. 37 may be written

$$a_{ik} = \frac{B_{ki}}{B} \tag{51}$$

in which

$$B = \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix} \tag{52}$$

is the determinant of the system of Eqs. 47 and B_{ki} its cofactors.

Evaluating the product AB of the determinants of the inverse systems of Eqs. 37 and 47 by substituting Eq. 48 into Eq. 34 gives

$$c_{ik} = \frac{a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn}}{A} \tag{53}$$

Reference to Eq. 46 shows that

$$c_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \tag{54}$$

Hence for the determinants of these inverse systems

$$\begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix} \times \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \tag{55}$$

The determinants have inverse values.

9. Conditions for the existence of solutions

The conditions under which a system of simultaneous equations such as the set 37 can have solutions may be seen from the form of these solutions as expressed by Eq. 44. For arbitrary values of $y_1 \cdots y_n$, a necessary and sufficient condition for the existence of these solutions is that the determinant A shall not be zero. If A is zero, in general no solutions exist.

They may, however, still exist in case the determinant in the numerator of Eq. 44 is also zero, as it is if the y's satisfy the conditions

$$y_i = \alpha_1 a_{i1} + \alpha_2 a_{i2} + \cdots + \alpha_n a_{in} \tag{56}$$

in which $\alpha_1, \cdots \alpha_n$ are arbitrary factors. The column of y's in the numerator of Eq. 44 is then a linear combination of the other columns of this determinant and can, by repeated application of rule IV, Art. 1, be reduced to a column of zeros.

When the y's are expressed by the relations 56, the Eqs. 37 can be rewritten in the form

$$\left.\begin{array}{l} a_{11}(x_1 - \alpha_1) + a_{12}(x_2 - \alpha_2) + \cdots + a_{1n}(x_n - \alpha_n) = 0 \\ a_{21}(x_1 - \alpha_1) + a_{22}(x_2 - \alpha_2) + \cdots + a_{2n}(x_n - \alpha_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}(x_1 - \alpha_1) + a_{n2}(x_2 - \alpha_2) + \cdots + a_{nn}(x_n - \alpha_n) = 0 \end{array}\right\} \qquad [57]$$

A special case of this sort occurs when all the y's are zero. Then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, and Eqs. 57 become identical with Eqs. 37 for $y_1 = y_2 = \cdots = y_n = 0$. This is called the corresponding *homogeneous* set of equations. For these, the Cramer rule as expressed by Eq. 44 yields the solutions in indeterminate form except when $A \neq 0$, but then the solutions are all zero. They are spoken of as the trivial solutions because their existence is at once evident upon inspection of the homogeneous equations.

Nontrivial solutions to the homogeneous set of Eqs. 57 exist only if the determinant is zero, but Cramer's rule, Eq. 44, is of no use in determining them. In order to see how this difficulty might be overcome, it is helpful to consider the Eqs. 37 for the special case that one of the y's, for example, y_i, alone is different from zero. Then, if it is assumed for the moment that the determinant is not zero and Cramer's rule or Eq. 43 is applied, it is found that

$$x_k = \frac{A_{ik}y_i}{A} \qquad [58]$$

In this special case the ratio of any two unknowns is given by

$$\frac{x_r}{x_s} = \frac{A_{ir}}{A_{is}} \qquad [59]$$

which is independent of the values of both y_i and the determinant A. It may be inferred, therefore, that Eq. 59 holds also when both y_i and A are zero.

The correctness of this conclusion is demonstrated in a rigorous fashion in Art. 7, Ch. III. Meanwhile it is interesting to note that when the homogeneous set of equations has nontrivial solutions, these are not uniquely determined by Eq. 59, in which only the ratios of the unknowns are given. Any value can be assigned to one of them, and the remaining unknowns are then expressed in terms of this one.

The general conditions for the existence of solutions may be discussed as follows. The fact that the inhomogeneous equations 37 have solutions only when the determinant is not zero simply amounts to stating that these equations must be independent. If one is a linear combination of the others (in this case the determinant vanishes), then, speaking in physical terms, the data are insufficient to yield an explicit answer.

In case the right-hand members of the Eqs. 37 are zero and all the equations are independent ($A \neq 0$), the system is overspecified from a physical point of view. The situation is like a deadlock, and nothing can happen; that is, only zero values for the unknowns can satisfy the equations. If one of the equations is a linear combination of the others ($A = 0$), this one may be discarded and one of the terms in each remaining equation, for example, that with x_n, transposed to the right-hand side. These $(n - 1)$ equations may then be solved for the $(n - 1)$ remaining unknowns in terms of x_n, provided the determinant of this reduced set is not zero. If it is zero, this method fails, but so does the corresponding form of solution expressed by Eq. 59.

This kind of failure in the method of solution indicates that two independent sets of solutions exist, but it is difficult to obtain a clear picture of this situation without the aid of such appropriate geometrical interpretations as are given in Ch. III. The present discussion is completed in that chapter. The material of the following article, however, is helpful in summarizing some of the characteristics of the determinant which are pertinent to the present problem.

10. The rank of a determinant

If in the determinant A, Eq. 1, there exists among the elements of each column the *same* linear relation

$$\alpha_1 a_{1k} + \alpha_2 a_{2k} + \cdots + \alpha_n a_{nk} = 0 \qquad (k = 1, 2, \cdots n) \qquad [60]$$

in which the α's are arbitrary nonzero factors, the elements of any row are expressible as linear combinations of the corresponding elements of the remaining rows. If some of the factors $\alpha_1 \cdots \alpha_n$ are zero, this fact still holds for the elements of some of the rows. By repeated modification of the determinant according to rule IV, Art. 1, any one of these rows can be reduced to a row of zeros. Hence it is seen that a determinant is zero if a relation of the form given by Eq. 60 exists in which at least one of the α's is different from zero.

Conversely, if the determinant is known to be zero, it is surely possible to find a relation of the form of Eq. 60, as is clear if Eq. 60 is written out for all the k-values, thus

$$\left.\begin{array}{l} a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{n1}\alpha_n = 0 \\ a_{12}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{n2}\alpha_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{nn}\alpha_n = 0 \end{array}\right\} \qquad [61]$$

This is simply the homogeneous system of equations in which the determinant has its columns written as rows. It is called the *transposed* set of equations. According to property VIII, Art. 2, the transposed determinant is zero if the given one is. Hence, by the method discussed in the previous article, a set of nonzero α-values can be found to satisfy Eqs. 61.

It is then also possible to find a set of nonzero β-values satisfying the relations

$$\beta_1 a_{i1} + \beta_2 a_{i2} + \cdots + \beta_n a_{in} = 0 \qquad (i = 1, 2, \cdots n) \qquad [62]$$

since they are simply a set of solutions to the untransposed system of homogeneous equations.

It follows that if the elements in a row of a determinant are expressible as linear combinations of the corresponding elements of the remaining rows, the elements in a column of this determinant are similarly expressible.

When a determinant of order n is not zero, so that neither the relations 60 nor 62 exist (except for zero α- and β-values), it is said to have the *rank* $r = n$. If the determinant is zero but at least one of its $(n - 1)$-rowed minors is different from zero, one set of α- and β-values exist which satisfy the Eqs. 60 and 62. The determinant is then said to have the rank $r = n - 1$. If the determinant and *all* its $(n - 1)$-rowed minors are zero but at least one of its $(n - 2)$-rowed minors is different from zero, it has the rank $r = n - 2$. In this case it is possible to find *two* independent sets of α- and β-values satisfying Eqs. 60 and 62, and the homogeneous system of equations has two independent sets of solutions, as is discussed further in Ch. III. Corresponding statements for the condition that the rank is $n - 3$, $n - 4$, etc., are obvious. Finally, if the rank of a determinant is zero, all its elements are zero.

The definition of rank facilitates the discussion of possible solutions to a system of linear equations. Thus the nonhomogeneous Eqs. 37 have solutions only when the rank of their determinant is n. The corresponding homogeneous equations have nontrivial solutions only if the rank is less than n, and they have p sets of independent solutions if the rank is $n - p$. A more lucid exposition of this line of reasoning is found in the chapters immediately following.

PROBLEMS

1. Determine the rank of each of the following determinants:

(a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & 9 & 20 & 18 \\ 8 & 19 & 40 & 37 \\ 13 & 20 & 47 & 34 \\ 20 & 22 & 59 & 31 \end{vmatrix}$ (c) $\begin{vmatrix} 5 & 0 & -2 & 9 \\ 0 & 5 & -11 & 7 \\ -2 & -11 & 25 & -19 \\ 9 & 7 & -19 & 26 \end{vmatrix}$

(d) $\begin{vmatrix} 12 & -5 & 8 \\ 4 & 0 & 1 \\ -4 & 3 & -4 \end{vmatrix}$ (e) $\begin{vmatrix} 39 & 24 & 12 & 5 \\ 24 & 21 & 2 & 2 \\ 12 & 2 & 10 & 3 \\ 5 & 2 & 3 & 1 \end{vmatrix}$ (f) $\begin{vmatrix} 6.5 & 6.5 & 4 \\ 6.5 & 17 & 8 \\ 4 & 8 & 4 \end{vmatrix}$

2. Transform each of the above determinants to the triangular form, thus finding their values and checking the answers to Prob. 1.

3. For each determinant in Prob. 1 whose rank is less than its order, find relations of the form of Eqs. 60 and 62.

4. Using determinants (e) and (f) of Prob. 1, write down corresponding sets of simultaneous equations, denoting the right-hand members by $y_1, y_2, \cdots$ as in Eqs. 37. Solve these equations by means of Cramer's rule.

5. Repeat the solutions of the equations in Prob. 4 by means of a systematic elimination process. Compare the total number of multiplications and additions with those required in the solutions using Cramer's rule.

6. Evaluate the following determinant according to the pattern shown in Eq. 20.

$$\begin{vmatrix} 2 & 1 & 4 & 3 \\ 6 & -1 & 2 & -4 \\ 3 & -2 & 5 & 1 \\ -5 & 6 & 4 & -1 \end{vmatrix}$$

Repeat the evaluation through reduction to the triangular form and compare the total numbers of multiplications and additions required in the two methods. Derive a formula giving the total numbers of multiplications and of additions required for the evaluation of an nth order determinant by the method involving its reduction to the triangular form.

7. Determine the solutions to a set of equations (like Eqs. 37) having the ac-

$$\begin{vmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -0.866 & 0.289 & 0.289 & 0.289 \\ 0 & -0.258 & 0.408 & 0.408 \\ 0 & 0 & -0.707 & 0.707 \end{vmatrix}$$

companying determinant. Compare the set of equations representing the solutions with the given equations and note any obvious mutual relations existing between these two sets of equations.

8. Given the two sets of equations

$$\sum_{k=1}^{n} a_{ik}x_k = y_i \qquad (i = 1, 2, \cdots n)$$

and

$$\sum_{s=1}^{n} b_{ks}y_s = x_k \qquad (k = 1, 2, \cdots n)$$

which are solutions of each other, show that the corresponding determinants A and B have inverse values; that is, $AB = 1$. A proof may be based upon the rule for forming the product of two determinants.

9. In the following special nth order determinant

$$D_n = \begin{vmatrix} \alpha & 1 & 0 & 0 \cdots 0 \\ 1 & \alpha & 1 & 0 \cdots 0 \\ 0 & 1 & \alpha & 1 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 \cdots 0 & 1 & \alpha \end{vmatrix}$$

all the elements of the principal diagonal are equal to α; those on the diagonals immediately above and below the principal diagonal are unity, and all the rest are zero. Derive the following recursion formula:

$$D_n = \alpha D_{n-1} - D_{n-2}$$

applicable for $n = 1, 2, \cdots$ with the definitions: $D_0 = 1$ and $D_1 = \alpha$. From this recursion formula obtain an explicit expression for the determinant of order n which reads

$$D_n = \frac{\sinh (n+1)\gamma}{\sinh \gamma}, \quad \text{with } \gamma = \cosh^{-1}\frac{\alpha}{2}$$

10. If the first and the last elements on the principal diagonal of the determinant in Prob. 9 are replaced by $\alpha/2$, show that the resulting determinant has the value

$$D_n = \sinh (n-1)\gamma \cdot \sinh \gamma$$

while if *only the first or last* of these elements is $\alpha/2$, the value is given by

$$D_n = \cosh n\gamma$$

11. Consider the determinant

$$D = \begin{vmatrix} d_1 & 1 & 0 & 0 & 0 & 0 \cdots \\ -1 & d_2 & 1 & 0 & 0 & 0 \cdots \\ 0 & -1 & d_3 & 1 & 0 & 0 \cdots \\ 0 & 0 & -1 & d_4 & 1 & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = K(d_1, \cdots d_n)$$

and show that this function $K(d_1, \cdots d_n)$, called a simple *continuant*, possesses the recursion formula

$$K(d_1, \cdots d_n) = d_n K(d_1, \cdots d_{n-1}) + K(d_1, \cdots d_{n-2})$$

with

$$K(d_1) = d_1, \qquad \text{and} \qquad K(0) = 1$$

12. Denote by D_{11} the cofactor formed through canceling the first row and column in the determinant given in Prob. 11. Make use of the results of Prob. 11 to show that

$$\frac{D}{D_{11}} = \frac{K(d_1, \cdots d_n)}{K(d_2, \cdots d_n)} = d_1 + \frac{1|}{|d_2} + \frac{1|}{|d_3} + \frac{1|}{|d_4} + \cdots$$

which is known as a continued fraction.

13. Show that

$$K(d_1, \cdots d_n) = K(d_n, \cdots d_1)$$

and that the recursion formula given in Prob. 11 may alternatively be written

$$K(d_1, \cdots d_n) = d_1 K(d_2, \cdots d_n) + K(d_3, \cdots d_n)$$

14. Show that the partial derivative of a determinant with respect to one of its elements equals the cofactor of that element. In symbols: $\partial A/\partial a_{sk} = A_{sk}$.

15. Consider the cofactors A_{sk} and A_{sr} corresponding to the elements a_{sk} and a_{sr} in the same row of a determinant A. Show that the sum $(A_{sk} + A_{sr})$ is equal to A_{sk} with the column involving the elements a_{sr} replaced by one with elements $a_{sr} + a_{sk}(-1)^{k-r}$, or to A_{sr} with the column involving the elements a_{sk} replaced by one in which the elements are $a_{sk} + a_{sr}(-1)^{k-r}$.

16. Using the type of reasoning involved in the previous problem, show that the fourth-order determinant may be written as the following sum of two third-order determinants:

$$\begin{vmatrix} (a_{11}a_{22} - a_{12}a_{21}) & a_{23} & a_{24} \\ (a_{11}a_{32} - a_{12}a_{31}) & a_{33} & a_{34} \\ (a_{11}a_{42} - a_{12}a_{41}) & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} & (a_{13}a_{24} - a_{14}a_{23}) \\ a_{31} & a_{32} & (a_{13}a_{34} - a_{14}a_{33}) \\ a_{41} & a_{42} & (a_{13}a_{44} - a_{14}a_{43}) \end{vmatrix}$$

or as a variety of obvious modifications of these forms.

17. Express in determinant form the condition that the three straight lines defined by

$$a_{11}x + a_{12}y + a_{13} = 0$$

$$a_{21}x + a_{22}y + a_{23} = 0$$

$$a_{31}x + a_{32}y + a_{33} = 0$$

shall intersect at a common point.

18. In the XY-plane the origin O, and the two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ determine a triangle. Show that the area of this triangle is expressible by means of the determinant

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

19. Using the result of the previous problem show that the area of a triangle determined by the points (x_1,y_1), (x_2,y_2) and (x_3,y_3) is expressible as

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Write the condition for which these three points lie on the same straight line.

20. $ax + by + cz + d = 0$ is the equation of a plane. Its intercepts on the coordinate axes are: $x = -d/a$, $y = -d/b$, $z = -d/c$. Let n denote the normal from the origin to the plane. Its direction cosines are:

$$\cos(n,x) = \frac{n}{-\left(\frac{d}{a}\right)} \qquad \cos(n,y) = \frac{n}{-\left(\frac{d}{b}\right)} \qquad \cos(n,z) = \frac{n}{-\left(\frac{d}{c}\right)}$$

Since the sum of the squares of these cosines equals unity, one has

$$n = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

Consider a point $P_0(x_0,y_0,z_0)$ for which $ax_0 + by_0 + cz_0 + d = D_0$. Subtracting this equation from the original one gives $a(x - x_0) + b(y - y_0) + c(z - z_0) + D_0 = 0$, from which it is clear that the length of the normal dropped from the point P_0 to the plane is

$$n_0 = \frac{D_0}{\sqrt{a^2 + b^2 + c^2}}$$

These results and that of the previous problem are to be made use of to show: (a) That the equation of a plane passing through the three points $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$ and $P_3(x_3,y_3,z_3)$ may be written in the form

$$D(x,y,z) = \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

(b) That the cofactors of the first three elements of the first row, that is,

$$\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \quad \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

are equal to the projections of the area of the triangle $P_1P_2P_3$ upon planes normal to the X-, Y-, Z-axes respectively and that the square root of the sum of the squares of these cofactors equals the area of this triangle. (c) That $D_0 = D(x_0,y_0,z_0)$ divided by this square root equals the normal distance of a point $P_0(x_0,y_0,z_0)$ from the plane, and hence that the volume of the tetrahedron whose vertexes are the points $P_0P_1P_2P_3$ is equal to one-sixth the value of the determinant D_0.

21. Three planes passing through the origin are represented by the equations

$$a_1x + a_2y + a_3z = 0$$
$$b_1x + b_2y + b_3z = 0$$
$$c_1x + c_2y + c_3z = 0$$

Express in determinant form the condition for which these planes intersect in the same straight line and find the expressions for the direction cosines of this line. Given any two planes, what are the direction cosines of their intersection?

22. Write the following determinant as a polynomial in λ:

$$P(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

and obtain expressions for the coefficients of the polynomial in terms of the determinant A and its cofactors. Indicate the forms of these expressions for an nth degree polynomial.

23. Prove that

(a) $$\begin{vmatrix} (a_{11} + b_{11}) & c_{12} & \cdots & c_{1n} \\ (a_{21} + b_{21}) & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ (a_{n1} + b_{n1}) & c_{n2} & \cdots & c_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & c_{12} & \cdots & c_{1n} \\ a_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & c_{12} & \cdots & c_{1n} \\ b_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix}$$

(b) If $\qquad c_{ik} = a_{ik} + jb_{ik} \qquad i, k = 1, 2, 3 \qquad j = \sqrt{-1}$

then

$$|c_{ik}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & b_{12} & b_{13} \\ a_{21} & b_{22} & b_{23} \\ a_{31} & b_{32} & b_{33} \end{vmatrix} - \begin{vmatrix} b_{11} & a_{12} & b_{13} \\ b_{21} & a_{22} & b_{23} \\ b_{31} & a_{32} & b_{33} \end{vmatrix} - \begin{vmatrix} b_{11} & b_{12} & a_{13} \\ b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & a_{33} \end{vmatrix}$$

$$+j\left\{\begin{vmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ a_{31} & a_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}\right\}$$

(c) If $$d_{ik} = a_{ik} - jb_{ik} \qquad i, k = 1, 2, 3$$

then

$$|d_{ik}| = \boxed{c_{ik}} \qquad \text{complex conjugate of } |c_{ik}|$$

24. Let

$$c_{ik} = a_{ik} + jb_{ik} \qquad i, k = 1, 2, \cdots n \qquad j = \sqrt{-1}$$

Prove that $|c_{ik}|$ is a complex number with the following law of formation:

(a) $|c_{ik}|$ is equal to the sum of 2^n determinants of order n, 2^{n-1} being real and 2^{n-1} pure imaginary.

(b) The real determinants have an even, or zero, number of "b" columns. The imaginary ones have an odd number of "b" columns.

(c) If m is the number of "b" columns in any determinant in the expansion, the sign and complex character of the determinant are given by j^m. For a given value of m, there are several determinants which contain m "b" columns; their number is given by

$$\frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}$$

(d) By using the above properties show incidentally that

$$1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots + \frac{n!}{n!} = 2^n$$

25. If $y_{ik} = y_{ik}(t)$, for $i, k = 1, 2, \cdots n$, are n^2 single-valued, differentiable functions of the independent variable t, show that

$$\frac{d}{dt}|y_{ik}| = \begin{vmatrix} y'_{11} & y_{12} & \cdots & y_{1n} \\ y'_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y'_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} & y'_{12} & \cdots & y_{1n} \\ y_{21} & y'_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n1} & y'_{n2} & \cdots & y_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} y_{11} & y_{12} & \cdots & y'_{1n} \\ y_{21} & y_{22} & \cdots & y'_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n1} & y_{n2} & \cdots & y'_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} y'_{11} & y'_{12} & \cdots & y'_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y'_{21} & y'_{22} & \cdots & y'_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y'_{n1} & y'_{n2} & \cdots & y'_{nn} \end{vmatrix}$$

in which

$$y'_{ik} = \frac{dy_{ik}}{dt}$$

26. In terms of the n independent functions

$$y_k = y_k(t) \quad \text{for } k = 1, 2, \cdots n$$

(differentiable up to and including the nth order), construct the following determinant (the so-called Wronskian of those functions):

$$\Delta = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y'_1 & y'_2 & y'_3 & \cdots & y'_n \\ y''_1 & y''_2 & y''_3 & \cdots & y''_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \qquad y_k^{(h)} = \frac{d^h}{dt^h}(y_k)$$

If these functions are connected by a linear relation of the form

$$A_1y_1 + A_2y_2 + A_3y_3 + \cdots + A_ny_n = 0$$

in which the A's are constants, show that the above determinant vanishes identically.

Hint. Differentiate the linear relation successively $n - 1$ times so as to obtain

$$\begin{aligned} A_1y'_1 + \cdots + A_ny'_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots& \\ A_1y_1^{(n-1)} + \cdots + A_ny_n^{(n-1)} &= 0 \end{aligned}$$

Together with the original relation, one then has a set of n equations. From these, the value of any function y_k, for example, y_1, and its first $(n - 1)$ derivatives can be obtained. Substitution into the Wronskian, followed by an expansion according to columns, leads to the desired result.

27. Using the determinant, Δ, as defined in Prob. 26, show that

$$\frac{d}{dt}\Delta = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y'_1 & y'_2 & y'_3 & \cdots & y'_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)} & y_2^{(n-2)} & y_3^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

Hint. Use the result of Prob. 25 and observe the resulting structure of the rows.

28. If (with reference to the situation given in Prob. 27) there exists a set of n relationships (differential equations) of the form

$$p_0y_r^{(n)} + p_1y_r^{(n-1)} + p_2y_r^{(n-2)} + \cdots + p_ny_r = 0$$

for

$$r = 1, 2, \cdots n$$

in which the coefficients $p_0, p_1, p_2, \cdots p_n$ are constant or variable, show that

(a) $$\frac{d}{dt}\Delta = -\frac{p_1}{p_0}\Delta$$

(b) $$\Delta = \Delta_0 e^{-\int (p_1/p_0)\,dt}$$

in which Δ_0 is the integration constant.

Hint. Give r the values $1, 2, \cdots n$ and obtain, from each equation, the value of $y_r^{(n)}$. Substitute these values into the last row of the expression in Prob. 27.

29. Given

$x_k = x_k(t)$ for $k = 1, 2, \cdots n$ (a system of n unknown functions of t)

$y_k = y_k(t)$ for $k = 1, 2, \cdots n$ (a system of n known functions of t)

and

a_{ik} for $i, k = 1, 2, \cdots n$ (a collection of n^2 constants)

These quantities are related by the following **system of first-order, first-degree, linear** equations

$$a_{1k}\dot{x}_1 + a_{2k}\dot{x}_2 + \cdots + a_{nk}\dot{x}_n = y_k \quad \textbf{for } k = 1, 2, \cdots n$$

Show that a solution of this system (the particular one) is given by

$$x_k = \frac{A_{1k}\int y_1\,dt + A_{2k}\int y_2\,dt + \cdots + A_{nk}\int y_n\,dt}{|a_{ik}|}$$

in which the A_{ik}'s are cofactors of the determinant $|a_{ik}|$.

30. Show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{n-1} \\ 1 & 3 & 3^2 & \cdots & 3^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & n & n^2 & \cdots & n^{n-1} \end{vmatrix} = 1! \times 2! \times 3! \cdots (n-1)!$$

Hint. Reduce the determinant to the diagonal form. Use Barlow's tables of squares (pages 202 to 206) for the powers of integer numbers and observe the law of formation.

31.

$$u_k = u_k(x_1 \cdots x_n) \quad \text{for } k = 1, 2, \cdots n$$

are n single-valued differentiable functions of the independent variables $x_1, x_2, \cdots x_n$.

The "Jacobian" of these functions is, by definition, the following functional determinant:

$$J = \left(\frac{u_1 u_2 \cdots u_n}{x_1 x_2 \cdots x_n}\right) = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Suppose the variables $x_1 \cdots x_n$ are changed to the new independent variables $z_1 \cdots z_n$ according to the equations of transformation

$$x_k = x_k(z_1 \cdots z_n) \quad \text{for } k = 1, 2, \cdots n$$

The original u_k functions are now

$$u_k = u_k(z_1 \cdots z_n) \quad \text{for } k = 1, 2, \cdots n$$

and their Jacobian with respect to the variables $z_1 \cdots z_n$, for example, J_1, differs from J only in that the variables x are replaced by corresponding z's.

(a) Show that the above Jacobians are connected by the relation

$$J_1 = \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} & \cdots & \frac{\partial x_1}{\partial z_n} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} & \cdots & \frac{\partial x_2}{\partial z_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial z_1} & \frac{\partial x_n}{\partial z_2} & \cdots & \frac{\partial x_n}{\partial z} \end{vmatrix} \times J$$

(b) Extend the above result so as to consider subsequent transformations of the form

$$z_j = z_j(r_1, r_2, \cdots r_n) \quad \text{for } j = 1\ 2, \cdots n$$

$$r_p = r_p(t_1, t_2, \cdots t_n) \quad \text{for } p = 1, 2, \cdots n$$

(c) What happens with the last Jacobian when any intermediate functional determinant is identically zero? *Hint.* Apply the rule for differentiation which reads:

$$\frac{\partial u_r}{\partial z_s} = \sum_{i=1}^{i=n} \frac{\partial u_r}{\partial x_i} \frac{\partial x_i}{\partial z_s} \qquad r, s = 1, 2, \cdots n$$

32. Let

$$a_{jk} = a_{jk}(x_1, x_2, \cdots x_n) \quad \text{for } j, k = 1, 2, \cdots n$$

be a system of n^2 differentiable functions of the independent variables $x_1, x_2 \cdots x_n$. Through the introduction of a new set of independent variables $\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n$ by means of the functional relations

$$x_k = x_k(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) \quad \text{for } k = 1, 2, \cdots n$$

the system of functions a_{jk} in the old variables goes over into the transformed system $\bar{a}_{jk}$ in the new variables.

Accepting the result that a_{jk} goes over into $\bar{a}_{jk}$ in accordance with the law of transformation

$$\bar{a}_{pq} = \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} a_{jk} \frac{\partial x_j}{\partial \bar{x}_p} \frac{\partial x_k}{\partial \bar{x}_q} \qquad (p, q = 1, 2, \cdots n)$$

prove that the determinant $|a_{jk}|$ is transformed as indicated by

$$|\bar{a}_{pq}| = |a_{jk}| \times J^2$$

in which

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial \bar{x}_1} & \frac{\partial x_1}{\partial \bar{x}_2} & \cdots & \frac{\partial x_1}{\partial \bar{x}_n} \\ \frac{\partial x_2}{\partial \bar{x}_1} & \frac{\partial x_2}{\partial \bar{x}_2} & \cdots & \frac{\partial x_2}{\partial \bar{x}_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial \bar{x}_1} & \frac{\partial x_n}{\partial \bar{x}_2} & \cdots & \frac{\partial x_n}{\partial \bar{x}_n} \end{vmatrix}$$

33. The expression for the three-dimensional volume element in a general system of co-ordinates is given by

$$dV = \sqrt{|g_{jk}|}\, d\bar{x}_1\, d\bar{x}_2\, d\bar{x}_3$$

in which

$$g_{jk} = \sum_{\nu=1}^{\nu=n} \frac{\partial x_\nu}{\partial \bar{x}_j} \frac{\partial x_\nu}{\partial \bar{x}_k} \qquad \nu = 1, 2, \cdots n \qquad j = 1, 2, \cdots n \qquad k = 1, 2, \cdots n$$

If the co-ordinate system is orthogonal, the g_{jk} system has the property

$$g_{jk} \begin{cases} \neq 0 & \text{for } j = k \\ = 0 & \text{for } j \neq k \end{cases}$$

Check the values of $\sqrt{|g_{jk}|}$ for the different co-ordinate systems and laws of co-ordinate transformation given in the following table:

Name	Equations of Transformation	$\sqrt{\lvert g_{jk}\rvert}$
Cartesian	$x_1 = \bar{x}_1$ $x_2 = \bar{x}_2$ $x_3 = \bar{x}_3$	1
Circular cylindrical	$x_1 = \bar{x}_1 \cos \bar{x}_2$ $x_2 = \bar{x}_1 \sin \bar{x}_2$ $x_3 = \bar{x}_3$	$\bar{x}_1$
Elliptic cylindrical	$x_1 = c \cosh \bar{x}_1 \cos \bar{x}_2$ $x_2 = c \sinh \bar{x}_1 \sin \bar{x}_2$ $x_3 = \bar{x}_3;\ c = \text{const.}$	$c^2(\cosh^2 \bar{x}_1 - \cos^2 \bar{x}_2)$
Parabolic cylindrical	$x_1 = \frac{1}{2}(\bar{x}_1{}^2 - \bar{x}_2{}^2)$ $x_2 = \bar{x}_1\bar{x}_2$ $x_3 = \bar{x}_3$	$(\bar{x}_1{}^2 + \bar{x}_2{}^2)$
Bipolar cylindrical	$x_1 = \dfrac{a \sinh \bar{x}_1}{\cosh \bar{x}_1 - \cos \bar{x}_2}$ $x_2 = \dfrac{a \sin \bar{x}_2}{\cosh \bar{x}_1 - \cos \bar{x}_2}$ $x_3 = \bar{x}_3;\ a = \text{const.}$	$\dfrac{a^2}{(\cosh \bar{x}_1 - \cos \bar{x}_2)^2}$
Spheroidal	$x_1 = c \cosh \bar{x}_1 \cos \bar{x}_2;\ c = \text{const.}$ $x_2 = c \sinh \bar{x}_1 \sin \bar{x}_2 \cos \bar{x}_3$ $x_3 = c \sinh \bar{x}_1 \sin \bar{x}_2 \sin \bar{x}_3$	$c^3(\cosh^2 \bar{x}_1 - \cos^2 \bar{x}_2) \sinh \bar{x}_1 \sin \bar{x}_2$
Spherical	$x_1 = \bar{x}_1 \cos \bar{x}_2 \sin \bar{x}_3$ $x_2 = \bar{x}_1 \sin \bar{x}_2 \sin \bar{x}_3$ $x_3 = \bar{x}_1 \cos \bar{x}_3$	$\bar{x}_1{}^2 \sin \bar{x}_3$

34. Given the multiple integral

$$I = \int\int \cdots \int F(x_1, x_2, \cdots x_n)\, dx_1\, dx_2 \cdots dx_n$$

in which $x_1, x_2, \cdots x_n$ are the independent variables. If new variables $\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n$

are introduced in accordance with the relations

$$x_k = x_k(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) \quad \text{for } k = 1, 2, \cdots n$$

it can be shown that the above integral becomes

$$I = \int\int \cdots \int JF(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)\, d\bar{x}_1\, d\bar{x}_2 \cdots d\bar{x}_n$$

in which J is the determinant given in Prob. 32.

Compute the value of the determinant J for each set of transformation functions given in the second column of the table in Prob. 33.

35. Given the following system of $m + n$ linear equations involving the $m + n$ unknowns x_λ for $\lambda = 1, 2, \cdots n$, and e_ρ for $\rho = 1, 2, \cdots m$, with $n > m$:

$$\sum_{\mu=1}^{\mu=n} A_{\lambda\mu} x_\mu - e x_\lambda + \sum_{\rho=1}^{\rho=m} e_\rho a_{\rho\lambda} = 0 \quad \text{for } \lambda = 1, 2, \cdots n$$

$$\sum_{\mu=1}^{\mu=n} a_{\rho\mu} x_\mu = 0 \quad \text{for } \rho = 1, 2, \cdots m$$

(a) Write in determinant form, according to the above order of these equations, the condition for the existence of nontrivial solutions.

(b) Show that this determinant is a polynomial in e of the degree $n - m$.

(c) Using Laplace's development with respect to the last m rows, show that the total number of mth-order minors which can be formed is given by

$$\frac{(n + m)(n + m - 1)(n + m - 2) \cdots (n + 1)}{m\,!} \qquad (m < n)$$

and that only

$$\frac{n(n - 1)(n - 2) \cdots (n - m + 1)}{m\,!}$$

of these are not necessarily zero (the rest being identically zero).