I

Review of Differential Calculus

The mathematics of physiology serves two functions. The first is what most people usually think of as mathematics, a set of tools for providing numerical answers to problems. The second is that of a language by which concepts can be easily communicated and handled. As in any language, there is a certain basic vocabulary to be learned. This vocabulary consists largely of definitions that must be learned, just as in learning French, one learns that *poulet* means chicken. There is no logical way to derive this. It is a definition.

Some of the chapters begin with a mathematical introduction that describes the new mathematical concepts introduced in that chapter and which also contains their definitions. Try to think of these definitions as a new vocabulary and learn them as you would learn a vocabulary.

1. Dimensions

At many places throughout this book, the reader will observe that a dimensional equation has been written beneath the usual symbolic equation. It is hoped he will develop the habit of doing this himself at least twice in the solution of each problem: when the physical problem is stated in mathematical terms, and in the solution to the problem. Checking for dimensional balance should be a routine part of the solution of any problem. Remember, if it does not balance dimensionally, it is not correct. There are a few simple rules for manipulating dimensions.

Rule 1: Only quantities of like dimensions can be added or subtracted.

Rule 2: Dimensions multiply and divide in the same manner as numbers.

Example. To find the cost of 3 eggs plus 4 apples, given that E is the cost of eggs per dozen and A the cost of one apple,

$$C \text{ cents} = \frac{1}{4} \text{ dozen } E \frac{\text{cents}}{\text{dozen}} + 4 \text{ apples } A \frac{\text{cents}}{\text{apple}}$$

In the first term dozen cancels dozen, and in the second term apple cancels apple. Both are now in units of cents and can be added.

Rule 3: The easy way to convert from one set of dimensions to another is to write the conversion factor as a fraction whose value is 1. For example, to convert $12\frac{1}{2}$ feet to inches, construct the fraction 12 inches divided by 1 foot, and multiply this by $12\frac{1}{2}$ feet:

$$\frac{12 \text{ inches}}{1 \text{ foot}} \times 12\tfrac{1}{2} \text{ feet} = 150 \text{ inches}$$

The constructed fraction is equal to 1. Similarly, to convert 3 hours into seconds, construct the fraction 60 min divided by 1 hour, which is equal to 1, and a second fraction, 60 sec divided by 1 min, which is equal to 1. Multiply 3 hours by these two fractions, each of which is numerically one:

$$\frac{60 \text{ min}}{1 \text{ hour}} \quad \text{and} \quad \frac{60 \text{ sec}}{1 \text{ min}}$$

$$3 \text{ hours} \frac{60 \text{ min}}{1 \text{ hour}} \frac{60 \text{ sec}}{1 \text{ min}} = 10{,}800 \text{ sec}$$

Rule 4: Exponents must be dimensionless. When dimensioned quantities appear in exponents, they must combine with other dimensioned quantities so that the product or quotient is dimensionless. One cannot have a term such as 2^t where t is time. One can have

$$2^{t_1/t_2}$$

where the exponent is a ratio of two times (in the same units) and therefore dimensionless.

In dimensional equations we shall use the symbol * to indicate a dimensionless quantity.

When dimensional equations are used in this book they are usually written in terms of typical units rather than dimensional abbreviations. Other units having the same dimensionality can be substituted provided that the same substitution is made throughout the equation. For example,

$$C = \frac{Q}{V}$$

$$\frac{\text{moles}}{\text{liter}} = \text{moles}\,\frac{1}{\text{liter}}$$

$$\frac{\text{mmoles}}{\text{cm}^3} = \text{mmoles}\,\frac{1}{\text{cm}^3}$$

Except in this chapter, where other units are used for illustrative purposes, the units used in this book are metric.

2. The Concept of a Functional Relationship

The term "a function of" might be called a "depend-upon" relation. The postage required to send a package depends upon the distance the package must go and the weight of the package. A mathematician says that postage is a function of distance and weight. He indicates this relationship with the symbolism $P(D, W)$ and calls P a dependent variable because it depends upon the two independent variables D and W. The term "independent" implies that the independent variable can be chosen arbitrarily. You tell me a distance and weight and I will tell you, by some rule called the functional relationship, the postage required.

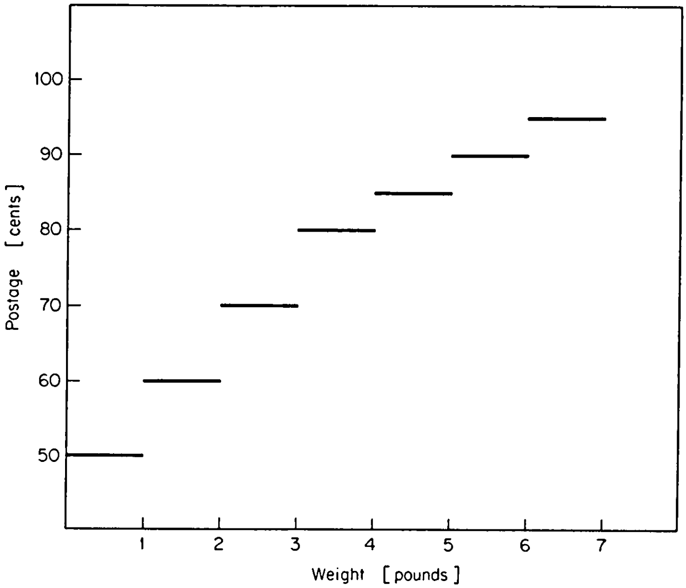

Figure 1-1. A graph of hypothetical postal rates for a package as a function of weight.

A dependent variable can be a function of one independent variable (Figure 1-1) or two independent variables (as in this case), or of many independent variables. Most of the problems we will encounter in the early parts of this book involve only a single independent variable. It is therefore easy to draw graphs that depict functional relationships. Conventionally, the independent variable is the horizontal axis of the graph.

Let us illustrate the concept of a functional dependence for some other simple cases. Consider the area A cm^2 of a circle of radius r cm. We know from plane geometry that

$$A = \pi r^2$$
$$\text{cm}^2 = * \text{ cm}^2$$

and we graph this functional relation $A(r)$ in Figure 1-2. If we wish to designate the area corresponding to a radius of 2 cm, we write $A(2\text{ cm})$, by which it is understood that the independent variable will take on the value 2 cm even though the independent variable r is not specifically written in this notation. We might even leave out the symbol cm if this is made clear by context, and simply write $A(2)$.

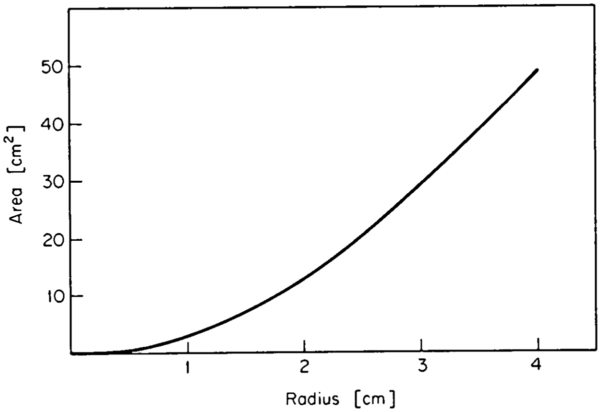

Figure 1-2. The area of a circle as a function of its radius.

The two foregoing examples of functional relationships are clearly defined by rules. In the first case, given distance and weight we go to a table at the post office and read off the corresponding value for the postage. In the second case, we do a simple arithmetic operation of squaring the radius and multiplying it by π. Other functional relations are empirical. For example, we can make a graph of the weight of a baby versus its age, which might resemble Figure 1-3. There is no obvious way to find an algebraic expression that represents the weight of the baby as a function of age. Nevertheless, it is a clearly defined number and it is perfectly appropriate to write W(age).

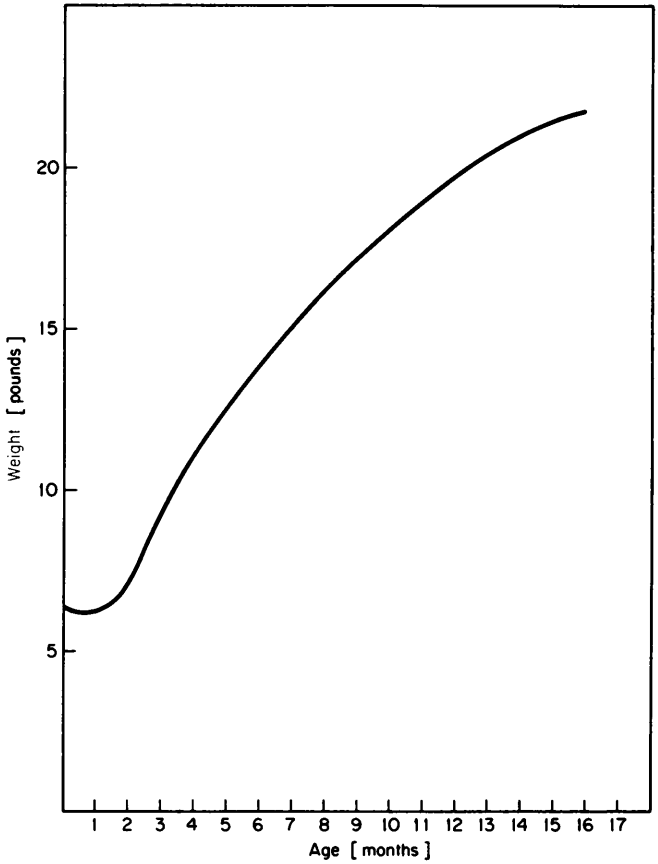

Figure 1-3. **The hypothetical weight of a baby plotted against his age in months.**

The last two examples, the area of the circle and the weight of the baby, have a property that mathematicians refer to as being *continuous functions.* The term "continuous" means simply a smooth relationship. Given two values of the independent variable, the radius, for example, that are very close to each other, the two corresponding values of the dependent variable will then also be very close to each other. This is not true, for example, of the postage required to send a package a specified distance, since by post office rules a package that weighs a trifle more than 6 pounds goes at the 7-pound rate, whereas a package that weighs a trifle less than 6 pounds goes at the 6-pound rate. No matter how close to 6 pounds each of these packages becomes, the rates do not get closer together but remain discretely at the 6-pound rate and the 7-pound rate. Thus, postage is a discontinuous function even though it is continuous between say, 6.01 pounds and 6.99 pounds, where the rate would not change. Discontinuous functions are usually discontinuous at a limited number of points. Although it is possible to define functions that are discontinuous everywhere, they do not arise in this book.

3. The Derivative

One of the important things we want to know about a functional relationship such as $A(r)$ is how much A changes when r changes a little bit (in other words, their relative rates of change). If, for example, r changes by 0.1 cm from 3.0 to 3.1 cm,

$$A(3.1)\ \text{cm}^2 = 30.19\ \text{cm}^2$$

$$A(3.0)\ \text{cm}^2 = 28.27\ \text{cm}^2$$

$$\frac{\text{change in } A}{\text{change in } r} = \frac{1.92\ \text{cm}^2}{0.1\ \text{cm}} = 19.2\ \text{cm} = \text{relative rate of change}$$

Thus, A changes 19 cm times as fast as r for a small change in r located around the point $r = 3.0$. Now, had we done the same calculation with r starting at 4 cm and going to 4.1 cm, we would find that A would change at a different rate relative to r:

$$A(4.1) = (4.1)^2 = 52.80\ \text{cm}^2$$

$$A(4.0) = (4.0)^2 = 50.25\ \text{cm}^2$$

in this case by an amount $2.55\ \text{cm}^2$ or 25.5 cm times as fast as r. So that in general we observe that in defining the relative rates of change of A and r we must allow for the possibility that this relative rate is not uniform but changes as r changes. We can easily draw a picture by constructing a little triangle around the point $r = 3$ cm on the graph of r versus A in which the horizontal leg of the triangle is the change in r and the vertical leg of the triangle the change in A (Figure 1-4). The relative rates of change are then given by the ratios of the two sides of the triangle, and we see that if we construct the triangle in a variety of places along the graph, the ratio of the two sides will change. Our original question was, how fast does A change relative to r around the point $r = 3$? The quantity we have actually calculated is not quite this. It is, in fact, a sort of average of this rate between the point $r = 3$ cm and the point $r = 3.1$ cm. We would perhaps have gotten a slightly different answer if we had calculated the relative rates of change between $r = 3$ cm and $r = 3.01$ cm, which we proceed to do:

$$\frac{A(3.01) - A(3.00)}{3.01 - 3.00} = \frac{28.46 - 28.27}{0.01} = 19.0\ \text{cm}$$

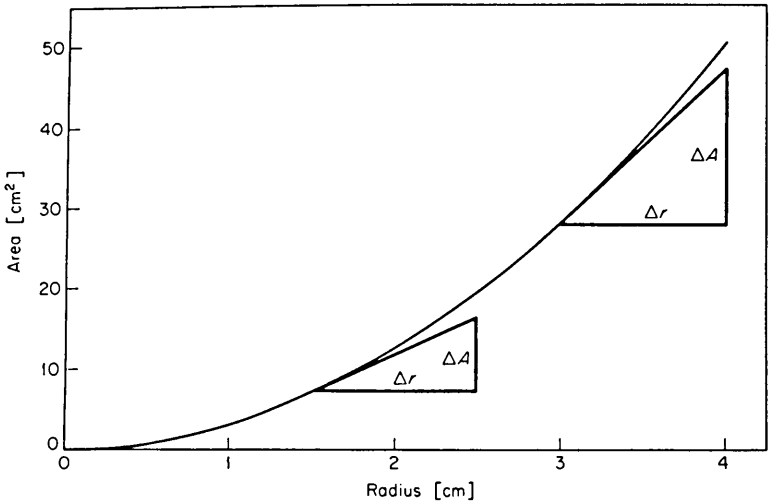

Figure 1-4. The definition of a derivative in terms of tangents to a line.

We note that this relative rate is not much different from the one we calculated previously, but it is slightly different and that in order to define the precise rate of change around the point $r = 3$ we should really do the calculation with an extremely small change in r.

The process we describe here should be studied carefully, as it is fundamental to the rest of this book.

In order to find the rate of change of A relative to the rate of change of r, we calculate the value of A at the point r and at an adjacent point $r + \Delta r$ where Δr is understood to be very small and will eventually be made infinitesimal or, to use the conventional terminology, Δr will be allowed to go to zero. We have already done the calculation of $A(r)$, but let us repeat it in a symbolic fashion rather than with numbers. The calculation of $A(r + \Delta r)$ proceeds along exactly the same line except that $r + \Delta r$ takes the place of r:

$$A(r) = \pi r^2$$

$$A(r + \Delta r) = \pi(r + \Delta r)^2 = \pi(\Delta r)^2 + 2\pi r\, \Delta r + \pi r^2$$

We now calculate the difference ΔA between the two areas thus obtained and divide this difference by the quantity Δr, just as we did in the numerical example above,

$$\frac{\Delta A}{\Delta r} = \frac{A(r + \Delta r) - A(r)}{\Delta r} = \pi 2r + \Delta r \pi$$

The quantity thus obtained, the ratio of the two small quantities ΔA and Δr, is called a differential. We now consider what happens as Δr gets very, very small, or as it is formally stated, in the limit as Δr goes to zero. This procedure is done so frequently that a special notation, called a *derivative*, is used for it:

$$\frac{dA}{dr} = \operatorname*{limit}_{\Delta r \to 0} \frac{A(r + \Delta r) - A(r)}{\Delta r} = 2\pi r$$

We have worked out in somewhat tedious detail the derivative of a very simple functional relationship. Let us now define it formally, recognizing that this is a definition. *The derivative of a function $f(x)$ relative to its independent variable is given by*

$$\frac{df(x)}{dx} = \operatorname*{limit}_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1.1}$$

when Δx goes to zero.

In most beginning calculus texts the derivative is first defined as the slope of the hypotenuse of one of the little triangles in Figure 1-4. Although this definition has a certain intuitive appeal, for the purposes of this book the formal definition is far more satisfactory.

When one wishes to indicate the value of a derivative at a particular value of the independent variable, this value is enclosed in parentheses. Thus the value of df/dx evaluated at x equal to 4 is $df(4)/dx$.

Returning to the area example, we have

$$\frac{dA(3)}{dr} = 2\pi 3 \text{ cm} = 18.8 \text{ cm}$$

Note that the dimensions of the derivative dA/dr are the same as the dimensions of A/r. This is a general rule. Dimensionally, df/dx is the same as f/x.

The derivative of a function as defined above is the rate of change of the function f relative to the rate of change of the independent variable x for infinitesimal changes in x. Frequently we want to know how much f changes for a small but not infinitesimal change in x. To a very good approximation, indicated by the symbol $\approx$, this change is given by

$$\Delta f \approx \Delta x \frac{df}{dx} \tag{1.2}$$

The error in this approximation is the difference between the true value of $f(x + \Delta x)$ and the triangular approximation we find by drawing a tangent line to the curve of f versus x (Figure 1-5).

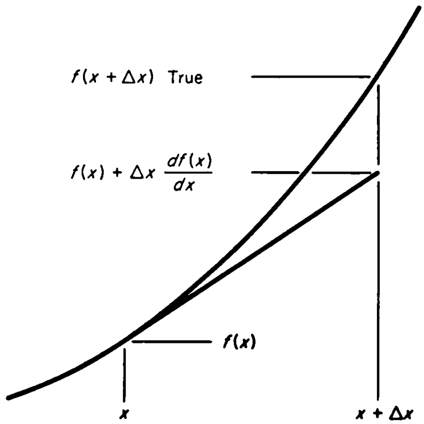

Figure 1-5. How true rate of change of a function is approximated by Δx multiplied by its derivative.

Using this approximation, let us find the approximate value of $A(3.1)$ given that $A(3.00)$ is 28.27 cm^2:

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA(3)}{dr} = 2\pi 3.0 = 18.84 \text{ cm}$$

$$\Delta r = 0.1 \text{ cm}$$

so that

$$A(3.1) \approx A(3.0) + \Delta r \frac{dA(3)}{dr} = 28.27 + 0.1 \times 18.8$$

$$= 30.15 \text{ cm}^2$$

The exact value is 30.19 cm^2. Had we chosen a smaller interval Δr, the approximation would have been better.

In the previous example we have calculated the derivative of

$$A = \pi r^2$$

To calculate the derivative of any function we proceed along identical lines, calculating symbolically the value of the function at some value x of the independent variable and at some slightly different value $x + \Delta x$ of the independent variable. We then compute the difference between the values thus calculated, divide by the change Δx of the independent variable, and take the limit as this change goes to zero. Thus to find the derivative of

$$y = x^3$$

proceed as follows.

$$y(x) = x^3$$

$$y(x + \Delta x) = (x + \Delta x)^3 = x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\frac{y(x + \Delta x) - y(x)}{\Delta x} = 3x^2 + 3x \Delta x + (\Delta x)^2$$

$$\operatorname*{limit}_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} = 3x^2$$

In principle, we can always calculate derivatives in this way. But this is not the way mathematicians like to do things. Instead they derive a set of rules which, though tedious to develop, then make future problems simpler. We therefore state a set of rules that are useful in calculating derivatives with the hope that the reader will either remember (from his elementary calculus) how these are derived, or will look at Appendix I.

Rule: The derivative of a constant is zero. This rule requires a moment's explanation. A constant can be a function of a variable. It just happens to be a function that never changes its value. Thus, for example, one might say that the number of hours in a calendar day is a function of the day of the year; but it is a function that never changes from the value 24. Therefore the difference between its value calculated at one time (the independent variable) and at another time, slightly different, is zero. Therefore its derivative is zero.

Rule: The derivative of a constant multiplied by a function is the constant multiplied by the derivative of the function.

Rule: The derivative of the sum of two functions is the sum of their individual derivatives:

$$y = f(x) + g(x)$$

$$\frac{dy}{dx} = \frac{df}{dx} + \frac{dg}{dx} \tag{1.3}$$

Rule: The derivative of a product of two functions is the first multiplied by the derivative of the second plus the second multiplied by the derivative of the first:

$$y = f(x)g(x)$$

$$\frac{dy}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx} \tag{1.4}$$

Rule: The derivative of the quotient of two functions

$$y = \frac{f(x)}{g(x)}$$

is the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator:

$$\frac{dy}{dx} = \left(g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}\right)\frac{1}{g^2(x)} \tag{1.5}$$

Note as a special case of this rule that the derivative of a reciprocal

$$y = \frac{1}{f(x)}$$

is

$$\frac{dy}{dx} = -\frac{1}{f^2(x)}\frac{df}{dx}$$

Rule: The derivative of $y = x^n$ with respect to x is

$$\frac{dy}{dx} = nx^{n-1} \tag{1.6}$$

These rules are used separately or in combination to reduce a function whose derivative is required to a combination of functions whose derivatives are known. Thus, for example, to find the derivative of the function

$$y = \frac{x^2 - 2x}{x - 1}$$

we proceed in the following way, using the quotient rule

$$\frac{dy}{dx} = \frac{1}{(x-1)^2}\left[(x-1)\frac{d}{dx}(x^2 - 2x) - (x^2 - 2x)\frac{d}{dx}(x-1)\right]$$

$$= \frac{1}{(x-1)^2}[(x-1)(2x-2) - (x^2 - 2x)]$$

$$= 2 - \frac{x(x-2)}{(x-1)^2}$$

Sometimes we have a function that depends upon a second function: A is a function of B and B is a function of C. We ask for the derivative of A

with respect to C; in other words, how much does A change if C changes a little bit? To find this, we must go through an intermediate step. We know that for a small change in C, B will change by approximately

$$\Delta B = \Delta C \frac{dB}{dC}$$

We know that for a change in B, A will change as follows:

$$\Delta A = \Delta B \frac{dA}{dB}$$

Thus if we substitute the change in B in the equation above, we find the result that the rate of change of A relative to C is the product of the rates of change of A relative to B and B relative to C:

$$\Delta A = \Delta C \frac{dA}{dB} \frac{dB}{dC}$$

$$\frac{dA}{dC} = \frac{dA}{dB} \frac{dB}{dC} \tag{1.7}$$

This is called the *chain rule*. It is easily remembered by considering the derivatives as fractions in which the two dB quantities cancel.

4. The Function That Is Its Own Derivative

In general, the derivative of a function is a different function. Thus the derivative of x^3 is $3x^2$. The derivative of

$$y = \sqrt{x}$$

is

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

and so on. There is, however, one rather peculiar function and its usefulness is the result of its peculiarity. It is its own derivative. That function is

$$y = e^x, \qquad e \approx 2.7182 \tag{1.8}$$

$$\frac{dy}{dx} = e^x \tag{1.9}$$

Why e^x has this property, and why this strange number occurs, is explained in Appendix I. It is based on the representation of e^x by a series with an infinite number of terms, which is

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

When this series is differentiated by differentiating each term

$$\frac{d}{dx} e^x = 0 + \frac{1}{1} + \frac{2x}{1 \cdot 2} + \frac{3x^2}{1 \cdot 2 \cdot 3} + \frac{4x^3}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

the resulting series is seen to be identical to the series for e^x.

Accepting the fact that e^x is its own derivative, we proceed, by means of the chain rule, to find the derivative of e to a function of x. Let p be a function of x

$$y = e^{p(x)}$$

$$\frac{dy}{dp} = e^{p(x)}$$

$$\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = e^{p(x)}\frac{dp}{dx} = y\frac{dp}{dx}$$

and in the special case where the function is a constant, say alpha, multiplied by x, we have

$$y = e^{\alpha x}$$

$$\frac{dy}{dx} = e^{\alpha x}\frac{d(\alpha x)}{dx} = \alpha e^{\alpha x} = \alpha y$$

The function $y = e^{\alpha x}$ occurs so frequently that we would like to describe its properties in considerable detail.

First, let us make a plot of

$$y = e^{\alpha x} \tag{1.10}$$

on a conventional rectangular graph (Figure 1-6).

To illustrate the point that $\alpha e^{\alpha x}$ is the derivative of $e^{\alpha x}$, let us take two points from the graph and numerically compute the derivative.

Let us choose $\alpha = 1.5$, $x = 1.4$, $\Delta x = 0.1$; at

$$x = 1.4, \qquad \alpha x = 2.10, \qquad e^{\alpha x} = 8.0$$

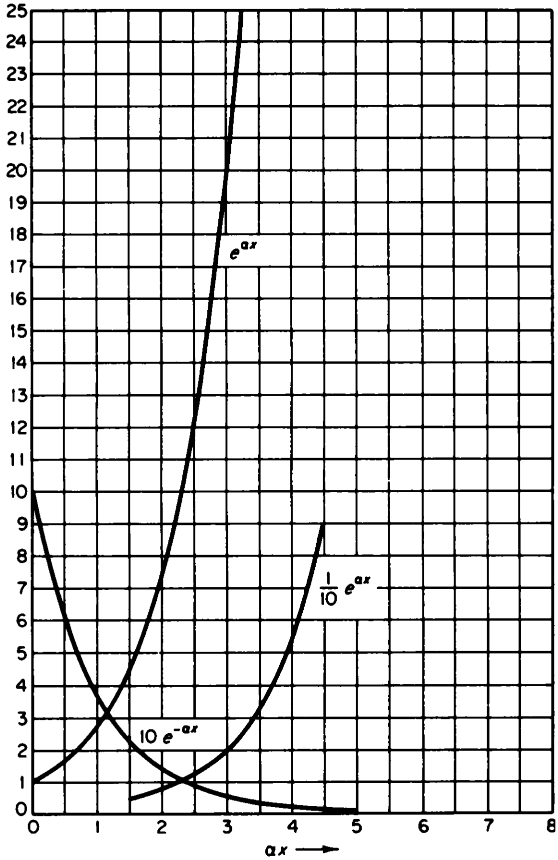

Figure 1-6. **Plots of the value of the function $e^{\alpha x}$ and $e^{-\alpha x}$ as a function of αx.**

and at

$$x + \Delta x = 1.5, \qquad \alpha x = 2.25, \qquad e^{\alpha x} = 9.3$$

$$\frac{\Delta y}{\Delta x} = \frac{y(x + \Delta x) - y(x)}{\Delta x} = \frac{9.3 - 8.0}{0.1} = 13$$

Using the relation

$$\frac{dy}{dx} = \alpha e^{\alpha x} = 1.5e^{2.10} = 12.0$$

We see that finding the derivative numerically does not quite yield perfect agreement between dy/dx and $\Delta y/\Delta x$. However, this is a result of taking a finite step in x and of reading the graph inaccurately. Had the step been

taken much smaller (that is, had it been possible to do so from this graph), the agreement would have been much better. In fact, let us do so, by expanding the graph around the point $\alpha x = 2.1$ in Figure 1-7 and using a Δx of 0.02:

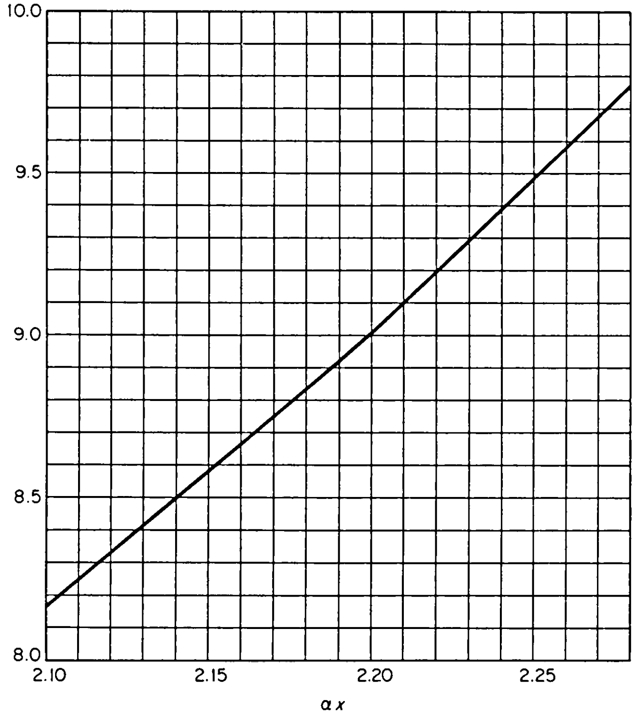

Figure 1-7. An expanded scale plot of $e^{\alpha x}$ as a function of αx.

at

$$x = 1.40, \qquad \alpha x = 2.10, \qquad e^{\alpha x} = 8.16$$

and at

$$x = 1.42, \qquad \alpha x = 2.13, \qquad e^{\alpha x} = 8.41$$

$$\frac{\Delta y}{\Delta x} = \frac{y(x + \Delta x) - y(x)}{\Delta x} = \frac{0.25}{0.02} = 12.5$$

$$\frac{dy}{dx} = \alpha e^{\alpha x} = 1.5e^{2.10} = 12.24$$

We note that we get considerably better agreement between the value of the derivative and the numerical differential.

The function $y = e^{\alpha x}$ occurs so often that special graph paper has been devised which plots it as straight lines by distorting the vertical scale. This paper is called *semilogarithmic* graph paper. In Figure 1-8 we have plotted on semilogarithmic paper the values of $e^{\alpha x}$ for various values of alpha.

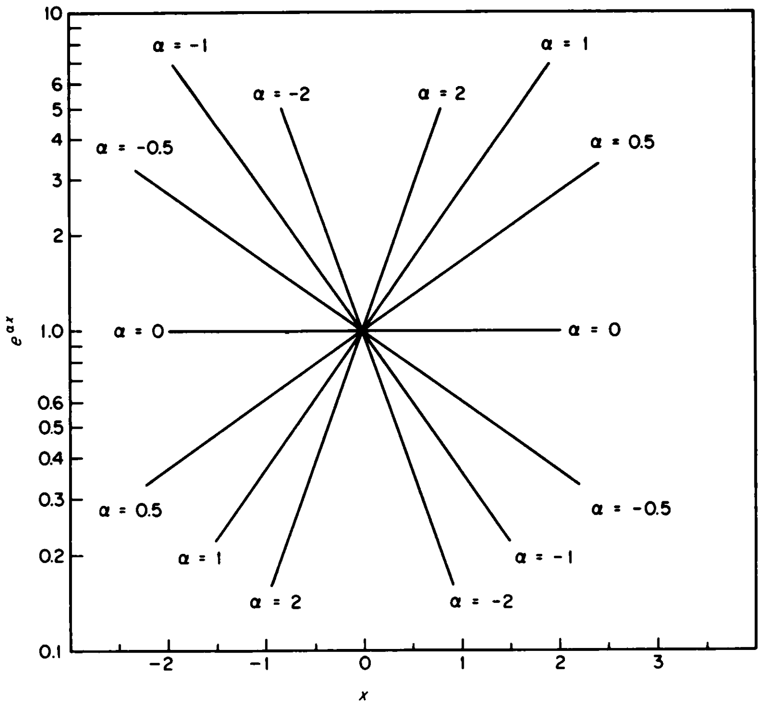

Figure 1-8. $e^{\alpha x}$ plotted semilogarithmically as a function of x for various values of α.

5. Differential Equations

The statement that e^x is its own derivative can be written as an equation

$$\frac{dy}{dx} = y \tag{1.11}$$

This kind of equation is one of a great class of equations that are encountered in physical problems called *differential equations.* They should, perhaps, be called *derivative* equations. However, tradition rules here, and we shall retain the terminology differential equation. The differential equation (1.11), since it is the definition of e^x, has the solution

$$y = e^x \tag{1.12}$$

It happens, however, that it also has other solutions. In fact, any constant multiplied by e^x is also a solution. Let A be an arbitrary constant. Let

$$y = Ae^x \tag{1.13}$$

Then

$$\frac{dy}{dx} = A\frac{de^x}{dx} = Ae^x = y \tag{1.14}$$

We make use of this property to fit solutions of the type Ae^x to specified conditions of the physical problem. If, for example, we have the relations

$$\frac{dy}{dx} = y \qquad \text{and} \qquad y(0) = 3$$

we can choose $A = 3$ to satisfy the "initial condition" of $y(0) = 3$.

$$y = 3e^x = \frac{dy}{dx} \qquad \text{and} \qquad y(0) = 3$$

Suppose the differential equation is

$$\frac{dy}{dx} = ky$$

Our previous discussion suggests a solution

$$y = Ae^{kx}$$

Let us try this solution by differentiating it and inserting the derivative into the differential equation.

$$\frac{dy}{dx} = Ake^{kx} = ky$$

We find that, indeed, it does satisfy the differential equation. If we add the condition $y(2) = 4$,

$$4 = Ae^{k2}$$

we find

$$A = \frac{4}{e^{2k}}$$

which, given k, we can evaluate.

Plotted on semilogarithm paper, as in Figure 1-9, the relation $y = Ae^{kx}$ also yields a straight line. The value of the vertical coordinate at $x = 0$ is A, in this case 2.6. The slope of the line is related to k. The easy way to measure this slope is to find the range of x required to double or halve the value of y. For the line shown in Figure 1-9, y decreases by half in the interval between

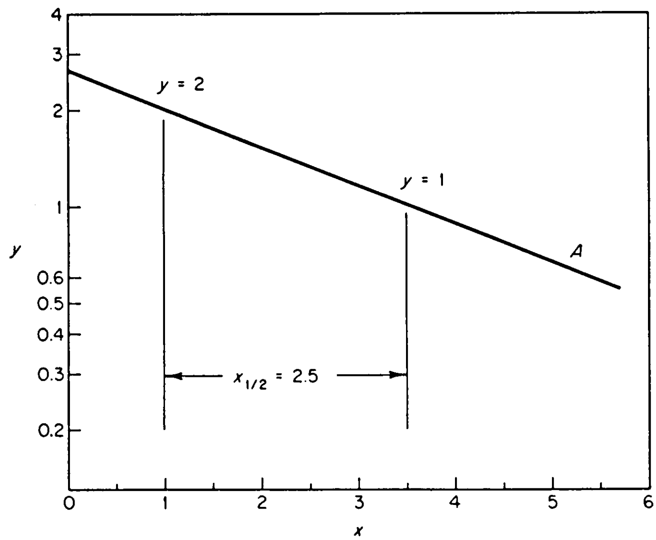

Figure 1-9. The method of estimating the half period of decay of an exponential from a semilogarithmic plot.

$x = 1$ and $x = 3.5$. Thus, the half range $x_{1/2}$ is 2.5, and since

$$e^{-0.69} = \tfrac{1}{2}$$

$$kx_{1/2} = -0.69 \tag{1.15}$$

$$k = \frac{-0.69}{2.5} = -0.276$$

Note that we could have chosen any two convenient points on the graph to determine $x_{1/2}$. We chose the points at which y was 2 and 1. We could equally well have chosen y equal to 1.5 and 0.75. They would have yielded the same value for k. Had the line sloped upward, we would have used the doubling range x_2 and the relation

$$e^{+0.69} = 2, \qquad kx_2 = 0.69 \tag{1.16}$$

The foregoing introduction has been intentionally brief. If the reader finds it inadequate, he should refer to his college calculus text or to either (18) or (23).

Problems

1. If $W = G(x)$, define the derivative

$$\frac{dW}{dx} = \text{limit} \quad ?$$

2. $W = x^2 - 2x + C$ where C is a constant; find

$$W(2) \qquad \text{and} \qquad \frac{dW(2)}{dx}$$

3. Given

$$U(x) = W(x)V(x) \qquad S(x) = \frac{V(x)}{W(x)}$$

$$W(2) = 3 \qquad \frac{dW(2)}{dx} = 4$$

$$V(2) = 4 \qquad \frac{dV(2)}{dx} = 2$$

find

$$\frac{dU(2)}{dx} \qquad \text{and} \qquad \frac{dS(2)}{dx}$$

4. If $y(x)$ is a smooth function of x and $y(3) = 4$ and $y(3.1) = 4.3$, find the approximate value of $dy(3)/dx$.

5. Given

$$\frac{dy(3)}{dx} = 2 \qquad y(3) = 12$$

find the approximate value of $y(3.1)$.

6. If $u = f(x)$ and $V = g(u)$, find dV/dx.

7. In Problem 6 let

$$f(x) = x^4 + 2, \qquad g(u) = u^2$$

find dV/dx by the chain rule and also by substituting

$$V = g(u) = u^2 = x^8 + 4x^4 + 4$$

8. Given

$$\frac{dC(t)}{dt} = kC(t), \qquad C(0) = 2 \text{ moles/liter}$$

find $C(t)$.

9. Given

$$\frac{dC(t)}{dt} = kC(t), \qquad C(1) = 2 \text{ moles/liter}$$

find $C(t)$.

10. Given

$$y = Ae^{-kt}, \qquad y(4) = \tfrac{1}{2}y(2)$$

find k.

11. Find the decay constant α of $y = Ae^{-\alpha t}$ for the following data:

t [sec]	y
1	0.0096
2	0.0048
3	0.0024
4	0.0012

12. Derive the relations corresponding to Eqs. (1.15) and (1.16) for the 1/10 interval $\tau_{1/10}$ or 10 times interval τ_{10}.

13. If the following results occurred in problems, what must the dimension of K have been in each case?

(a) e^{-kt^2}, t in seconds

(b) $C = \dfrac{F}{K}t$, $F\dfrac{\text{liters}}{\text{sec}}$, t sec, $C\dfrac{\text{moles}}{\text{liter}}$

(c) $R = (F + K)e^{-(F/V)t}$, $F\dfrac{\text{liter}}{\text{sec}}$

14. The velocity of light is 3 times 10^{10} or 30,000,000,000 cm/sec. Find the velocity of light in furlongs per fortnight given that a furlong is one-eighth of a mile, a mile is 5280 feet, a foot is 12 inches, an inch is 2.54 cm, and a fortnight is 2 weeks. Use the unit conversion method in Section 1.

15.

$$\frac{dC}{dt} = \frac{R}{V} - \frac{F}{V}C$$

Write a dimensional equation corresponding to the foregoing if C is a concentration, t time, R a rate of generation of a substance, V a volume, F a rate of liquid flow.

16.

$$\frac{d^2C}{dt^2} + B\frac{dC}{dt} + DC = E$$

Write a dimensional equation for the foregoing, given that C is a concentration and t is time.

17. Find the derivatives of the following.

(a) $y = x^4 + 3x^3 + x + 2$ (b) $y = \dfrac{x^3 - x}{x^3 + x}$

(c) $y = e^{x^2}$ (d) $y = xe^{\alpha x}$

If the reader has difficulty with these problems, additional review of elementary calculus is indicated.