CHAPTER I

On the Methodology of Mathematical Models

To understand the role played by mathematical models in science we must first have some understanding of the nature of the scientific method. The scientific method may initially be described as a cyclic process through which human beings learn from experience. As evidence accumulates, theories in better and better agreement with the actual functioning of nature can be formulated.

The basic cycle of the scientific method may be divided into three steps: induction, deduction, and verification. *Induction* is the step which carries the scientist from factual observations to the formation of theories. These theories may be very close to facts in that they "simply summarize observed facts," or they may be as abstract as are those of modern theoretical physics. The inductive step is necessarily creative, and, although various rules to aid the theoretical scientist have been proposed, these are at best uncertain guides and not by any means guarantees of success. The scientist is able to investigate a process under only a small number of conditions, after which he must attempt to explain it in its complete generality. Thus, even an inspired theory is in part an inspired guess.

Once the theory is formulated precisely, the tools of logic and mathematics are available to deduce consequences from it. It is due to the availability of the second, deductive, step that the formation of theories becomes of importance to the scientist. For it is during the process of *deduction* that the scientist discovers a number of consequences of his theories which may not have been immediately obvious to him. In some cases the chain of deduction may take many years, and the results may be quite unexpected.

Once a number of interesting consequences have been deduced from the theory, they must be put to the test of experimental *verification*. In some cases the newly deduced facts may correspond to events already observed, whereas in other instances new observations and experiments will be required to test the predictions. In the former case one speaks of the theory as having served to explain known facts; in the latter we have succeeded in predicting novel occurrences.

Our confidence in a theory builds up as more and more of its predictions turn out to be true. On the other hand, it happens frequently in the history of science that further testing persuades us to reject a previously accepted theory. Very often the rejection of one theory directly stimulates the formulation of another, improved theory—one which explains both those old facts on which the discarded theory was based and those new facts which have led to the rejection of the old theory.

In this book we consider eight mathematical models illustrating theories from a variety of different branches of the social sciences. Each chapter is organized in a manner paralleling the usual application of the scientific method. First a problem is stated from a branch of the social sciences, then a mathematical model (theory) is formed, then a number of consequences are deduced from this theory, and finally the results are interpreted. We hope that this procedure will illustrate the manner in which theories are formed. Indeed it should illustrate the entire cycle of the scientific method, with the exception of the gathering of facts. That is, we start with a collection of facts already given to us and end at the point where we are ready to put the predictions of our theory to further observational tests.

Although most of our chapters illustrate the basic cycle, the entire cycle can best be observed in Chapter V. After forming a mathematical model for a small-groups experiment and developing it in some detail, we present experimental evidence indicating that the original model may not be adequate, and we show how the very act of disconfirmation suggests an improved model.

It is important to contrast the pure mathematical theory with its interpreted version that serves as a model. Let us illustrate this in terms of the ecology model of Chapter III. From the point of view of pure mathematics, we are confronted with a pair of simple differential equations. These equations are neither true nor false, since they have no factual content. Rather, they are abstract forms which may be studied, and from which we can deduce certain "if ... then ..." statements. For example, we can show that if certain quantities of any sort happen to obey the laws of nature expressed in our equations, then these quantities must forever be on the trajectory determined by their initial values.

Next we find that certain species of animal seem to multiply in a manner that roughly may be considered to obey these differential equations. We thus supply an interpretation, letting x and y stand for actual numbers of the two species and interpreting t as time measured in a convenient unit. We have thereby automatically supplied interpretations for all the results that may be deduced from these equations. Each of these deductions, interpreted in the indicated manner, must either correspond to a known fact or serve as a prediction of an unknown fact.

Closely related to the formation of scientific theories is the problem of how scientists arrive at their basic concepts. Of particular interest are the so-called theoretical concepts, those concepts which are reasonably far removed from terms that describe our immediate experience. The history of science points to the conclusion that the most useful theoretical concepts are formed simultaneously with the most useful theories. That is, the only test of the fruitfulness of a given concept is the fact that we are able to form fruitful theories in its terms. It is therefore not surprising that in most of the following chapters it is difficult to separate the formation of a concept from the formulation of the mathematical model. However, Chapter II comes as close to pure concept formation as one can ever hope to find in a scientific context.

Indeed, in Chapter II the primary problem is not so much that of forming a mathematical model as it is that of developing a mathematical tool (a "distance" between rankings) which may later on be used in building a variety of different models. Such a technique, known as *explication*, is becoming very popular for the formation of precise concepts. One begins with an imprecisely expressed idea and hopefully arrives at a quite precise and fruitful concept. The general procedure is as follows: First one lays down the conditions of adequacy that a precise definition must meet; then one searches out the simplest definition that will meet all of them. Chapter II carries out such a procedure and indicates in some detail the types of problems with which one is confronted in the formation of mathematical concepts.†

Very often one finds that a single vague, intuitive idea leads to a number of distinct and precise concepts, each of which may turn out to be fruitful in different applications. This is well illustrated in terms of our intuitive idea of a structure being "in balance" or "in equilibrium." This idea receives different precise formulations in Chapters III, IV, and VIII, where we discuss the equilibrium or balance of a pair of species, a market, and a political structure, respectively.

Let us now group mathematical models used in the sciences into types. One fundamental distinction is whether or not a model is of a so-called "deterministic" nature. The distinction can be illustrated in terms of classical physics. For example, Newton's laws are of a deterministic nature; i.e., if one has sufficient information available concerning the past, one can predict the entire future of the system. On the other hand the models of statistical mechanics are non-deterministic and therefore probabilistic in nature; that is, no matter how much information one has about the past, one can predict only the probabilities of certain future occurrences, and usually the amount of information available loses its value as time passes. Deterministic models will be found in Chapters III, IV, VI and IX. Probabilistic models are treated in Chapters V and VII. The reader should compare the predictions that one deduces from the two types of models.

† The application of the concept of distance developed in Chapter II was to consensus rankings by a group of experts, which is a procedural matter rather than a scientific theory. However, once such concepts are available, they often prove useful in new contexts. For example, J. Berger is working on a theory of small groups of experiments which makes essential use of this concept of distance.

It should in particular be noted that mere differences in the sorts of predictions that one can hope to make with each type of model do not imply that one sort of prediction is less useful than the other.

All the above-mentioned models share the feature of being "predictive" in nature. They may be contrasted with the models in Chapters II and VIII, which are primarily "descriptive." The model in Chapter II allows us to compare different rankings of the same set of objects, whereas the model in Chapter VIII allows us to classify social structures.

It is sometimes useful to distinguish theories by their levels of abstraction. We may illustrate this distinction either in terms of the deterministic theories or in terms of the probabilistic ones. Chapters III and IV both treat deterministic differential equation models. But the model of Chapter III is much closer to the level of observations. In this ecological model the variables represent directly observable quantities; namely, numbers of animals in various species. In contrast, Chapter IV attempts to describe the "underlying machinery" that operates a market. Such quantities as are needed to describe the utility of various goods for various individuals are not directly observable.

Similarly, of the two probabilistic models, the one in Chapter VII is less abstract than the one in Chapter V. The former deals with such directly observable quantities as the number of customers in a waiting line; the latter attempts to reconstruct the manner in which a subject in an experiment arrives at his decisions. The subject's mental state is not observable in as simple and easy a way as is service time, which is directly measurable. Of course, none of these models achieve the level of abstraction that characterizes theories in modern physics. In some truly abstract models the connection with experience is established only very indirectly, after long chains of deductions. These levels are as yet rarely reached in the social sciences.

One of the best examples of a more abstract model in the social sciences is a model applicable to simple learning experiments that has been developed by W. K. Estes. We do not treat this in the present volume, as it was treated in some detail in a previous book by the present authors.†

Since many scientists seem to be under the impression that the use of mathematics is closely tied to the existence of numerical concepts, or at least of concepts dealing with space, it is significant that two nonnumerical and nongeometrical models are included in our collection. Both Chapters II and VIII employ techniques that a classical mathematician would not have recognized as mathematics, namely, abstract distances (metric spaces) and graph theory. It is thanks to the ever-broadening conception of the nature of mathematics that such models are available to us today. It is entirely possible that the greatest successes in the very complex areas of

† John G. Kemeny and J. Laurie Snell, *Finite Markov Chains* (Princeton, N. J.: D. Van Nostrand Co., 1960), Chap. VII, Sec. 5.

the social sciences will be made possible by nonnumerical models produced by modern mathematics.

Finally let us ask questions concerning the various uses to which mathematical models can be put. As indicated above, the two primary uses lie in explaining known facts and in predicting facts not as yet known. Of course, each of our chapters illustrates both of these tendencies to some extent; for if the models are correctly formed they will explain the facts on which they were built, and if they contain any element of novelty at all they will make predictions concerning the future.

We will examine one clear-cut example of each type of use of a model. Let us first refer to Chapter III as an example of clear-cut scientific explanation: A cyclic pattern has been observed in the numbers of animals in certain species in closed ecological systems. The simple model developed in the chapter offers an elementary explanation for this cyclic behavior.

A good example of predicting the future is provided by Chapter VII. The "second problem" treated in this chapter has received highly detailed and extensive study because it enables various industries to predict their needs for staff and equipment. For example, it is through methods illustrated in this chapter that the Bell Telephone Company has planned the number and type of trunk lines that it must make available to furnish adequate service to its customers. It is the task of the model to predict how long the average customer will have to wait until a telephone trunk line becomes free, given information concerning the telephone habits of the customers and specifications as to the number of available trunk lines. By carrying out a number of such computations, the company may find the minimum number of trunk lines with which it can furnish what its directorate considers satisfactory service.

The previous example illustrates the fact that, although some predictions are made purely out of scientific curiosity, in many instances models with strong predictive powers may be used as planning devices. This feature is also well illustrated in Chapters VI and IX. The technique discussed in the latter chapter, dynamic programming, has in the last fifteen years proven invaluable as a planning device for industry. The former chapter contains a novel idea, suggested as a planning device for the first time in this book.

We hope that one by-product of the present book will be a reinforcement of the general impression that mathematics has broad and fruitful applications. Although this point has been conceded in the physical sciences, and to some extent in the biological sciences, there are still many skeptics as far as the social sciences are concerned. To some extent this skepticism is due to the very legitimate objection that the social sciences are vastly more complex than the physical or biological sciences. However, this seems to indicate only that their mathematical techniques will have to be more sophisticated as well. It also indicates that the time which it will take to develop nontrivial models for the social sciences may have to be substantial, even in the present age of rapid scientific progress.

But to some extent the objections are based on a misunderstanding of the nature of mathematics. Mathematics is best viewed as the study of abstract relations in the broadest sense of that word. From this point of view it is not surprising that mathematics is applicable to any well-defined field. Whatever the nature of the phenomena studied in a given social science, their various components do bear certain relations to each other, and once one succeeds in formulating these abstractly and precisely, one is in a position to apply the full machinery of mathematical analysis. Of course, it is to be expected that often the mathematical model so formed will be one not previously studied by mathematicians. Therefore, one may look forward to the day when the social sciences will be as major a stimulus for the development of new mathematics as physics has been in the past.

On the other hand, it has been the good luck of both science and mathematics that mathematical models which were developed by mathematicians purely for their aesthetic satisfaction have subsequently proved to be extremely useful. An outstanding example of this in the physical sciences was the invention of "imaginary numbers," which, as their very name indicates, were supposed to have no connection with reality. As it has turned out, however, these numbers play a crucial role in modern physics. For example, all of electromagnetic theory is based on their use.

It is not even too surprising to find that the same abstract mathematical model may serve a variety of different purposes. We find good illustrations of this point in two of our chapters. In Chapter VII the same mathematical model is applied on the one hand to the growth of a population, and on the other to the problem of people waiting in line to be served. An even more spectacular illustration of this type of strange coincidence may be found in Chapter VI. The mathematical tools used there were developed by pure mathematicians who were interested in giving a probabilistic generalization of some results from the theory of potentials in classical physics. That these results should turn out to be applicable also to practical planning problems in economics indicates the tremendous flexibility of an abstract model.

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