Perturbation Theory Anomalies

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CONTENTS

1. INTRODUCTION AND REVIEW OF PERTURBATION THEORY

The purpose of these lectures is to study some recent developments in renormalized perturbation theory. The subject of renormalized perturbation theory is, of course, old and well developed; for large classes of Lagrangian field theory models, renormalization procedures and proofs of renormalizability to all orders of perturbation theory have been given.[1] The new aspects which we will discuss here have been brought into focus by recent work on current algebras, which makes heavy use of Ward identities and of Bjorken limits involving currents, both of which are statements about time-ordered products of the type which commonly occur in Lagrangian field theories. Since some of these statements are obtained by rather naive formal manipulations of highly divergent quantities, it is natural to ask whether the manipulations are indeed correct. There is at present no general way to answer this question, even in specific Lagrangian field theory models, since no general methods for calculation in field theory exist. However, it is possible, and is quite illuminating, to examine the question within the framework of renormalized perturbation theory, where definite methods of calculation exist and concrete answers can be obtained. This track has been pursued by a number of authors during the past several years, and will be described below in detail. The results show that, in perturbation theory, there are many cases in which the usual naive manipulations break down, leading to modifications, or anomalies, in Ward identities and Bjorken limits. These anomalies and their properties are an interesting mathematical physics question in their own right, as well as having important implications for certain current algebra calculations.

1.1 Review of Quantum Electrodynamics and Renormalization Theory

We will begin our study of perturbation theory anomalies by reviewing the usual renormalization theory, in the familiar case of quantum electrodynamics (QED). We will find that QED exhibits many of the anomalies which will interest us, and the generalizations to other cases of physical interest, such as the quark model with massive vector gluon and the σ-model, involve questions of detail rather than of general principle. The Lagrangian for QED is[2]

$$\mathcal{L}(x) = \overline{\psi}(x)(i\gamma\cdot\Box - m_0)\psi(x) - \tfrac{1}{4}F_{\mu\nu}(x)\, F^{\mu\nu}(x) \qquad (1)$$
$$e_0\overline{\psi}(x)\;\gamma_\mu\,\psi(x)\;A^\mu(x)$$

with $\psi(x)$ the electron field, $A^\mu(x)$ the photon field, $F_{\mu\nu}(x) = \partial A_\mu(x)/\partial x^\nu - \partial A_\nu(x)/\partial x^\mu$ the electromagnetic field strength tensor, $\gamma\cdot\Box = \gamma^\mu\partial/\partial x^\mu$ and with $-e_0$ and m_0, respectively, the electron bare charge and bare mass. From the Lagrangian, we find the equations of motion for the fields,

$$\begin{aligned}(i\gamma\cdot\Box - m_0)\,\psi(x) &= e_0\gamma_\mu A^\mu(x)\;\psi(x), \\ \partial F^{\mu\nu}(x)/\partial x^\nu &= e_0\; j^\mu(x), \\ j^\mu(x) &= \overline{\psi}(x)\;\gamma^\mu\psi(x).\end{aligned} \qquad (2)$$

Using the equation of motion for $\psi(x)$ to calculate $\partial j^\mu(x)/\partial x^\mu$ and assuming that no subtlety arises when we apply the chain rule of differentiation to the product of two field operators at the same space time point, we find

$$\begin{aligned}\partial j^\mu(x)/\partial x^\mu &= \partial\overline{\psi}(x)/\partial x^\mu\,\gamma^\mu\,\psi(x) + \overline{\psi}(x)\gamma^\mu\,\partial\psi(x)/\partial x^\mu \qquad (3)\\ &= \overline{\psi}(x)[\,im_0 + ie_0\gamma_\mu A^\mu(x)\,]\,\psi(x) + \overline{\psi}(x)[\,-im_0 - ie_0\gamma_\mu A^\mu(x)\,]\psi(x)\\ &= 0,\end{aligned}$$

which is the usual equation of current conservation. Finally, the canonical anticommutation relations of the spinor fields are

$$\{\psi_\alpha(\underset{\sim}{x}, t),\; \psi^\dagger_\beta(\underset{\sim}{y}, t)\} = \delta_{\alpha\beta}\,\delta^3(\underset{\sim}{x} - \underset{\sim}{y})\;, \qquad (4)$$

with $\alpha, \beta = 1, \ldots, 4$ the labels of the spinor components, and, if we use the Feynman gauge, the canonical commu-

tation relations of the photon fields are

$$[A_\mu(\underset{\sim}{x}, t), \partial A_\nu(\underset{\sim}{y}, t)/\partial t] = -ig_{\mu\nu}\delta^3(\underset{\sim}{x}-\underset{\sim}{y}). \qquad (5)$$

Eqs. (1) - (5) are the basic equationsof QED.

The usual method for dealing with these equations is, of course, to work in momentum space and to expand in a perturbation series in powers of e_0. This leads to the familiar Feynman rules, which we summarize as follows:

(i) For each internal electron line with momentum p we include a factor $i(\not{p}-m_0+i\varepsilon)^{-1}$ and for each vertex a factor $-ie_0\gamma_\mu$. For each internal photon line of momentum q we include afactor $-ig_{\mu\nu}(q^2+i\varepsilon)^{-1}$.

(ii) There is a factor $\int d^4\ell/(2\pi)^4$ for each internal integration over loop variable ℓ and a factor -1 for each fermion loop.

(iii) For each external photon line there is a factor $\varepsilon_\mu\sqrt{Z_3}$, where ε_μ is the photon polarization four-vector and Z_3 is the photon wave-function renormalization. For each external electron line entering (leaving) the graph there is a factor $\sqrt{Z_2}\,u(p,s)[\sqrt{Z_2}\,\bar{u}(p,s)]$, and similar factors for external positron lines, with Z_2 the electron wave-function renormalization. Disconnected bubbles and self-energy insertions on external electron and photon lines are excluded.

Using these rules, we can construct the invariant Feynman amplitude $\mathcal{M}$ for any process, to any order of perturbation theory. As is well known, divergences are encountered which must be removed by a renormalization procedure, which we will now briefly sketch. Let us define the electron propagator $S'_F(p)$, the photon propagator $D'_F(q)^{\mu\nu}$ and the vertex part $\Gamma_\mu(p,p')$ by

$$iS'_F(p) = \int d^4x\, e^{ip\cdot x} < 0|\, T(\psi(x)\overline{\psi}(0))\,|0>, \tag{6a}$$

$$iD'_F(q)^{\mu\nu} = \int d^4x\, e^{iq\cdot x} <0|\, T(A^\mu(x)A^\nu(0))\,|0>, \tag{6b}$$

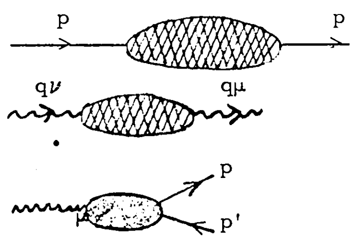
$$S'_F(p)\,\Gamma_\mu(p,p')\,S'_F(p') \tag{6c}$$
$$= -\int d^4x\, d^4y\, e^{ip\cdot x} e^{-ip'\cdot y} <0|\, T(\psi(x)j_\mu(0)\overline{\psi}(y))\,|0>.$$

They have the following diagrammatic representations:

$iS'_F(p)$

$iD'_F(q)^{\mu\nu}$

$\Gamma_\mu(p,p')$

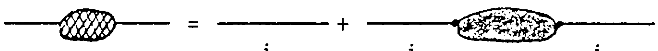


The vertex part is proper; that is, it cannot be divided into two disjoint graphs by cutting a single line. The electron and photon propagators can be expressed in terms of the proper electron and photon self-energy parts, defined by

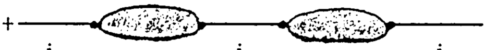
$-i\,\Sigma(p) =$  (the blob includes proper diagrams only with the two electron ends removed)

$ie_0^2\,\Pi(q)^{\mu\nu}$ =  (the blob includes proper diagrams only with the two photon ends removed)

For the electron propagator, we find



$$iS'_F(p) = \frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}[-i\Sigma(p)]\frac{i}{\not p - m_0}$$



$$+ \frac{i}{\not p - m_0}[-i\Sigma(p)]\frac{i}{\not p - m_0}[-i\Sigma(p)]\frac{i}{\not p - m_0} + \ldots$$

which sums to

$$S'_F(p) = \frac{1}{\not p - m_0 - \Sigma(p)} \tag{7}$$

Similarly, writing

$$D'_F(q)^{\mu\nu} = -g^{\mu\nu}\, D'_F(q) + \text{longitudinal terms} \tag{8a}$$

and using the fact that current conservation requires $\Pi(q)^{\mu\nu}$ to have the form

$$\Pi(q)^{\mu\nu} = (-q^2 g^{\mu\nu} + q^\mu q^\nu)\,\Pi(q^2) \tag{8b}$$

we find that a similar summation gives us

$$D'_F(q) = \frac{1}{q^2\,[1 + e_0^2\,\Pi(q^2)]}\ . \tag{9}$$

The reason for defining the propagators and vertex part is that all divergences reside in these quantities. To see this, we note that the superficial degree of divergence of a graph is given by

$$D = 4k - 2b - f, \tag{10a}$$

b = number of internal photon lines,

f = number of internal electron lines,

k = number of internal-momentum integrations.

Letting

n = number of vertices

B = number of external boson lines, (10b)

F = number of external fermion lines,

and using the topological relations

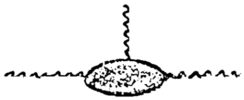
$$F + 2f = 2n, \tag{10c}$$

$$B + 2b = n,$$

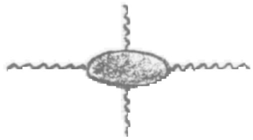
we can rewrite D in terms of the numbers of external lines above,

$$D = 4 - \frac{3}{2} F - B. \tag{11}$$

The condition for a graph to converge is that $D < 0$ for the graph itself and for all subgraphs contained inside the graph. From Eq. (11), we learn that the potentially dangerous types of graphs are (a) the electron proper self energy part $\Sigma(p)$ ($D=1$), (b) the proper vertex part $\Gamma_\mu(p, p')$ ($D=0$), (c) the photon proper self-energy part $\Pi(q^2)$ ($D=2$), (d) the proper vertex of three photons



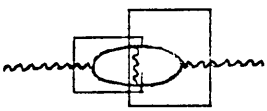
and the proper photon-photon scattering amplitude



The three photon vertex vanishes because the photon is odd under charge conjugation. (Furry's theorem.) Although $D=2$ for the photon proper-self-energy part, two powers of momentum are used up to form the $q^2 g_{\mu\nu} - q_\mu q_\nu$ term in Eq. (8b), and thus $\Pi(q^2)$ has only an effective divergence $D_{eff} = 0$. Similarly, in the case of photon-photon scattering, <u>four</u> powers of internal momenta are used up in forming electromagnetic field strengths $F_{\mu\nu}(q) \sim (q_\mu \varepsilon_\nu - q_\nu \varepsilon_\mu)$ for each of the four external photons. Hence this graph has $D_{eff} = -4$ and is highly convergent.

Having shown that divergences are always associated with self-energy and vertex parts, we can now state a procedure for studying, and then removing, the divergences of an arbitrary graph. Given an arbitrary graph, let us define the <u>skeleton</u> as the new graph obtained by contracting all self-energy and vertex parts into points. [In doing this, we write $m_0 = m - \delta m$, with m the physical mass, and treat δm as part of the electron-self-energy part. Hence the electron propagators appearing in the skeleton are all of the form $(\not{p}-m)^{-1}$.] Clearly, the skeleton graph is always finite.

If we can devise a procedure for making the self-energy and vertex parts finite, we can then obtain a finite value for our original graph by making appropriate insertions of the finite self-energy and vertex parts on the skeleton. For skeletons with three legs or more, the insertions can always be made in a non-overlapping way, and the recipe is simple to implement. For skeletons with two legs (electron and photon self-energy parts themselves) there are situations which necessarily involve overlapping vertex insertions, such as

 ;

this is what makes proofs of renormalizability so difficult.

The recipe for making the self-energy and vertex parts finite is the following. First, we choose $\delta m = m - m_0$ such that

$$\delta m - \Sigma(p) = 0 \quad \text{at} \quad \not p = m, \tag{12}$$

guaranteeing that S'_F has a pole when $\not p$ is equal to the physical electron mass m,

$$S'_F \to \frac{Z_2}{\not p - m} \quad \text{for} \quad \not p \to m. \tag{13}$$

Z_2 is the electron wave-function renormalization constant. Similarly, we define a photon wave function renormalization constant by examining the behavior of the photon propagator near mass shell,

$$D'_F(q)^{\mu\nu} \to \frac{-Z_3\, g^{\mu\nu}}{q^2} + \text{longitudinal terms, for } q^2 \to 0, \quad (14)$$

and a vertex renormalization constant by examining the zero momentum transfer limit of the vertex with the electron lines on mass shell,

$$\Gamma_\mu(p,p)\big|_{\not{p}=m} = Z_1^{-1}\,\gamma_\mu . \quad (15)$$

The renormalization recipe consists of rescaling the self-energy and vertex functions and the bare charge so as to define renormalized functions $\tilde{S}'_F(p)$, $\tilde{D}'_F(q)^{\mu\nu}$ and $\tilde{\Gamma}_\mu(p,p')$ and a renormalized (physical) charge e,

$$\begin{aligned} S'_F(p) &= Z_2\, \tilde{S}'_F(p) \\ D'_F(q)^{\mu\nu} &= Z_3\, \tilde{D}'_F(q)^{\mu\nu}, \\ \Gamma_\mu(p,p') &= Z_1^{-1}\, \tilde{\Gamma}_\mu(p,p') \end{aligned} \quad (16)$$

$$e_0 = \frac{Z_1\, e}{Z_2\sqrt{Z_3}} .$$

The remarkable fact is that Eqs. (16), when used to express the tilde functions in terms of the physical charge and mass e and m, lead to finite values of the tilde functions for all values of the four-momenta p and p'. That is, all of the infinities can be removed into the renormalization constants Z_1, Z_2 and Z_3. It further turns out that, as a consequence of current conservation, one has

$$Z_1 = Z_2 \ . \tag{17}$$

When Eqs. (16) are substituted into a skeleton graph with external legs removed [item (iii) of our Feynman rules specifies that self-energy insertions in external lines are to be omitted] , they give the graph obtained from the skeleton by making renormalized self-energy and vertex insertions, times a product of renormalization factors, which is clearly

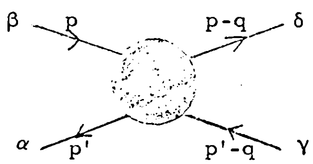
$$\left(\frac{Z_1}{Z_2\sqrt{Z_3}}\right)^n \quad Z_1^{-n} \; Z_2^f \; Z_3^b = Z_2^{f-n} \; Z_3^{b-\frac{1}{2}n} \ . \tag{18}$$

Upon using the topological relations of Eq. (10c), Eq. (18) reduces to

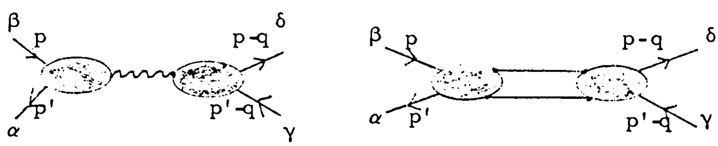
$$Z_2^{-\frac{1}{2}F} \quad Z_3^{-\frac{1}{2}B} \ , \tag{19}$$

which is exactly canceled by the product of external line factors $\sqrt{Z_2}$ and $\sqrt{Z_3}$ specified in item (iii) of our Feynman rules. Thus, our Feynman rules, with the rescalings of Eq. (16), always lead to a finite renormalized matrix element $\mathcal{M}$.

The proof that the rescalings do really make the self-energy and vertex parts finite is based on mathematical induction: one assumes that the procedure makes all graphs of order n-2 in e finite, and then demonstrates the convergence of the rescaled graphs of order n. We will briefly sketch the proof in the cases of the vertex and electron-self-energy parts, since the arguments involved are simple and involve concepts which will be useful to us later on in our discussion of anomalies. The proof in the case of the photon self-energy part is made complicated by the overlapping divergence problem, and will be omitted.[3] In order to prove renormalizability of the vertex part, we first formulate an integral equation which it satisfies. To do this, let us define an electron-positron scattering kernel $K(p', p, q)_{\alpha\beta,\gamma\delta}$, which represents all diagrams of the form



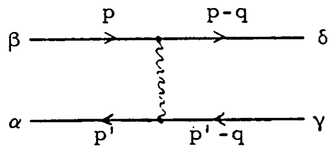
with external leg propagators removed and with disconnected diagrams and diagrams of the following two classes omitted,



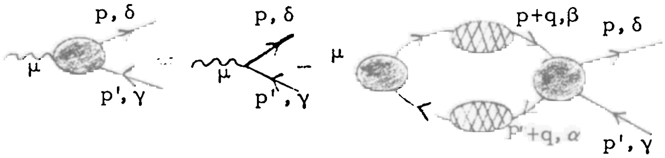
The lowest order contribution to K is

$$K^{(0)}(p', p, q)_{\alpha\beta,\gamma\delta} = \frac{ie_0^2}{q^2}(\gamma_\mu)_{\alpha\gamma}(\gamma^\mu)_{\delta\beta}, \qquad (20)$$

coming from the diagram



In terms of this kernel, we can write an integral equation for the vertex part, according to the following diagrammatic representation:



We get

$$\Gamma_\mu(p,p')_{\delta\gamma} = (\gamma_\mu)_{\delta\gamma} + \int \frac{d^4q}{(2\pi)^4} [\, i\, S'_F(p+q)\, \Gamma_\mu(p+q,p'+q) \times iS'_F(p'+q)\,]_{\beta\alpha} K(p'+q,p+q,q)_{\alpha\beta,\gamma\delta}, \tag{21}$$

or, in a condensed notation,

$$\Gamma = \gamma - \int \Gamma\, S'_F\, S'_F\, K. \tag{22}$$

If we define a rescaled kernel $\tilde{K}$ by

$$\tilde{K} \equiv Z_2^2\, K, \tag{23}$$

then Eq. (22) can be rewritten in terms of rescaled quantities as

$$\tilde{\Gamma}_\mu = Z_1\, \gamma_\mu - \int \tilde{\Gamma}_\mu\, \tilde{S}'_F\, \tilde{S}'_F\, \tilde{K}. \tag{24}$$

We wish to show that when the theory has been made finite to order n-2, the renormalization constant defined by Eq. (15) makes $\tilde{\Gamma}$ finite in order n. Since $\tilde{K}$ begins in order e^2 [see Eq. (20)] , we can get the order n contribution to $\tilde{\Gamma}$ by substituting the order $n-2$ contributions to $\tilde{\Gamma}$, $\tilde{S}'_F$ and $\tilde{K}$ into the right hand side of Eq. (24):

$$\tilde{\Gamma}_\mu^{(n)} = Z_1^{(n)} \gamma_\mu - \int \tilde{\Gamma}_\mu^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{K}^{(n-2)} \tag{25}$$

The definitions of the rescaled vertex, Eq. (15) and (16), tell us that

$$\tilde{\Gamma}_\mu^{(n)}(p,p)\big|_{\not{p}=m} = 0 \tag{26}$$

which implies that

$$Z_1^{(n)} \gamma_\mu = \int \tilde{\Gamma}_\mu^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{K}^{(n-2)}\big|_{\not{p}=\not{p}'=m}, \tag{27}$$

determining $Z_1^{(n)}$ in terms of known quantities. Note that the statement that the right hand side of Eq. (26) is proportional to a (divergent) constant times γ_μ involves no assumptions: the right hand side is a dimensionless Lorentz vector function of m, and therefore must be proportional to γ_μ, the only Lorentz vector in the problem when $\not{p}$ and $\not{p}'$ have both been replaced by m. Substituting Eq. (27) into Eq. (25), we get

$$\begin{aligned}\tilde{\Gamma}_\mu^{(n)} = & \int \tilde{\Gamma}_\mu^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{K}^{(n-2)}\big|_{\not{p}=\not{p}'=m} \\ & - \int \tilde{\Gamma}_\mu^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{S}_F'^{(n-2)} \tilde{K}^{(n-2)}.\end{aligned} \tag{28}$$

Now, using the induction hypothesis, it is an easy matter to show that all subintegrations in Eq. (24) involving some, but not all of the internal lines, are finite. Thus the only divergence is a logarithmic one connected with the overall subintegration in which the four-momenta passing through all internal lines become large simultaneously. This overall divergence is, however made finite by the single differencing of Eq. (28), and hence $\tilde{\Gamma}_\mu^{(n)}$ is seen to be convergent. We have therefore established that the vertex part Γ_μ is multiplicatively renormalizable. The important thing to note about our argument is that it has proceeded entirely from the integral equation of Eq. (22), but in no way depended on specific properties of the vector-vertex γ_μ. Since the pseudoscalar and axial-vector vertices in quantum electrodynamics satisfy similar integral equations,

$$\begin{aligned} \Gamma_\mu^5 &= \gamma_\mu \gamma_5 - \int \Gamma_\mu^5 S_F' S_F' K, \\ \Gamma^5 &= \gamma_5 - \int \Gamma^5 S_F' S_F' K, \end{aligned} \tag{29}$$

it follows by an inductive argument identical to the one given above that these vertices are also multiplicatively renormalizable,

$$\begin{aligned} \Gamma_\mu^5(p, p') &= Z_A^{-1} \tilde{\Gamma}_\mu^5 (p, p'), \\ \Gamma^5(p, p') &= Z_D^{-1} \tilde{\Gamma}^5 (p, p'), \end{aligned} \tag{30}$$

with Z_A and Z_D divergent constants.

Next, let us turn our attention to the electron propagator $S_F'(p)$. Although overlapping divergences are present here, the overlapping divergence problem can be circumvented by using an important connection between the propagator and the vertex part known as the Ward identity. To obtain the Ward identity, we multiply Eq. (6c) by $(p-p')^\mu$, giving

$$
\begin{aligned}
&(p-p')^\mu S_F'(p)\,\Gamma_\mu(p,p')S_F'(p') \qquad (31)\\
&= -(p-p')^\mu \int d^4x d^4y\, e^{ip\cdot x}\, e^{-ip'\cdot y} <0|\,T(\psi(x) j_\mu(0)\overline{\psi}(y))\,|0> \\
&= -(p-p')^\mu \int d^4x\, d^4y\, e^{i(p'-p)\cdot x}\, e^{-ip'\cdot y} <0|\,T(\psi(0) j_\mu(x)\overline{\psi}(y))\,|0> \\
&= \int d^4x\, d^4y\, e^{i(p'-p)\cdot x}\, e^{-ip'\cdot y}\, i\,\frac{\partial}{\partial x_\mu} <0|\,T(\psi(0) j_\mu(x)\overline{\psi}(y))\,|0> \\
&= i\int d^4x\, d^4y\, e^{i(p'-p)\cdot x}\, e^{-ip'\cdot y} <0|\,T(\psi(0)\frac{\partial}{\partial x_\mu} j_\mu(x)\,\overline{\psi}(y)) \\
&+ \delta(x_0)(T[j_0(x),\psi(0)]\,\overline{\psi}(y)) + \delta(x_0-y_0)\,T(\psi(0)[j_0(x),\overline{\psi}(y)])\,|0>.
\end{aligned}
$$

[In going from the first line to the second line on the right-hand side of Eq. (31), we have set $x \rightarrow -x$, $y \rightarrow y-x$, and used translation invariance of the T-product. In going to the next line we have integrated by parts, and in the final line we have used the standard formula for the time derivative of a T-product.] Using Eq. (3) to evaluate the first term in the final line, using the canonical commutation relations to evaluate the two remaining terms, and comparing with Eq. (6a), we get

$$(p-p')^\mu S_F'(p)\,\Gamma_\mu(p,p')S_F'(p') = S_F'(p')-S_F'(p), \qquad (32)$$

or multiplying by $S_F'(p)^{-1}\ldots S_F'(p')^{-1}$,

$$(p-p')^\mu\,\Gamma_\mu(p,p') = S_F'(p)^{-1} - S_F'(p')^{-1} \qquad (33)$$

which is the Ward identity. Now, when $\not{p}' = m$, Eqs. (7) and (12) tell us that $S_F'(p')^{-1} = 0$. Hence

$$S_F'(p)^{-1} = (p-p')^{\mu}\,\Gamma_{\mu}(p,p')\big|_{\not{p}'=m}, \tag{34}$$

which immediately implies that $Z_1 S_F'(p)^{-1}$ is finite to order n, since $Z_1\,\Gamma_{\mu}$ is. Furthermore, examining Eq. (34) in the neighborhood of $\not{p} = m$, we have

$$\begin{aligned} Z_2^{-1}(\not{p}-m) \approx S_F'(p)^{-1}\big|_{\not{p}\approx m} &= (p-p')^{\mu}\,\Gamma_{\mu}(p,p')\big|_{\substack{\not{p}'=m\\ \not{p}\approx m}} \\ &\approx Z_1^{-1}(\not{p}-m), \end{aligned} \tag{35}$$

which tells us that $Z_1 = Z_2$, as claimed. We see, then, that the Ward identity plays a very useful role in discussing the renormalization of QED. In deriving the current conservation condition in Eq. (3) and the Ward identity in Eq. (16), we have made with impunity the type of dangerous manipulations referred to in the opening paragraphs. It turns out, however, that these manipulations are justified, and the Ward identity and other consequences of vector current conservation are valid in all orders of perturbation theory. In other words, in QED with only the vector current $j_{\mu}(x)$ considered, there are no Ward identity anomalies.

2. THE VVA TRIANGLE ANOMALY

To get our first example of a Ward identity anomaly, we will consider the axial-vector current $j_\mu^5(x)$ in quantum electrodynamics,

$$j_\mu^5(x) = \overline{\psi}(x)\,\gamma_\mu \gamma_5 \psi(x). \tag{36}$$

This current plays no role in pure quantum electrodynamics, but appears when the weak interaction between electrons and neutrinos is taken into account within the framework of the local current-current theory. {To see this, we note that in the current-current theory without intermediate boson, the leptonic weak interactions are described by the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = (G/\sqrt{2})\, j_\lambda^\dagger\, j^\lambda, \tag{37}$$

where $G \approx 10^{-5}/M_{\text{proton}}^{\;2}$ is the Fermi constant and where

$$j^\lambda = \overline{\nu}_\mu \gamma^\lambda (1-\gamma_5)\mu + \overline{\nu}_e \gamma^\lambda (1-\gamma_5) e \tag{38}$$

is the leptonic current. In addition to the usual terms describing muon decay, Eq. (37) contains the terms

$$\begin{aligned}(G/\sqrt{2})[&\overline{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu\, \overline{\nu}_\mu \gamma^\lambda (1-\gamma_5)\mu \\ &+ \overline{e}\gamma_\lambda(1-\gamma_5)\,\nu_e \overline{\nu}_e \gamma^\lambda (1-\gamma_5) e],\end{aligned} \tag{39}$$

which describe elastic neutrino-lepton scattering. It is frequently convenient to rewrite Eq. (39), by means of a Fierz transformation,[4] in the form (the so-called charge retention ordering)

$$(G/\sqrt{2})[\overline{\mu}\,\gamma_\lambda(1-\gamma_5)\,\mu\,\overline{\nu}_\mu\,\gamma^\lambda(1-\gamma_5)\,\nu_\mu$$
$$+\,\overline{e}\,\gamma_\lambda(1-\gamma_5)\,e\,\overline{\nu}_e\,\gamma^\lambda(1-\gamma_5)\,\nu_e], \qquad (40)$$

which clearly involves the muon and electron axial-vector currents as well as the corresponding vector currents.} Proceeding in analogy with our treatment of the vector current in the last section, we use the equations of motion to calculate the divergence of the axial-vector current,

$$\begin{aligned}\partial j^5_\mu(x)/\partial x_\mu &= \overline{\psi}(x)[\,im_0+ie_0\gamma_\mu A^\mu(x)\,]\,\gamma_5\psi(x)\\ &\quad+\overline{\psi}(x)\,\gamma_5[\,im_0+ie_0\gamma_\mu A^\mu(x)\,]\,\psi(x) \qquad (41)\\ &= 2im_0 j^5(x),\end{aligned}$$

with

$$j^5(x) = \overline{\psi}(x)\gamma_5\psi(x) \qquad (42)$$

the pseudoscalar current. Defining axial-vector and pseudoscalar vertex parts by analogy with Eq. (6c);

$$\begin{aligned}&S'_F(p)\;\Gamma^5_\mu(p,p')\;S'_F(p')\\ &\quad= -\int d^4x\,d^4y\;e^{ip\cdot x}e^{-ip'\cdot y}\langle 0|\,T(\psi(x)j^5_\mu(0)\overline{\psi}(y))\,|0\rangle,\\ &S'_F(p)\;\Gamma^5(p,p')\;S'_F(p') \qquad (43)\\ &\quad= -\int d^4x\,d^4y\;e^{ip\cdot x}e^{-ip'\cdot y}\langle 0|\,T(\psi(x)j^5(0)\overline{\psi}(y))\,|0\rangle,\end{aligned}$$

a derivation precisely analogous to that of Eq. (31) gives the naive axial-vector Ward identity

$$(p-p')^\mu\,\Gamma^5_\mu(p,p') = 2m_0\,\Gamma^5(p,p') + S'_F(p)^{-1}\gamma_5 + \gamma_5 S'_F(p')^{-1}. \qquad (44)$$

2.1 Ward Identity in Perturbation Theory

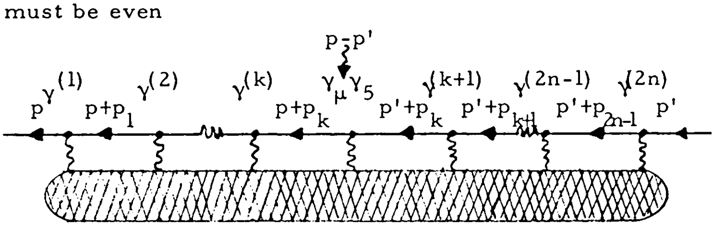
We wish to examine whether the naive formal manipulations which led to Eq. (44) are actually valid in perturbation theory. Defining vertex corrections Λ^5_μ and Λ^5 by

$$\begin{aligned}\Gamma^5_\mu &= \gamma_\mu\gamma_5 + \Lambda^5_\mu,\\ \Gamma^5 &= \gamma_5 + \Lambda^5\end{aligned} \tag{45}$$

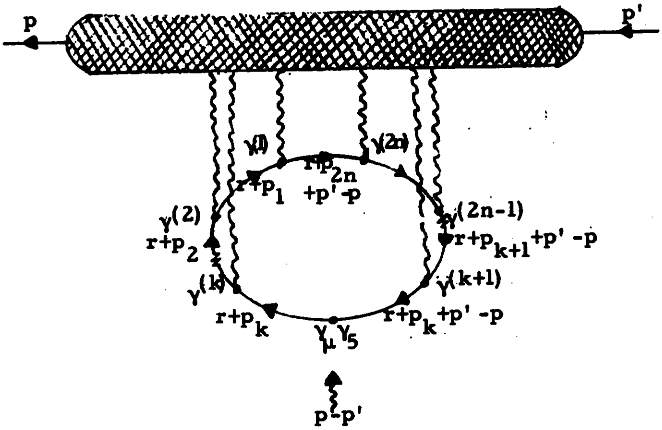
and using Eq. (7), we may rewrite Eq. (44) as

$$(p-p')^\mu \Lambda^5_\mu(p,p') = 2m_0\Lambda^5(p,p') - \Sigma(p)\gamma_5 - \gamma_5\Sigma(p'). \tag{46}$$

In order to derive Eq. (46), let us divide the diagrams contributing to $\Lambda^5_\mu(p,p')$ into two types: (a) diagrams in which the axial-vector vertex $\gamma_\mu\gamma_5$ is attached to the fermion line beginning with external four-momentum p' and ending with external four-momentum p; (b) diagrams in which the axial-vector vertex $\gamma_\mu\gamma_5$ is attached to an internal closed loop. Because the axial-vector current is charge conjugation <u>even</u>,[5] Furry's theorem tells us that the number of photon lines emerging from the closed loop must be even



(a)



(b)

A typical contribution of type (a) has the form

$$\sum_{k=1}^{2n-1}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{\not p+\not p_j-m_0}\right]\gamma^{(k)}\frac{1}{\not p+\not p_k-m_0}\gamma_\mu\gamma_5\frac{1}{\not p'+\not p_k-m_0}$$
$$\times\prod_{j=k+1}^{2n-1}\left[\gamma^{(j)}\frac{1}{\not p'+\not p_j-m_0}\right]\gamma^{(2n)}(\dots), \tag{47}$$

where we have focused our attention on the line to which the $\gamma_\mu\gamma_5$ vertex is attached and have denoted the remainder of the diagram by $(\dots)$. Multiplying Eq. (47) by $(p-p')^\mu$ and making use of the identity

$$\frac{1}{\not p+\not p_k-m_0}(\not p-\not p')\gamma_5\frac{1}{\not p'+\not p_k-m_0}=\frac{1}{\not p+\not p_k-m_0}(2m_0\gamma_5)$$
$$\times\frac{1}{\not p'+\not p_k-m_0}+\frac{1}{\not p+\not p_k-m_0}\gamma_5+\gamma_5\frac{1}{\not p'+\not p_k-m_0} \tag{48}$$

gives, after a little algebraic rearrangement,

$$\sum_{k=1}^{2n-1}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{\not p+\not p_j-m_0}\right]\gamma^{(k)}\frac{1}{\not p+\not p_k-m_0}2m_0\gamma_5$$

$$\times \frac{1}{\not p' + \not p_k - m_0} \prod_{j=k+1}^{2n-1} \left[\gamma^{(j)} \frac{1}{\not p' + \not p_j - m_0} \right] \gamma^{(2n)} (\ldots)$$
$$-(\ldots) \prod_{j=1}^{2n-1} \left[\gamma^{(j)} \frac{1}{\not p + \not p_j - m_0} \right] \gamma^{(2n)} \gamma_5$$
$$-\gamma_5 \prod_{j=1}^{2n-1} \left[\gamma^{(j)} \frac{1}{\not p' + \not p_j - m_0} \right] \gamma^{(2n)} (\ldots). \tag{49}$$

The first, second, and third terms in Eq. (49) are, respectively, the type-(a) piece of Λ^5 and the pieces of $-\Sigma(p)\gamma_5$ and $-\gamma_5\Sigma(p')$ corresponding to the type-(a) piece of Λ_μ^5 in Eq. (47). Summing over all type-(a) contributions to Λ_μ^5, we get

$$(p-p')^\mu \Lambda_\mu^{5(a)}(p,p')$$
$$=2m_0\Lambda^{5(a)}(p,p') - \Sigma(p)\gamma_5 - \gamma_5\Sigma(p'). \tag{50}$$

We turn next to contributions to Λ_μ^5 of type (b). A typical term is

$$\int d^4r \, \mathrm{Tr}\{ \sum_{k=1}^{2n} \prod_{j=1}^{k-1} \left[\gamma^{(j)} \frac{1}{r + \not p_j - m_0} \right] \gamma^{(k)} \frac{1}{r + \not p_k - m_0} \gamma_\mu \gamma_5$$
$$\times \frac{1}{r + \not p_k + \not p' - \not p - m_0} \prod_{j=k+1}^{2n} \left[\gamma^{(j)} \frac{1}{r + \not p_j + \not p' - \not p - m_0} \right] \} (\ldots). \tag{51}$$

Multiplying by $(p-p')^\mu$ and using Eq. (48) gives

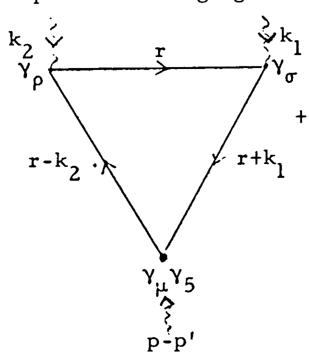
$$\int d^4r \, \mathrm{Tr}\{ \sum_{k=1}^{2n} \prod_{j=1}^{k-1} \left[\gamma^{(j)} \frac{1}{r + \not p_j - m_0} \right] \gamma^{(k)} \frac{1}{r + \not p_k - m_0} 2m_0 \gamma_5$$
$$\times \frac{1}{r + \not p_k + \not p' - p - m_0} \prod_{j=k+1}^{2n} \left[\gamma^{(j)} \frac{1}{r + \not p_j + \not p' - \not p - m_0} \right] \}$$
$$\times (\ldots) + \int d^4r \, \mathrm{Tr}\{ \gamma^5 \prod_{j=1}^{2n} \left[\gamma^{(j)} \frac{1}{r + \not p_j - m_0} \right]$$
$$-\gamma_5 \prod_{j=1}^{2n} \left[\gamma^{(j)} \frac{1}{r + \not p_j + \not p' - \not p - m_0} \right] \} (\ldots). \tag{52}$$

The first term in Eq. (52) is the type-(b) contribution to Λ^5 corresponding to Eq. (51), while making the change of variable $r \rightarrow r + p' - p$ in the integration in the second term causes the second and third terms to cancel. This gives, when we sum over all type-(b) contributions,

$$(p-p')^{\mu}\Lambda_{\mu}^{5(b)}(p,p') = 2m_0\Lambda^{5(b)}(p,p'). \qquad (53)$$

The Ward identity of Eq. (44) is finally obtained by adding Eqs. (50) and (53).

Clearly, the only step in the above derivation which is not simply an algebraic rearrangement is the change of integration variable in the second term of Eq. (52). This will be a valid operation provided that the integral is at worst superficially logarithmically divergent, a condition that is satisfied by loops with four or more photons, that is, loops with $n \geq 2$. However, when the loop is a triangle graph with only two photons emerging,



+ diagram with photon lines interchanged we have $n = 1$, and the integral in Eq. (52) appears to be quadratically divergent. Actually, since $\mathrm{tr}\{\gamma_5 \gamma^{(1)} \not{r}\, \gamma^{(2)} \not{r}\} = 0$, the integral in the $n = 1$ case is superficially linearly divergent. Since it is well known that translation of a linearly divergent Feynman integral is not necessarily a valid operation,[6] we suspect that Eq. (53) breaks down for the triangle graph.

To see that this really does happen, we make use of an explicit expression for the triangle graph calculated by Rosenberg.[7] The sum of the triangle illustrated above and the corresponding graph with the two photons interchanged is

$$\frac{-ie_0^2}{(2\pi)^4} R_{\sigma\rho\mu} \equiv 2\int \frac{d^4 r}{(2\pi)^4}(-1)\,\mathrm{Tr}\{\frac{i}{\not{r}+\not{k}_1-m_0}(-ie_0\gamma_\sigma) \times \frac{i}{\not{r}-m_0}(-ie_0\gamma_\rho)\frac{i}{\not{r}-\not{k}_2-m_0}\gamma_\mu\gamma_5\} . \tag{54}$$

This expression is linearly divergent, but as in the case of the photon self-energy part, current conservation requires that the photons couple through their field-strength tensors $k_2^\xi \varepsilon_2^\rho - k_2^\rho \varepsilon_2^\xi$, $k_1^\eta \varepsilon_1^\sigma - k_1^\sigma \varepsilon_1^\eta$, using up two powers of momentum and leaving a convergent integral with $D_{\mathrm{eff}} = -1$. The simplest way to make use of current conservation in practice is to first write down the most general form for the axial-tensor $R_{\sigma\rho\mu}$, consistent with the requirements of parity and Lorentz invariance. A little thought shows that this is

$$
\begin{aligned}
R_{\sigma\rho\mu}(k_1, k_2) = {} & A_1 k_1^{\tau} \epsilon_{\tau\sigma\rho\mu} + A_2 k_2^{\tau} \epsilon_{\tau\sigma\rho\mu} \\
& + A_3 k_{1\rho} k_1^{\xi} k_2^{\tau} \epsilon_{\xi\tau\sigma\mu} + A_4 k_{2\rho} k_1^{\xi} k_2^{\tau} \epsilon_{\xi\tau\sigma\mu} \\
& + A_5 k_{1\sigma} k_1^{\xi} k_2^{\tau} \epsilon_{\xi\tau\rho\mu} + A_6 k_{2\sigma} k_1^{\xi} k_2^{\tau} \epsilon_{\xi\tau\rho\mu},
\end{aligned}
\tag{55}
$$

with the $A_j (j=1, \ldots, 6)$ Lorentz scalar functions of k_1 and k_2. The requirement of Bose symmetry, $R_{\sigma\rho\mu}(k_1, k_2) = R_{\rho\sigma\mu}(k_2, k_1)$ implies that

$$
\begin{aligned}
A_1(k_1, k_2) &= -A_2(k_2, k_1) \\
A_3(k_1, k_2) &= -A_6(k_2, k_1) \\
A_4(k_1, k_2) &= -A_5(k_2, k_1).
\end{aligned}
\tag{56}
$$

Imposing the condition of current conservation, $k_1^{\sigma} R_{\sigma\rho\mu} = k_2^{\rho} R_{\sigma\rho\mu} = 0$ gives us relations between A_1, A_2 and the remaining A' s,

$$
\begin{aligned}
A_1 &= k_1 \cdot k_2 \, A_3 + k_2^2 \, A_4, \\
A_2 &= k_1^2 \, A_5 + k_1 \cdot k_2 \, A_6.
\end{aligned}
\tag{57}
$$

Now $A_{3,4,5,6}$ each appear in Eq. (55) multiplied by three powers of external photon four-momentum, and therefore will each involve a highly convergent Feynman integral with $D_{\mathrm{eff}} = 1\text{-}3 = -2$. On the other hand, the scalars A_1 and A_2 each multiply just one power of momentum, and therefore are represented by formally logarithmically divergent Feynman integrals with $D_{\mathrm{eff}} = 1\text{-}1=0$. But current conservation saves the day, since it allows us to calculate A_1 and A_2 directly from the convergent quantities $A_{3,4,5,6}$. Introducing Feynman parameters and doing the r integration in the standard manner, we find

$$\begin{aligned} A_3(k_1,k_2) &= -16\pi^2\, I_{11}(k_1,k_2), \\ A_4(k_1,k_2) &= 16\pi^2\,[\, I_{20}(k_1,k_2) - I_{10}(k_1,k_2)\,], \end{aligned} \tag{58}$$

with

$$\begin{aligned} I_{st}(k_1,k_2) = \int_0^1 dx \int_0^{1-x} dy\; x^s y^t [\, & y(1-y)k_1^2 \\ & + x(1-x)k_2^2 + 2xy k_1\cdot k_2 - m_0^2]^{-1}. \end{aligned} \tag{59}$$

In order to check the Ward identity, we will also need an expression for the triangle graph with $\gamma_\mu\gamma_5$ replaced by $2m_0\gamma_5$. Defining

$$\begin{aligned} \frac{-ie_0^2}{(2\pi)^4}\, 2m_0 R_{\sigma\rho} \equiv 2\int \frac{d^4r}{(2\pi)^4}(-1)\,\mathrm{Tr}\Big\{ & \frac{i}{\not r + \not k_1 - m_0}(-ie_0\gamma_\sigma) \\ & \times \frac{i}{\not r - m_0}(-ie_0\gamma_\rho)\,\frac{i}{\not r - \not k_2 - m_0}\, 2m_0\gamma_5\Big\}, \end{aligned} \tag{60}$$

we find by straightforward calculation that

$$\begin{aligned} R_{\sigma\rho} &= k_1^\xi\, k_2^\tau\, \epsilon_{\xi\tau\sigma\rho} B_1 \\ B_1 &= 8\pi^2 m_0 I_{00}(k_1,k_2). \end{aligned} \tag{61}$$

We are now ready to calculate the divergence of the axial-vector triangle diagram. If the Ward identity holds, we should find

$$-(k_1+k_2)^\mu R_{\sigma\rho\mu} = 2m_0 R_{\sigma\rho}, \tag{62}$$

but from Eqs. (55) - (61) we find, instead,

$$-(k_1+k_2)^{\mu} R_{\sigma\rho\mu} = 2m_0 R_{\sigma\rho} + 8\pi^2 k_1^{\xi} k_2^{\tau} \epsilon_{\xi\tau\sigma\rho}. \qquad (63)$$

We see that the <u>axial-vector Ward identity fails in the case of the triangle graph</u>. The failure is a result of the fact that the integration variable in a linearly divergent Feynman integral cannot be freely translated.

2.2 Impossibility of Eliminating the Anomaly by a Subtraction

The question now immediately arises, whether it is possible to redefine $R_{\sigma\rho\mu}$ by a subtraction or in some other manner, so as to eliminate the Ward identity anomaly of Eq. (63), but without introducing any new types of anomalous behavior? A subtraction term, in order to preserve the expected behavior of $R_{\sigma\rho\mu}$, must have the following properties: (i) It must be a three-index Lorentz axial-tensor. (ii) It must be symmetric under interchange of the photon variables (k_1,σ) and (k_2,ρ). (iii) It must be a polynomial in the momentum variables k_1 and k_2. This requirement follows from generalized unitarity, which says that discontinuities of $R_{\sigma\rho\mu}$ with respect to external variables are related to Feynman amplitudes for intermediate state processes by the Cutkosky rules. Since taking a discontinuity renders the Feynman integrals in Eq. (54) convergent, the discontinuities of $R_{\sigma\rho\mu}$ have no anomalies, and $R_{\sigma\rho\mu}$ satisfies generalized unitarity by itself. Thus the subtraction term must have vanishing discontinuities, i.e., it must be a polynomial. (iv) If we set $k_1 = \xi Q$, $k_2 = -\xi Q + \xi R + S$, with Q, R, S arbitrary, and let $\xi \to \infty$, the subtraction must diverge at worst as ξ times a power of $\ln \xi$. This requirement follows from Weinberg's theorem,[8] which states that when the external momenta of a Feynman graph approach infinity as above, the largest power appearing is $\xi^{D_{MAX}}$, where D_{MAX} is the maximum of the superficial divergences of the graph and of its subgraphs. For the triangle, $D_{MAX} = 1$. Since $R_{\sigma\rho\mu}$ already has asymptotic behavior consistent with Weinberg's theorem, so must the subtraction. We note for future reference that when $R = 0$,

$$R_{\sigma\rho\mu}(k_1 = \xi Q,\ k_2 = -\xi Q + p' - p) \to -8\pi^2 \xi Q^\tau \epsilon_{\tau\sigma\rho\mu} + O(\ln \xi). \quad (64)$$

(v) The subtraction must have the dimensionality of a mass. (vi) The subtraction must satisfy the requirements of vector current conservation.

It is easy to see that it is in fact impossible to find a subtraction satisfying these six conditions. The first five conditions are only satisfied by a term of the form

$\varepsilon_{\tau\sigma\rho\mu}(k_1-k_2)^\tau$, but this term does not satisfy condition (vi)! Thus, while it is possible to define a subtracted triangle diagram $R'_{\sigma\rho\mu}$ which has a normal axial-vector Ward identity,

$$R'_{\sigma\rho\mu} = R_{\sigma\rho\mu} + 4\pi^2 \varepsilon_{\tau\sigma\rho\mu}(k_1-k_2)^\tau ,$$
$$-(k_1+k_2)^\mu R'_{\sigma\rho\mu} = 2m_0 R_{\sigma\rho}, \tag{65}$$

it violates vector current conservation,

$$k_1^\sigma R'_{\sigma\rho\mu} = -4\pi^2 k_1^\sigma k_2^\tau \varepsilon_{\tau\sigma\rho\mu},$$
$$k_2^\rho R'_{\sigma\rho\mu} = 4\pi^2 k_2^\rho k_1^\tau \varepsilon_{\tau\sigma\rho\mu} . \tag{66}$$

All we have succeeded in doing is substituting one divergence anomaly for another. Similarly, if we introduce projection operators in order to force all divergences to have the correct values, as in

$$R''_{\sigma\rho\mu} = (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) R_{\sigma\rho}{}^\nu + \frac{q_\mu}{q^2} 2m_0 R_{\sigma\rho} , q=-(k_1+k_2),$$
$$R'''_{\sigma\rho\mu} = (g_{\sigma\xi} - \frac{k_{1\sigma}k_{1\xi}}{k_1^2})(g_{\rho\eta} - \frac{k_{2\rho}k_{2\eta}}{k_2^2}) R^{\xi\eta}_{\mu}{}' , \tag{67}$$

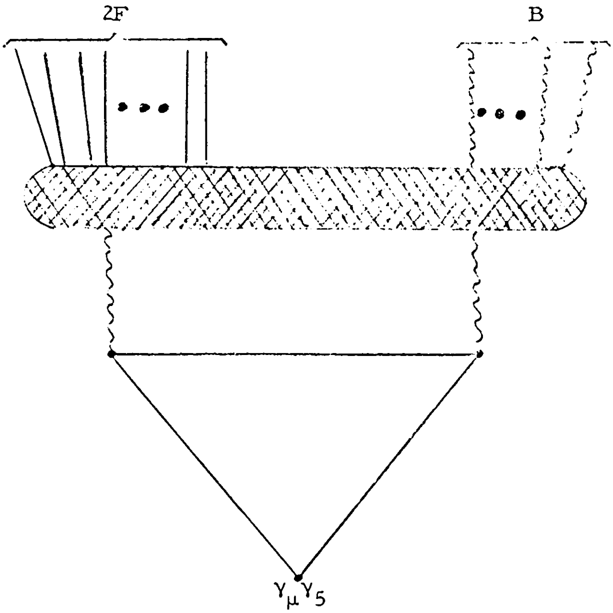
we introduce spurious kinematic singularities, in violation of condition (iii). We see that the situation is like the proverbial square peg being inserted in a round hole--we can fix the triangle diagram in one respect only by making it anomalous in some other respect. The anomaly really consists, not of Eq. (63) by itself, but rather of the impossibility of finding a redefined triangle diagram which simultaneously satisfies all of the six requirements above. Failure to appreciate this chameleon-like quality of the anomaly has resulted in erroneous claims in the literature that the anomaly can be eliminated.

In the remainder of these lectures, we will always use the expression $R_{\sigma\rho\mu}$ for the triangle, rather than the subtracted expression $R'_{\sigma\rho\mu}$ which has normal axial-vector but anomalous vector Ward identities. Since, *a priori*, it would appear that the vector Ward identity is no more sacred than the axial-vector Ward identity, the choice requires some words of justification. We note, first of all, that enforcing vector current conservation is essential if we want the triangle to describe the physical coupling between a neutrino-antineutrino pair and two photons. The reason is that since two photons can never be in a state with $J = 1$, the coupling of the $J=1$ state of the $\nu\bar{\nu}$ pair to two photons must vanish. Expressed in terms of $R_{\sigma\rho\mu}$, this requirement states that if ℓ_μ is an arbitrary spin-one polarization vector satisfying $\ell\cdot(k_1+k_2)=0$, and if (ε_1, k_1) and (ε_2, k_2) are photon variables satisfying $\varepsilon_1\cdot k_1 = \varepsilon_2\cdot k_2 = k_1^2 = k_2^2 = 0$, we must have $\ell^\mu \varepsilon_1^\sigma \varepsilon_2^\rho R_{\sigma\rho\mu} \equiv 0$. It has been shown by Rosenberg[7] that Eqs. (55) - (58) do satisfy this condition. On the other hand, the subtraction term in

Eq. (65) does not satisfy this condition, and hence $R'_{\sigma\rho\mu}$ does not, and so cannot describe a physical $\nu\nu$-pair--two photon coupling. Secondly, we will see below that the relations between the Ward identity anomaly and commutator anomalies take a particularly simple form when vector current conservation (and hence gauge invariance) are maintained. Finally, the most interesting application of the triangle anomaly, the derivation of a low energy theorem for π^0 decay, is independent of which definition of the triangle is used. So again, it proves convenient (although not essential in this case) to maintain gauge invariance.

2.3 Anomaly for General Axial-Vector Current Matrix Element

Let us now return to the diagrammatic analysis which we left off at Eq. (63). Clearly, the breakdown of the Ward identity for the basic triangle will also cause failure of the Ward identity for any graph of the type illustrated below, in which the two photon lines coming out of the triangle graph join onto a "blob" from which 2F fermion and B boson lines emerge.



From Eq. (63) for the divergence of the basic triangle graph, it is possible to show that the breakdown of the axial-vector Ward identity in the general case is simply described by replacing Eq. (41) for the axial-vector-current divergence (which we have shown to be incorrect) by[9]

$$\frac{\partial}{\partial x_\mu} j_\mu^5(x) = 2 i m_0 j^5(x) + \frac{\alpha_0}{4\pi} F^{\xi\sigma}(x)\, F^{\tau\rho}(x) \varepsilon_{\xi\sigma\tau\rho}. \qquad (68)$$

Eq. (68) is easily verified by using the following Feynman rules for the vertices of j_μ^5, j^5 and $(\alpha_0/4\pi)\, F^{\xi\sigma}\, F^{\tau\rho} \varepsilon_{\xi\sigma\tau\rho}$,

Operator	Vertex Factor	
$j^5_\mu(x)$	$\overset{p}{\longleftarrow}\bullet\overset{p'}{\longleftarrow}$	$\longleftrightarrow \quad \gamma_\mu\gamma_5$
$j^5(x)$	$\overset{p}{\longleftarrow}\bullet\overset{p'}{\longleftarrow}$	$\longleftrightarrow \quad \gamma_5$
$\frac{\alpha_0}{4\pi}\,F^{\xi\sigma}(x)F^{\tau\rho}(x)\,\epsilon_{\xi\sigma\tau\rho}$	$\overset{k_1\sigma}{\longrightarrow}\bullet\overset{k_2,\rho}{\longleftarrow}$	$\longleftrightarrow \quad \frac{-2\alpha_0}{\pi}k_1^\xi k_2^\tau \epsilon_{\xi\sigma\tau\rho}$.

$$[\,\alpha_0 = e_0^2/4\pi\,]$$

Using Eq. (68) we can easily see how the Ward identity for the axial-vector vertex is modified. Defining $\overline{F}(p,p')$ by

$$S_F'(p)\,\overline{F}(p,p')S_F'(p') = -\int d^4x\, d^4y\, e^{ip\cdot x}e^{-ip'\cdot y} \tag{69}$$
$$\times <0|\,T(\psi(x)\;F^{\xi\sigma}(0)\;F^{\tau\rho}(0)\;\epsilon_{\xi\sigma\tau\rho}\overline{\psi}(y))\,|0>,$$

then we find

$$(p-p')^\mu\,\Gamma^5_\mu(p,p') = 2m_0\Gamma^5(p,p') + S_F'(p)^{-1}\gamma_5 + \gamma_5 S_F'(p')^{-1}$$
$$-i(\alpha_0/4\pi)\;\overline{F}(p,p'), \tag{70}$$

which replaces Eq. (44).

2.4 Coordinate Space Calculation

So far we have worked exclusively in momentum space. However, the fact that Eq. (68) shows the anomaly to have a simple form in coordinate space suggests that a coordinate space derivation should be possible. To proceed in coordinate space, let us confine ourselves to the case of a c-number electromagnetic field and let us regard the axial-vector current j^5_μ as the limit of a nonlocal current in which the fields $\overline{\psi}$ and ψ are evaluated at separated space-time points,

$$j^5_\mu(x) = \lim_{\varepsilon\to 0} j^5_\mu(x,\varepsilon), \qquad (71a)$$

$$j^5_\mu(x,\varepsilon) = \bar\psi(x+\tfrac{\varepsilon}{2})\,\gamma_\mu\gamma_5\,\psi(x-\tfrac{\varepsilon}{2})\exp[-ie_0\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} d\ell\cdot A(\ell)]\,. \qquad (71b)$$

The line integral in Eq. (71b) is necessary in order to insure invariance of $j^5_\mu(x,\varepsilon)$ under the gauge transformation

$$\psi(x) \to e^{-ie_0 v(x)}\psi(x),$$

$$A^\mu(x) \to A^\mu(x) + \frac{\partial v(x)}{\partial x_\mu}. \qquad (72)$$

Expanding the exponential out to first order in ε and then using the equations of motion to calculate the divergence gives

$$j^5_\mu(x,\varepsilon) = \bar\psi(x+\tfrac{\varepsilon}{2})\,\gamma_\mu\gamma_5\psi(x-\tfrac{\varepsilon}{2})[1-ie_0\varepsilon_\lambda A^\lambda(x)] + \text{higher order},$$

$$\frac{\partial}{\partial x_\mu} j^5_\mu(x,\varepsilon) = \bar\psi(x+\tfrac{\varepsilon}{2})\gamma_\mu\gamma_5\,\psi(x-\tfrac{\varepsilon}{2})\{-ie_0\,\varepsilon_\lambda\frac{\partial}{\partial x_\mu}A^\lambda(x)$$

$$\underbrace{-ie_0[A^\mu(x-\tfrac{\varepsilon}{2})-A^\mu(x+\tfrac{\varepsilon}{2})]}_{ie_0\varepsilon_\lambda\frac{\partial}{\partial x_\lambda}A^\mu(x)}\} + 2im_0 j^5(x,\varepsilon) + \text{higher order}$$

$$= j^5_\mu(x,\varepsilon)\, ie_0\varepsilon_\lambda F^{\mu\lambda} + 2im_0 j^5(x,\varepsilon) + \text{higher order}. \qquad (73)$$

Taking the vacuum expectation of Eq. (73), we get the divergence equation for the generating functional describing the coupling, through a single closed loop, of the axial-vector current to an arbitrary number of external c-number photons,

$$\frac{\partial}{\partial x_\mu} \langle 0\,|\,j^5_\mu(x,\varepsilon)|0\rangle \qquad (74)$$

$$= ie_0\langle 0|j^5_\mu(x,\varepsilon)\,\varepsilon_\lambda|0\rangle F^{\mu\lambda} + 2im_0\langle 0|j^5(x,\varepsilon)|0\rangle + \text{higher order}.$$

The first term on the right hand side of Eq. (74) is formally of order ε, and is neglected in the naive derivation of Eq. (41). A careful calculation shows, however, that $\langle 0 | j^5_\mu(x,\varepsilon) | 0\rangle$ is of order ε^{-1} as $\varepsilon \to 0$, so that in fact the first term makes a finite contribution as $\varepsilon \to 0$. Carrying out the details[10] gives

$$
\begin{aligned}
& ie_0 \langle 0 | j^5_\mu(x,\varepsilon)\, \varepsilon_\lambda | 0\rangle F^{\mu\lambda}(x) \\
&= \frac{\alpha_0}{4\pi} \varepsilon_{\mu\lambda\xi\eta} F^{\mu\lambda}(x)\, F^{\xi\eta}(x) \;+\; O(\varepsilon),
\end{aligned}
\tag{75}
$$

in agreement with the vacuum expectation of Eq. (68). Thus, the anomaly can be obtained by the equation of motion approach, provided that one is careful in handling the singular operator products which appear in the axial-vector current.

3. CONSEQUENCES OF THE TRIANGLE ANOMALY

Let us now examine some of the consequences of the anomalous axial-vector divergence which we found in the previous section. We will see that the anomaly produces changes in certain standard results having to do with renormalization of the axial-vector vertex and with γ_5-symmetry. We will also find that Eq. (68) leads to a low energy theorem for the vacuum to two photon matrix element of the naive divergence $2im_0 j^5$, the generalizations of which have interesting physical implications in π^0 decay.

3.1 Renormalization of the Axial-Vector Vertex

Let us begin with an analysis of the behavior of the axial-vector vertex under renormalization. As we recall, according to Eq. (30) both the axial-vector vertex and the pseudoscalar vertex are multiplicatively renormalizable, with respective renormalization constants Z_A and Z_D. Let us now ask whether these renormalization constants are the same as, or are simply related to, the electron-wave function renormalization Z_3. We will first find the answer which follows from the naive axial-vector Ward identity of Eq. (44), and then see how it is changed when the anomaly is taken into account, as in Eq.(70).

In order to talk in a precise way about the infinite renormalization constants m_0, Z_2, Z_A and Z_D, we will follow the standard procedure of introducing a cutoff Λ into our Feynman rules, so that the renormalization constants become finite functions of Λ which diverge as $\Lambda \to \infty$. There are many different ways of introducing a cutoff which accomplish this. [One particular way is specified in detail in the next section.] As long as we deal only with low-energy theorem type questions, in which all external momenta remain small compared with the cutoff, the precise details of how the cutoff is introduced are irrelevant. In particular, no ambiguities in order of limit are involved in a calculation in which external momenta are allowed to approach zero while the cutoff approaches infinity. On the other hand, we will see that in Bjorken limit calculations, in which external momenta approach infinity, the question of whether the external momenta remain much smaller than the cutoff, or become much larger than the cutoff, as both approach infinity, becomes of crucial importance.

To proceed, we start from the naive axial-vector Ward identity of Eq. (44) and set $p = p'$, so that the axial-vector vertex term on the left-hand side vanishes, and then multiply through by the electron wave function renormalization Z_2. This gives

$$2m_0 Z_2 \Gamma^5(p,p) = -[Z_2 S'_F(p)^{-1}\gamma_5 + \gamma_5 Z_2 S_F(p)^{-1}] \qquad (76)$$
$$= -[\tilde{S}'_F(p)^{-1}\gamma_5 + \gamma_5 \tilde{S}'_F(p)^{-1}],$$

and since the right-hand side of Eq. (76) is finite (i.e. Λ-independent in the limit of large Λ) , we see that the left hand side, $2m_0 Z_2 \Gamma^5(p,p)$, is also finite. In Section 1 we saw that, for general p and p′, $\Gamma^5(p,p')$ is always made finite by multiplication by a renormalization constant Z_D. Hence we conclude that Z_D and $2m_0 Z_2$ are the same, up to a finite factor,

$$2m_0 Z_2 / Z_D = \text{finite}. \qquad (77)$$

Next, we substitute into the naive axial-vector Ward identity the expressions of Eqs. (16) and (30) for the renormalized electron propagator, axial-vector vertex and pseudoscalar vertex, and multiply through by Z_A, giving

$$(p-p')^\mu \tilde{\Gamma}^5_\mu(p,p') = \frac{Z_A}{Z_2}\left[\left(\frac{2m_0 Z_2}{Z_D}\right)\tilde{\Gamma}^5(p,p') + \tilde{S}'_F(p)^{-1}\gamma_5 + \gamma_5 \tilde{S}'_F(p')^{-1}\right] \qquad (78)$$

Let us now differentiate with respect to the cutoff Λ. The tilde quantities, by construction, are Λ-independent in the limit of large Λ, as is the ratio of renormalization constants in Eq. (77) and hence the entire square bracket in Eq. (78) . So we get simply

$$0 = \frac{\partial}{\partial\Lambda}\left(\frac{Z_A}{Z_2}\right)\left[\left(\frac{2m_0 Z_2}{Z_D}\right)\tilde{\Gamma}^5(p,p') + \tilde{S}_F'(p)^{-1}\gamma_5 + \gamma_5\tilde{S}_F'(p')^{-1}\right], \quad (79a)$$

which implies that

$$0 = \frac{\partial}{\partial\Lambda}\left(\frac{Z_A}{Z_2}\right),$$
$$Z_A/Z_2 = \text{finite}. \quad (80)$$

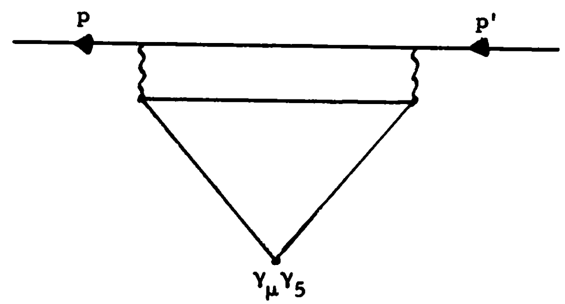
Eqs. (77) and (80) tell us that, up to arbitrary finite factors, the axial-vector and pseudoscalar vertex renormalizations are just Z_2 and $2m_0Z_2$, respectively.

When we replace the incorrect, naive Ward identity of Eq. (44) by the corrected Ward identity of Eq. (70), part of this conclusion must be modified. Referring back to the Feynman rules for the vertex of $F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}$, we see that when there is no net momentum transfer into the vertex, so that $k_1 = -k_2$, the antisymmetric tensor factor $k_1^\xi k_2^\tau \epsilon_{\xi\sigma\tau\rho}$ vanishes. Consequently, when $p = p'$, the additional term $\overline{F}(p,p')$ in Eq. (70) vanishes, and so Eqs. (76) and (77) are still valid. That is, even in the presence of the triangle anomaly, $2m_0Z_2\Gamma^5(p,p')$ is still finite.

On the other hand, the presence of the term $\overline{F}$ in Eq. (70) changes Eq. (79a) to read

$$0 = \frac{\partial}{\partial\Lambda}\left(\frac{Z_A}{Z_2}\right)\left[\left(\frac{2m_0 Z_2}{Z_D}\right)\tilde{\Gamma}^5(p,p') + \tilde{S}_F'(p)^{-1}\gamma_5 + \gamma_5\tilde{S}_F'(p')^{-1}\right]$$
$$+ \frac{\partial}{\partial\Lambda}\left[\left(\frac{-i\alpha_0}{4\pi}\right)Z_A\overline{F}(p,p')\right] = 0 \ . \quad (79b)$$

The presence of the extra term proportional to $\overline{F}$ in Eq. (79b) prevents us from drawing our previous conclusion of Eq. (80), that Z_A/Z_2 or $Z_2\Gamma_\mu^5(p,p')$ are finite. We expect that <u>even after multiplication by</u> Z_2, <u>there will still be divergent terms in the axial-vector vertex.</u> Such terms first appear in order α_0^2 of perturbation theory, as a result of the diagram



which, by use of Eq. (64), is easily seen to be logarithmically divergent. In heuristic terms, this divergence is not removed by multiplication by Z_2 because Z_2 is obtained from the theory with only vector currents present, and does not "know" about the existence of the axial-vector triangle anomaly. Introducing a cutoff by replacing the photon propagator $-ig_{\mu\nu}(q^2+i\varepsilon)^{-1}$ by $-ig_{\mu\nu}[(q^2+i\varepsilon)^{-1}-(q^2-\Lambda^2+i\varepsilon)^{-1}]$, we find that

$$Z_2\,\Gamma^5_\mu(p,p') = \gamma_\mu\gamma_5[1-\tfrac{3}{4}(\alpha_0/\pi)^2\,\ell n(\Lambda^2/m^2)] + \alpha_0\times\text{finite} + \alpha_0^2\times\text{finite} + O(\alpha_0^3), \tag{81}$$

or equivalently (up to an unspecified finite factor)

$$Z_A = Z_2[1+\tfrac{3}{4}(\alpha_0/\pi)^2\ell n(\Lambda^2/(m^2)+O(\alpha_0^3)]\,. \tag{82}$$

3.2 Radiative Corrections to $\nu_\ell\ \ell$ Scattering

As an application of Eq. (81), let us consider the radiative corrections to $\nu_\ell\ \ell$ scattering, where ℓ is a μ or an e. As we saw in Eq. (40), after Fierz transformation the terms in the local current-current Lagrangian which describe $\nu_\ell\ell$ scattering become

$$(G/\sqrt{2})[\bar{\mu}\,\gamma_\lambda(1-\gamma_5)\mu\,\bar{\nu}_\mu\gamma^\lambda(1-\gamma_5)\nu_\mu + \bar{e}\,\gamma_\lambda(1-\gamma_5)\,e\,\bar{\nu}_e\gamma^\lambda(1-\gamma_5)\nu_e] \tag{83}$$

The radiative corrections to Eq. (83) are simply obtained by calculating the radiative corrections to the charged-lepton currents $\bar{\mu}\gamma_\lambda(1-\gamma_5)\mu$ and $\bar{e}\gamma_\lambda(1-\gamma_5)e$, without any reference to the neutrino currents. Application of our Feynman rules shows that the effect of radiative corrections is to replace the matrix elements $\bar{u}_{(\mu)}\gamma_\lambda(1-\gamma_5)u_{(\mu)}$, $\bar{u}_{(e)}\gamma_\lambda(1-\gamma_5)u_{(e)}$ by

$$\begin{aligned} &\bar{u}_{(\mu)} Z_2^{(\mu)} [\Gamma_\lambda^{(\mu)} - \Gamma_\lambda^{5(\mu)}] u_{(\mu)}, \\ &\bar{u}_{(e)} Z_2^{(e)} [\Gamma_\lambda^{(e)} - \Gamma_\lambda^{5(e)}] u_{(e)}, \end{aligned} \tag{84}$$

with $\Gamma_\lambda^{(\mu, e)}$ and $\Gamma_\lambda^{5(\mu, e)}$ the proper vector and axial-vector vertices and with $Z_2^{(\mu, e)}$ the wave-function renormalization factors coming from item (iii) of the Feynman rules. From the usual vector-current Ward identity, we know that $Z_2^{(\mu)}\Gamma_\lambda^{(\mu)}$ and $Z_2^{(e)}\Gamma_\lambda^{(e)}$ are finite. On the other hand, Eq. (81) tells us that

$$\begin{aligned} Z_2^{(\mu, e)}\Gamma_\lambda^{5(\mu, e)} = &\gamma_\lambda\gamma_5[1-\tfrac{3}{4}(\alpha_0/\pi)^2 \ln(\Lambda^2/m^2)] \\ &+ \alpha_0 \times \text{finite} + \alpha_0^2 \times \text{finite} + O(\alpha_0^3), \end{aligned} \tag{85}$$

which means that, on account of the presence of axial-vector triangle diagrams, <u>the radiative corrections to $\nu_e e$ and $\nu_\mu \mu$ scattering diverge in the fourth order of perturbation theory</u>. This result contrasts sharply with the fact that the radiative corrections to muon decay or to the scattering reaction $\nu_\mu + e \to \nu_e + \mu$ are finite to all orders in perturbation theory.[11] The crucial difference between the two cases, of course, is that because of separate muon and electron-number conservation, the current $\bar{\mu}\gamma_\lambda(1-\gamma_5)e$ cannot couple into closed electron or muon loops, and thus the troublesome triangle diagram is not present.

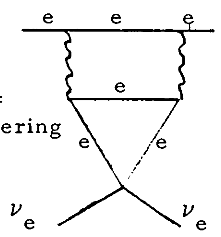
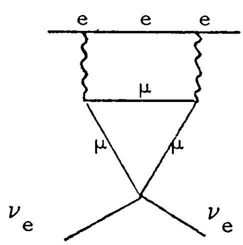
Two points of view can be taken towards the divergent radiative corrections in $\nu_\ell \ell$ scattering. One viewpoint is that we know, in any case, that the local current-current theory of leptonic weak interactions cannot be correct, since this theory leads at high energies to non-unitary matrix elements, and since it gives divergent results for higher-order weak-interaction effects.[12] Thus, it is entirely possible that the modifications in Eq. (83) necessary to give a satisfactory weak-interaction theory will also cure the disease of infinite radiative corrections in $\nu_\ell \ell$ scattering. The other viewpoint is that we should try to make the radiative corrections to $\nu_\ell \ell$ scattering finite, within the framework of a local weak-interaction theory. It turns out that this is possible, if we introduce $\nu_e \mu$ and $\nu_\mu e$ scattering terms into the effective Lagrangian so that Eq. (83) is replaced by

$$(G/\sqrt{2})[\bar{\mu}\gamma_\lambda(1-\gamma_5)\mu - \bar{e}\gamma_\lambda(1-\gamma_5)e] \times [\bar{\nu}_\mu \gamma^\lambda(1-\gamma_5)\nu_\mu - \bar{\nu}_e\gamma^\lambda(1-\gamma_5)\nu_e]. \tag{86}$$

This works because the troublesome extra term in Eq. (68) is independent of the bare mass m_0, so that it cancels between the muon and electron terms in Eq. (86), giving

$$\frac{\partial}{\partial x_\lambda}\left[\bar{\mu}\gamma_\lambda\gamma_5\mu - \bar{e}\gamma_\lambda\gamma_5 e\right] = 2im_0^{(\mu)}\bar{\mu}\gamma_5\mu - 2im_0^{(e)}\bar{e}\gamma_5 e \,. \qquad (87)$$

Application of the argument of Eqs. (76)-(79) then shows that the radiative corrections to Eq. (87) are finite. What has happened is that the e-triangle and μ-triangle contributions to the total $\nu_e e$ scattering amplitude contribute with opposite sign and regulate each other,

total $\nu_e e$ = scattering   + . . .

Experimentally, it will be possible to distinguish Eq. (86) from Eq. (83) by looking for elastic scattering of muon neutrinos from electrons; the present upper bound is still consistent with Eq. (86), but is getting very close.[13]

3.3 Connection Between γ_5 Invariance and a Conserved Axial-Vector Current in Massless Electrodynamics

Next, let us discuss the effects of the axial-vector triangle diagram in the case of massless spinor electrodynamics [Eq. (1) with $m_0 = 0$]. We will find that the triangle diagram leads to a breakdown of the usual connection between symmetries of the Lagrangian and conserved currents. As in our previous discussions, we begin by describing the standard theory, which holds in the absence of singular phenomena. Let $\{\Phi(x)\} = \{\Phi_1(x), \Phi_2(x), \ldots\}$ and $\{\partial_\lambda \Phi\}$ be a set of canonical fields and their space-time derivatives, and let us consider the field theory described by the Lagrangian density

$$\mathcal{L}(x) \equiv \mathcal{L}[\{\Phi\}, \{\partial_\lambda \Phi\}]. \tag{88}$$

To establish the connection between invariance properties of $\mathcal{L}$ and conserved currents, we make the infinitesimal, local gauge transformation on the fields,

$$\Phi_j(x) \to \Phi_j(x) + v(x) G_j[\{\Phi(x)\}], \tag{89}$$

and define the associated current J^α by

$$J^\alpha = -\delta \mathcal{L} / \delta(\partial_\alpha v). \tag{90}$$

Then, by using the Euler-Lagrange equations of motion of the fields, we easily find[14] that the divergence of the current is given by

$$\partial_\alpha J^\alpha = -\delta \mathcal{L}/\delta v. \tag{91}$$

In particular, if the gauge transformation of Eq. (89) with <u>constant gauge</u> function v, leaves the Lagrangian invariant, then $\delta \mathcal{L}/\delta v = 0$ and the current J^α is conserved. Thus, to any continuous invariance transformation of the

Lagrangian there is associated a conserved current. It is also easily verified that the charge $Q(t) = \int d^3x\, J^0(x, t)$ associated with the current J^α has the properties

$$dQ(t)/dt=0, \tag{92a}$$

$$[\, Q, \Phi_j(x)\,] = iG_j(x). \tag{92b}$$

Equation (92b) states that Q is the generator of the gauge transformation in Eq. (89), for constant v.

Let us now specialize to the case of massless electrodynamics, with Eq. (89) the gauge transformation

$$\psi\,(x) \to [1 + i\gamma_5 v(x)\,]\psi(x). \tag{93}$$

When v is a constant and $m_0=0$, this transformation leaves the Lagrangian of Eq. (1) invariant, so that according to Eq. (91), the associated current J^α should be conserved. But calculating J^α, we find

$$J^\alpha = -\delta\mathcal{L}/\delta(\partial_\alpha v) = \overline{\psi}\gamma^\alpha\gamma_5\psi\,, \tag{94}$$

which according to Eq. (68) has the divergence

$$\partial_\alpha J^\alpha = (\alpha_0/4\pi)F^{\xi\sigma}(x)F^{\tau\rho}(x)\epsilon_{\xi\sigma\tau\rho}. \tag{95}$$

Thus, Eq. (91), which was obtained by formal calculation using the equations of motion, breaks down in this case. We see that because of the presence of the axial-vector triangle diagram, <u>even though the Lagrangian (and all orders of perturbation theory) of massless electrodynamics are</u> γ_5

invariant, the axial-vector current associated with the γ_5 transformation is not conserved.

However, it is amusing that even though there is no conserved current connected with the γ_5 transformation, there is still a generator $\bar{Q}^5$ with the properties of Eq. (92). To see this, let us consider the quantity $\bar{j}^5$ defined by

$$\bar{j}^5_\mu(x) = j^5_\mu(x) - \frac{\alpha_0}{\pi} A^\xi(x) \frac{A^\tau(x)}{\partial x_\rho} \epsilon_{\xi\mu\tau\rho}; \tag{96}$$

referring to Eq. (68), we see that

$$\frac{\partial}{\partial x_\mu} \bar{j}^5_\mu(x) = 0. \tag{97}$$

Although $\bar{j}^5_\mu$ is conserved, it is explicitly gauge-dependent and therefore is not an observable current operator. But the associated charge

$$\bar{Q}^5 = \int d^3x\, \bar{j}^5_0(x) = \int d^3x[\psi^\dagger(x)\gamma_5\psi(x) + \frac{\alpha_0}{\pi} \underset{\sim}{A} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{A}] \tag{98}$$

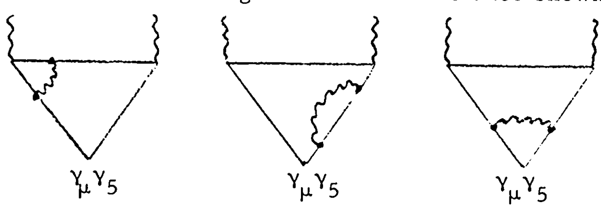
is gauge invariant and therefore observable. According to Eq. (97), $\bar{Q}^5$ is time-independent, and its commutator with $\psi(x)$ (calculated formally by use of the canonical commutation relations) is

$$[\bar{Q}^5, \psi(x)] = -\gamma_5\psi(x) = i[i\gamma_5\psi(x)]. \tag{99}$$

Also, as we will see below, because of an implicit photon field dependence of j^5_0 implied by Eq. (68), $\bar{Q}^5$ does commute with all the photon field variables. Thus, $\bar{Q}^5$ is the conserved generator of the γ_5 transformations.

3.4 Low Energy Theorem for $2im_0 j^5(x)$

Finally, we will show that the anomalous axial-vector divergence equation, Eq. (68), leads to an interesting low energy theorem for the vacuum to two photon matrix element of the naive divergence $2im_0 j^5$. First let us note that we have derived Eq. (68) by considering the triangle without radiative corrections, but have omitted the contributions of diagrams such as the ones shown:



These diagrams are also linearly divergent and hence may also have divergence anomalies of their own. Since the anomalous terms must be Lorentz pseudoscalars satisfying conditions analagous to the six conditions on possible subtraction terms listed in Subsection 2.2, one easily sees that they must have the same form as the lowest order triangle anomaly in Eq. (68). We take into account the possibility of divergence anomalies coming from radiative corrections to the triangle diagram by replacing Eq. (68) by

$$\frac{\partial}{\partial x_\mu} j_\mu^5(x) = 2im_0 j^5(x) + \frac{\alpha_0}{4\pi}(1+C)F^{\xi\sigma}\, F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}. \quad (100)$$

We will use Eq. (100) as the basis of our derivation of the low-energy theorem.

To proceed, we take the matrix element of Eq. (68) between the vacuum $|0>$ and the two photon state $<\gamma(k_1,\varepsilon_1)\ \gamma(k_2,\varepsilon_2)|$. Since the only pseudoscalar which can be formed from the four-momenta k_1, k_2 and the polarizations ε_1^*, ε_2^* of the two photons is $k_1^\xi\ k_2^\tau\ \varepsilon_1^{*\sigma}\varepsilon_2^{*\rho}\varepsilon_{\xi\tau\sigma\rho}$, the matrix element of each term in Eq. (100) contains this expression as a factor,

$$<\gamma(k_1,\varepsilon_1)\gamma(k_2,\varepsilon_2)|\left\{\begin{matrix} \partial\mu j_\mu^5 \\ 2im_0 j^5 \\ \frac{\alpha_0}{4\pi} F^{\xi\sigma}\ F^{\tau\rho}\ \varepsilon_{\xi\sigma\tau\rho} \end{matrix}\right\}|0> \qquad (101)$$

$$= (4k_{10}k_{20})^{-\frac{1}{2}} k_1^\xi k_2^\tau\ \varepsilon_1^{*\sigma}\varepsilon_2^{*\rho}\ \varepsilon_{\xi\tau\sigma\rho}\left\{\begin{matrix} F(k_1\cdot k_2) \\ G(k_1\cdot k_2) \\ H(k_1\cdot k_2)\end{matrix}\right\}$$

The matrix element of Eq. (100) can be rewritten in terms of the amplitudes F, G, H as

$$F(k_1\cdot k_2) = G(k_1\cdot k_2) + (1+C)\ H(k_1\cdot k_2). \qquad (102)$$

To derive the low energy theorem from Eq. (102), we make use of a remarkable kinematic property of the matrix element $\mathcal{M}_\mu = (4k_{10}k_{20})^{\frac{1}{2}}<\gamma(k_1,\varepsilon_1)\gamma(k_2,\varepsilon_2)|j_\mu^5|0>$. As we have noted in the previous section, the requirements of Lorentz invariance, gauge invariance and Bose statistics require this matrix element to have the general form

$$\begin{aligned}\mathcal{M}_\mu = \varepsilon_1^{*\rho}\,\varepsilon_2^{*\sigma}[\,& C_1 k_1^\tau \varepsilon_{\tau\sigma\rho\mu} + C_2 k_2^\tau \varepsilon_{\tau\sigma\rho\mu} \\ & + C_3 k_{1\rho} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\sigma\mu} + C_4 k_{2\rho} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\sigma\mu} \\ & + C_5 k_{1\sigma} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\rho\mu} + C_6 k_{2\sigma} k_1^\xi k_2^\tau \varepsilon_{\xi\tau\rho\mu}]\,,\end{aligned} \tag{103}$$

with

$$\begin{aligned} C_1 &= k_1\cdot k_2 C_3 + k_2^2 C_4, \\ C_2 &= k_1^2 C_5 + k_1\cdot k_2 C_6, \\ C_3 & (k_1, k_2) = -C_6(k_2, k_1), \\ C_4 & (k_1, k_2) = -C_5(k_2, k_1). \end{aligned} \tag{104}$$

The matrix element of the divergence of the axial-vector current is proportional to $(k_1+k_2)^\mu \mathcal{M}_\mu$. Using the algebraic identity satisfied by the six four-vectors $a, \ldots, f$,

$$\begin{aligned}&(af)\,|bcde| + (bf)\,|cdea| + (cf)\,|deab| \\ &\qquad + (df)\,|eabc| + (ef)\,|abcd| = 0, \\ &(af) \equiv a\cdot f, \quad |abcd| \equiv a^\xi b^\tau c^\sigma d^\eta\, \varepsilon_{\xi\tau\sigma\eta},\end{aligned} \tag{105}$$

with $a=f=k_1$, $b=k_2$, $c=k_1+k_2$, $d=e_1^*$ and $e=e_2^*$, we find that $(k_1+k_2)^\mu \mathcal{M}_\mu$ can be rearranged into the form

$$(k_1+k_2)^\mu \mathcal{M}_\mu = [\,C_3 - C_6\,]\, k_1\cdot k_2\, k_1^\xi k_2^\tau \varepsilon_1^{*\sigma} \varepsilon_2^{*\rho} \varepsilon_{\xi\tau\sigma\rho}. \tag{106}$$

[In obtaining Eq. (106) we have used the fact that for on-shell photons $k_1^2 = k_2^2 = 0$; in the off-shell case the additional terms

$$(k_2^2 C_4 - k_1^2 C_5)\, k_1^\xi k_2^\tau \varepsilon_1^{*\sigma} \varepsilon_2^{*\rho} \varepsilon_{\xi\tau\sigma\rho} \tag{107}$$

are present.] Comparing Eq. (106) with Eq. (101), we see that

$$F(k_1 \cdot k_2) \propto k_1 \cdot k_2, \qquad (108)$$
$$F(0) = 0$$

which gives us a low energy theorem relating the vacuum to two photon matrix element G of the naive divergence to the corresponding matrix element H of the operator $(\alpha_0/4\pi)\, F^{\xi\sigma} F^{\tau\rho} \varepsilon_{\xi\sigma\tau\rho}$,

$$G(0) = -(1+C)\, H(0). \qquad (109)$$

To lowest non-vanishing order in perturbation theory, C can be neglected (it represented possible radiative corrections to the triangle) and H(0) can be evaluated from the Feynman rules preceding Eq. (69), giving

$$H(0) = 2\alpha/\pi,$$
$$G(0) = -2\alpha/\pi. \qquad (110)$$

This result for G(0) could, of course, have been derived without all the fuss directly from the lowest order expression for G given in Eq. (61),

$$G(0) = \left[i\varepsilon_1^{*\sigma} \varepsilon_2^{*\rho} \left(\frac{-ie_0^2}{(2\pi)^4}\right) 2m_0 R_{\sigma\rho} \Big/ k_1^{\xi} k_2^{\tau} \varepsilon_1^{*\sigma} \varepsilon_2^{*\rho} \varepsilon_{\xi\tau\sigma\rho} \right]_{k_1=k_2=0}$$
$$= \frac{e_0^2}{(2\pi)^4} 2m_0 B_1 \Big|_{k_1=k_2=0} = \frac{-e_0^2}{2\pi^2} \underset{\text{(to lowest order)}}{=} -\frac{2\alpha}{\pi}. \qquad (111)$$

However, as we shall see in detail in the next section, the real significance of the low energy theorem is that Eq. (110) for $G(0)$ is exact, even when all radiative corrections are carefully taken into account.

4. ABSENCE OF RADIATIVE CORRECTIONS

We must now deal with the question, raised in the last section, of whether radiative corrections to the triangle modify the anomalous axial-vector divergence equation. That is, what is the value of the constant C in Eq. (100), and how is the low energy theorem of Eq. (110) modified by radiative corrections? We will find the remarkable result that $C = 0$ and that Eq. (110) is exact to all orders of perturbation theory. This conclusion can be understood heuristically by noting that radiative corrections to the basic triangle involve axial-vector loops with at least five vertices, which, unlike the lowest order axial-vector triangle, do satisfy the usual axial-vector Ward identities. Thus, when virtual photon momenta are held fixed, the complicated radiative correction diagrams have no divergence anomalies. Since the virtual photon four-momenta appear essentially as parameters on both sides of these Ward identities, one expects that as long as the virtual photon integrations are not too badly divergent, the Ward identities will continue to hold even after the integrations have been performed. The purpose of the present section is to support this heuristic argument with more detailed calculations, and, in particular, to show that no problems are caused by the usual renormalizable infinities in the radiative corrections to the triangle.[15]

4.1 General Argument

We begin by developing a general argument, valid to any order of perturbation theory, which shows that Eqs. (68) and (110) are exact. The basic idea is this: As we have seen in the preceding section, the multiplicative factor Z_2, which makes matrix elements of the naive divergence term in Eq. (68) finite, does not remove the divergences from matrix elements of the axial-vector current term on the left-hand side of Eq. (68). Thus, there is no simple rescaling which simultaneously makes all terms in Eq. (68) finite, and so it is simplest to deal with Eq. (68) directly, even though it involves unrenormalized (and hence divergent) fields, masses and coupling constants. In order to make our manipulations of these divergent quantities well defined, we construct a cutoff version of quantum electrodynamics by introducing a photon regulator field of mass Λ. The cutoff prescription allows the usual renormalization program to be carried out, so that the electron bare mass m_0 and wave function renormalization Z_2, and the axial-vector vertex renormalization Z_A, become specified functions of the renormalized charge and mass, and of the cutoff Λ. In the cutoff field theory it is straightforward to prove the validity of Eq. (68) for the unrenormalized quantities. We then derive the low energy theorem for the matrix element $\langle 2\gamma | 2im_0 j^5 | 0\rangle$, with the cutoff still present, and finally let the cutoff approach infinity to get a low energy theorem for the renormalized matrix element of the naive divergence.

We introduce the cutoff by modifying the usual Feynman rules for quantum electrodynamics which were stated in Section 1. Our new rules read as follows:

(i) For each internal electron line with momentum p we include a factor $i(\not{p}-m_0+i\epsilon)^{-1}$ and for each vertex a factor $-i\hat{e}_0\gamma_\mu$. For each internal photon line of momentum q, we replace the usual propagator $-ig_{\mu\nu}(q^2+i\epsilon)^{-1}$ by the regulated propagator

$$-ig_{\mu\nu}\left(\frac{1}{q^2+i\epsilon}-\frac{1}{q^2-\Lambda^2+i\epsilon}\right)=\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\,\frac{-\Lambda^2}{q^2-\Lambda^2+i\epsilon}\,. \tag{112}$$

(ii) Let $\Pi^{(2)}(q)_{\mu\nu}$ denote the two-vertex vacuum polarization loop illustrated below,

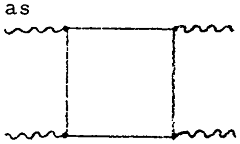
given by

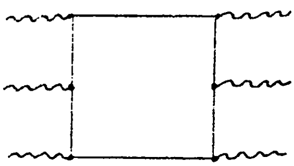
$$\Pi^{(2)}(q)_{\mu\nu} = i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma_\mu\frac{1}{\not{k}-m_0+i\epsilon}\gamma_\nu\frac{1}{\not{k}+\not{q}-m_0+i\epsilon}\right]. \tag{113}$$

Wherever $\Pi^{(2)}(q)_{\mu\nu}$ appears, we use its gauge-invariant, subtracted evaluation

$$\Pi^{(2)}(q)_{\mu\nu}=(-q^2 g_{\mu\nu}+q_\mu q_\nu)\,[\Pi^{(2)}(q^2)-\Pi^{(2)}(0)]\,. \qquad (114)$$

All vacuum polarization loops with four or more vertices, such as

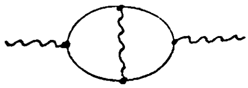




are calculated by imposing the current conservation condition; as we have seen, this suffices to make them finite without need for further subtractions.

(iii) As usual, there is a factor $\int d^4\ell/(2\pi)^4$ for each internal integration over loop variable ℓ and a factor -1 for each fermion loop.

(iv) We use the standard, iterative renormalization procedure outlined in Section 1 to fix the coupling $\hat{e}_0$ and the electron bare mass and wave function renormalization m_0 and Z_2, as functions of the renormalized charge and mass e and m and the cutoff Λ. For finite Λ, the quantities $\hat{e}_0$, m_0 and Z_2 will all be <u>finite.</u> The reason is that regulating the photon propagator (plus gauge invariance for loops) renders finite all vertex and electron self energy parts and all photon self-energy parts other than $\Pi^{(2)}_{\mu\nu}$, such as



The self-energy part $\Pi^{(2)}_{\mu\nu}$ has already been made finite by explicit subtraction. Note that the coupling $\hat{e}_0$ is not the same as the "bare charge" e_0 in Eq. (1), but rather is related to it by

$$\hat{e}_0^2 = \frac{e_0^2}{1 + e_0^2\,\Pi^{(2)}(0)} \tag{115}$$

That is, $\hat{e}_0$ is a so-called "intermediate renormalized" charge, obtained from the bare charge by removing only those divergences associated with the lowest order vacuum polarization loop and its iterations.

(v) We include wave-function renormalization factors $Z_2^{\frac{1}{2}}$ and $Z_3^{\frac{1}{2}} = e/\hat{e}_0$ for each external electron and photon line.

This simple set of rules makes all ordinary electrodynamics matrix elements finite. We may summarize the rules compactly by noting that they are the Feynman rules for the following regulated Lagrangian density:

$$\begin{aligned}
\mathcal{L}^R(x) &= \mathcal{L}_0^R(x) + \mathcal{L}_I^R(x), \\
\mathcal{L}_0^R(x) &= \bar{\psi}(x)(i\gamma\cdot\Box - m_0)\psi(x) - \tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \\
&\quad + \tfrac{1}{4}F^R_{\mu\nu}(x)\,F^{R\mu\nu}(x) - \tfrac{1}{2}\Lambda^2 A^R_\mu(x)\,A^{R\mu}(x), \\
\mathcal{L}_I^R(x) &= -\hat{e}_0\bar{\psi}(x)\gamma_\mu\psi(x)[A^\mu(x) + A^{R\mu}(x)]
\end{aligned} \tag{116}$$

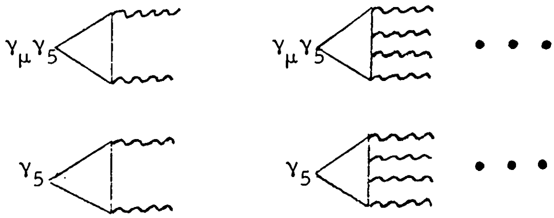
$$+C^{(2)}[F_{\mu\nu}(x)+F^{R}_{\mu\nu}(x)][F^{\mu\nu}(x)+F^{R\mu\nu}(x)],$$

where A^R_μ is the field of the regulator vector meson of mass Λ, and $F^R_{\mu\nu}(x) = \partial_\nu A^R_\mu(x) - \partial_\mu A^R_\nu(x)$ is the regulator field-strength tensor. The term containing $C^{(2)} \propto \Pi^{(2)}(0)$ is a logarithmically infinite counter term which performs the explicit subtraction of the two vertex vacuum polarization loop in Eq. (114). The regulator free-field Lagrangian density is included in Eq. (116) with the opposite sign from normal; hence, according to the canonical formalism, the regulator field is quantized with the opposite sign from normal. That is, we have

$$[A^R_\mu(\underset{\sim}{x}, t), \partial A^R_\nu(\underset{\sim}{y}, t)/\partial t] = i\, g_{\mu\nu}\delta^3(\underset{\sim}{x}-\underset{\sim}{y}), \tag{117}$$

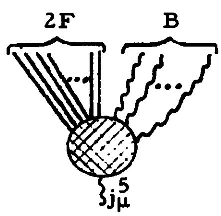
in contrast to Eq. (5). Since the sign of the bare propagator follows directly from the sign of the commutator in Eq. (117), the regulator bare propagator is opposite in sign from the photon bare propagator, as required by Eq. (112).

Having specified our cutoff procedure, we are now ready to introduce the axial-vector and pseudoscalar currents $j^5_\mu(x)$ and $j^5(x)$, and to study their properties. First we must check whether all matrix elements of these currents are finite when calculated in our cutoff theory. The answer is yes, that they are finite, and follows immediately from the fact that all of the basic fermion loops involving one axial-vector or one pseudoscalar vertex,

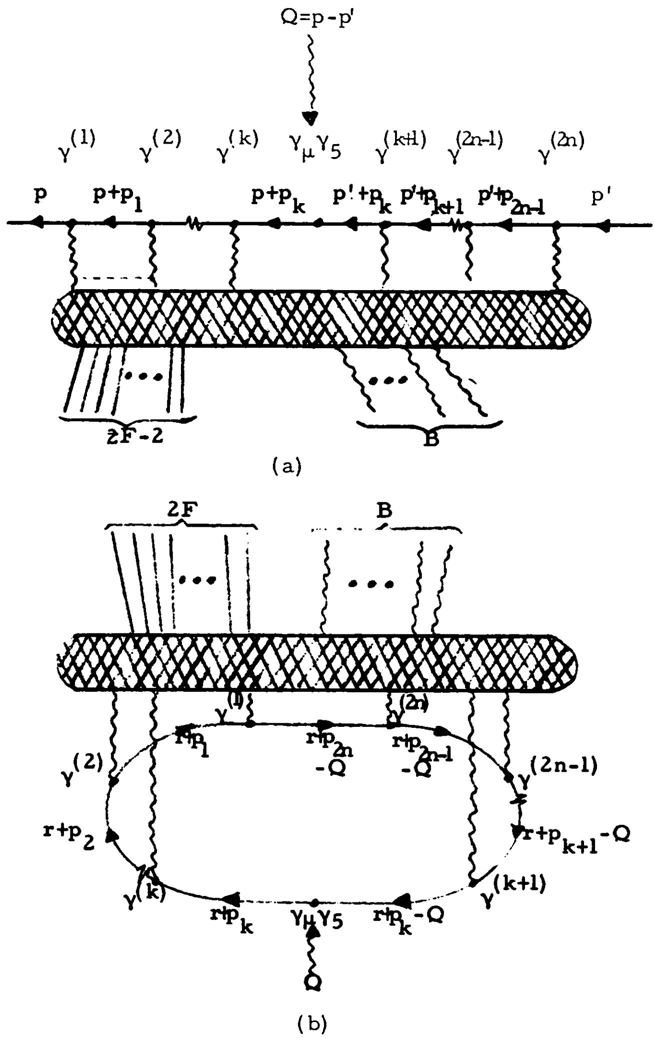


are made finite by the imposition of gauge invariance on the photon vertices, without the need for explicit subtractions. Thus, we can turn immediately to the problem of showing that Eq. (68) is exactly satisfied in our cutoff theory.

To do this, we consider an arbitrary Feynman amplitude involving j_μ^5 , with 2F external fermion and B external boson lines



Proceeding as in Section 2, **we divide the diagrams into** two categories, type (a) and type (b), according to whether the axial-vector vertex $\gamma_\mu\gamma_5$ is attached to one of the F fermion lines running through the diagram, or is attached to an internal closed loop; respectively. Typical type-(a) and type-(b) diagrams are drawn below:



(a)

(b)

For the type-(a) diagrams we find that, just as in Eqs.(47) -(49) of Section 2, the derivation of the Ward identity involves purely algebraic manipulations of the string of fermion propagators on the fermion line containing $\gamma_\mu\gamma_5$. Since the integrals over the four-momenta of the photon propagators joining the fermion propagator string to the shaded "blob" (i.e. the integrals over $p_1, \ldots, p_{2n-1}$) are all <u>convergent</u> in our regulated field theory, it is safe to do these algebraic manipulations <u>inside</u> the integrals. The first term on the right-hand side of Eq. (49) gives the type-(a) contribution to the Feynman amplitude for $2im_0 j^5$ corresponding to the type-(a) diagram for j_μ^5 which we started with. The two remaining terms in Eq. (49) give the usual "surface terms" which arise in Ward identities from the equal-time commutator of j_0^5 with the fields of the external fermions of momenta p and p'. Thus, as far as the type-(a) contributions to the Feynman amplitude are concerned, the divergence of j_μ^5 is simply $2im_0 j^5$, with no extra terms present. We next turn to a typical type-(b) contribution, which we may write as

$$L(Q;\gamma_\mu\gamma_5;p_1,\ldots,p_{2n-1})(\ldots),$$

$$L(Q;\Gamma;p_1,\ldots,p_{2n-1}) = \int d^4r\,\mathrm{Tr}\Big\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{\not r+\not p_j-m_0}\right]$$
$$\times\gamma^{(k)}\frac{1}{\not r+\not p_k-m_0}\,\Gamma\,\frac{1}{\not r+\not p_k-\not Q-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{\not r+\not p_j-\not Q-m_0}\right]\Big\}, \tag{118}$$

where we have focused our attention on the closed loop and have again denoted the remainder of the diagram by $(\ldots)$.

The same straightforward algebra as before shows that the divergence of Eq. (118) can be rewritten as

$$
\begin{aligned}
& L(Q; iQ^{\mu}\gamma_{\mu}\gamma_5; p_1, \ldots, p_{2n-1}) \\
& \quad = L(Q; 2im_0\gamma_5; p_1, \ldots, p_{2n-1}) \qquad (119) \\
& + i\int d^4r\, \mathrm{Tr}\{\gamma_5 \prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{\not{r}+\not{p}_j-m_0}\right] - \gamma_5 \prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{\not{r}+\not{p}_j-\not{Q}-m_0}\right]\}.
\end{aligned}
$$

As we have seen, for loops with $n \geq 2$, the residual integrals in Eq. (119) cancel and we get the Ward identity

$$L(Q; iQ^{\mu}\gamma_{\mu}\gamma_5; p_1, \ldots, p_{2n-1}) = L(Q; 2im_0\gamma_5; p_1, \ldots, p_{2n-1}). \quad (120)$$

Again, since the integrals over $p_1, \ldots p_{2n-1}$ are all <u>convergent</u> in the regulated field theory, the manipulations leading to Eq. (120) can all be performed <u>inside</u> these integrals. This means that the type-(b) pieces containing loops with $n \geq 2$ all agree with the usual divergence equation $\partial^{\mu} j_{\mu}^{5}(x) = 2im_0 j^5(x)$. Finally, we must consider the case of the axial-vector triangle, with $n = 1$. As is now familiar, this diagram has an anomalous Ward identity, which in our regulated electrodynamics adds to the normal axial-vector divergence equation the term

$$
\begin{aligned}
& (\hat{\alpha}_0/4\pi)[F^{\xi\sigma}(x)+F^{R\xi\sigma}(x)][F^{\tau\rho}(x)+F^{R\tau\rho}(x)]\epsilon_{\xi\sigma\tau\rho}, \qquad (121) \\
& \hat{\alpha}_0 = \hat{e}_0^2/4\pi.
\end{aligned}
$$

To summarize, our diagrammatic analysis shows that the axial-vector divergence equation in the regulated field theory is

$$\partial^{\mu} j_{\mu}^{5}(x)=2im_0 j^5(x)+(\hat{\alpha}_0/4\pi)F^{\xi\sigma}(x)F^{\tau\rho}(x)\,\epsilon_{\xi\sigma\tau\rho}$$
$$+(\hat{\alpha}_0/4\pi)[F^{\xi\sigma}(x)F^{R\tau\rho}(x)+F^{R\xi\sigma}(x)F^{\tau\rho}(x)+F^{R\xi\sigma}(x)F^{R\tau\rho}(x)]$$
$$\times\,\epsilon_{\xi\sigma\tau\rho}. \qquad (122)$$

Equation (122) is identical with Eq. (68), apart from the terms involving F^R which arise from our explicit inclusion of a regulator field and apart from the fact that Eq. (122) is written in terms of the intermediate renormalized charge and field strength. To see the full equivalence with Eq. (68), we note that the intermediate renormalized quantities used in this section are related to the unrenormalized ones used in Section 2 by

$$\hat{e}_0(F^{\xi\sigma})_{\substack{\text{intermediate}\\ \text{renormalized}}} = e_0(F^{\xi\sigma})_{\text{unrenormalized}}. \qquad (123)$$

The crucial point is that the coefficient of the anomalous term is exactly $\hat{\alpha}_0/4\pi$ and <u>does not involve an unknown power series in the coupling constant coming from higher orders in perturbation theory</u>.

The diagrammatic analysis which we have just given may be rephrased succinctly as follows: If we use the regulated Lagrangian density in Eq. (116) to calculate equations of motion, and then use the equations of motion to naively calculate the axial divergence, we find

$$\partial^\mu j^5_\mu(x) = 2im_0 j^5(x). \tag{124}$$

Extra terms on the right-hand side of Eq. (124) can only come from singular diagrams where the naive derivation breaks down. In the regulated field theory, all virtual photon integrations converge and therefore cannot lead to singularities giving additional terms in Eq. (124). Hence breakdown in Eq. (124), if it occurs at all, must be associated with the basic axial-vector loops, without internal radiative corrections. But, as we have seen, the axial-vector loops with four or more photons satisfy Eq. (124), so the basic triangle diagram is the only possible source of an anomaly.

Having derived our basic result, we turn next to the low-energy theorem for $2im_0j^5(x)$ which is implied by Eq. (122). Taking the matrix element of Eq. (122) between a state containing two physical photons and the vacuum state, using the definition

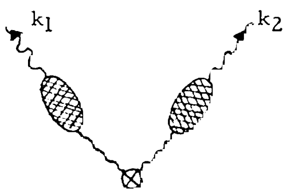
$$\langle\gamma(k_1,\epsilon_1)\,\gamma(k_2,\epsilon_2)|\left\{\begin{array}{l}\partial_\mu j^5_\mu \\ \\ 2im_0 j^5 \\ \\ \frac{\hat{\alpha}_0}{4\pi}(F^{\xi\sigma}+F^{R\xi\sigma})(F^{\tau\rho}+F^{R\tau\rho})\,\epsilon_{\xi\sigma\tau\rho}\end{array}\right\}|0\rangle_{\Lambda\ \text{finite}} \tag{125}$$

$$=(4k_{10}k_{20})^{-\frac{1}{2}}\,k_1^\xi\,k_2^\tau\,\epsilon_1^{*\sigma}\,\epsilon_2^{*\rho}\,\epsilon_{\xi\tau\sigma\rho}\left\{\begin{array}{c}F_\Lambda(k_1\cdot k_2)\\ G_\Lambda(k_1\cdot k_2)\\ H_\Lambda(k_1\cdot k_2)\end{array}\right\},$$

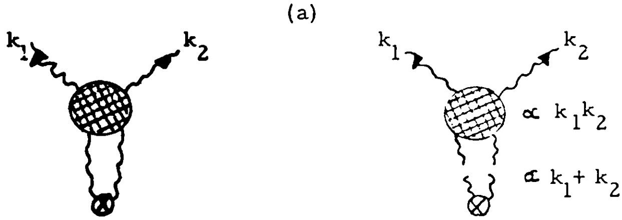
and proceeding exactly as in Eqs. (101)-(109), we find the low energy theorem

$$G_\Lambda(0) = -H_\Lambda(0). \tag{126}$$

We wish this time to calculate $H_\Lambda(0)$ to all orders in perturbation theory. There are two types of diagrams which contribute to $H_\Lambda(k_1 \cdot k_2)$, as illustrated below, where we have used the symbol $\otimes$ to denote the action of the operator $(\alpha_0/4\pi)(F^{\xi\sigma}+F^{R\xi\sigma})(F^{\tau\rho}+F^{R\tau\rho})\,\epsilon_{\xi\sigma\tau\rho}$:



(a)



(b)

In the diagrams (a), the field strength operators attach directly onto the external photon lines, without photon-photon scattering. The effect of the vacuum polarization parts and the external-line wave-function renormalizations is to change $\hat{\alpha}_0$ to α, giving

$$H_\Lambda(0)^{(a)} = 2\alpha/\pi\ . \tag{127}$$

In diagrams (b), there is a photon-photon scattering between ⊗ and the free photons. As a result of the antisymmetric tensor structure of the anomalous divergence term, the vertex ⊗ is proportional to k_1+k_2. Also, the diagram for the scattering of light by light is itself proportional to k_1k_2, since photon gauge invariance implies that the external photons couple through their field strength tensors. Thus, the diagrams (b) are proportional to $k_1k_2(k_1+k_2)$ and are of higher order than the terms which contribute to the low-energy theorem, giving us

$$H_\Lambda(0)^{(b)} = 0. \tag{128}$$

Combining Eqs. (126)-(128), we get an exact low-energy theorem for the operator $2im_0j^5$

$$G_\Lambda(0) = -2\alpha/\pi . \tag{129}$$

So far in our discussion we have kept the cutoff Λ finite, so that $G_\Lambda(0)$ is a matrix element calculated with our modified Feynman rules. However, we have seen that all matrix elements of $2im_0j^5$ become cutoff-independent in the limit $\Lambda \to \infty$. Defining a renormalized vacuum to two photon matrix element of the naive axial vector divergence, $\tilde{G}(k_1\cdot k_2)$, by taking the limit

$$\tilde{G}(k_1\cdot k_2) = \lim_{\Lambda\to\infty} G_\Lambda(k_1\cdot k_2), \tag{130}$$

we get from Eq. (129) the low energy theorem

$$\tilde{G}(0) = -2\alpha/\pi. \tag{131}$$

The observant reader will notice that our definition of the renormalized matrix element in Eq. (130) appears to differ from the skeleton diagram construction described in Section 1. According to the skeleton expansion, an arbitrary renormalized matrix element of the naive divergence (and the vacuum to two photon matrix element in particular) can be constructed by writing down the appropriate skeleton graphs and inserting the renormalized propagators and vertex functions $\tilde{S}_F'$, $\tilde{D}_F'$, $\tilde{\Gamma}_\mu$ and $\tilde{\Gamma}^5$. These quantities are defined as the Λ-independent limits

$$\begin{aligned} \tilde{S}_F' &= \lim_{\Lambda\to\infty} Z_2^{-1} S_F', \\ \tilde{D}_F' &= \lim_{\Lambda\to\infty} Z_3^{-1} D_F', \\ \tilde{\Gamma}_\mu &= \lim_{\Lambda\to\infty} Z_2 \Gamma_\mu, \\ m\tilde{\Gamma}^5 &= \lim_{\Lambda\to\infty} m_0 Z_2 \Gamma^5; \end{aligned} \tag{132}$$

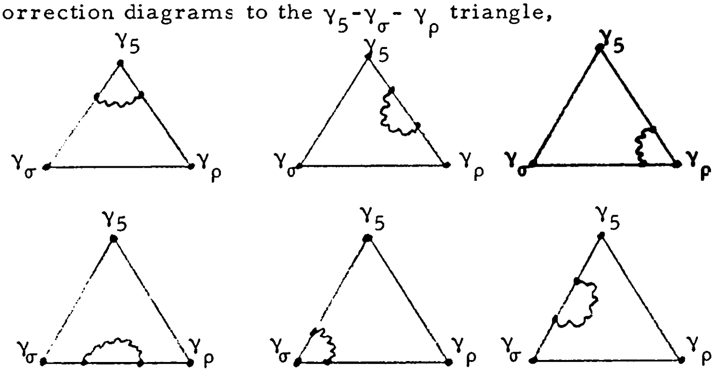
that is, the skeleton expansion construction consists of taking the $\Lambda\to\infty$ limit <u>first</u> in vertex parts and propagators, and then substituting onto the skeleton. In Eq. (130), however, these operations are performed in the reverse order; the cutoff dependent vertex parts and propagators are substituted onto the skeleton, all integrations are carried out, and then finally the $\Lambda\to\infty$ limit is taken. Can this interchange of order make any difference in the final value of the renormalized matrix element which is obtained? A simple inductive argument shows that the answer to this question is in the negative. Let us suppose that the two procedures give the same answer for all matrix elements of j^5 of order n-2 in perturbation theory. For all matrix elements of order n which have convergent skeletons, the two procedures must obviously agree. According to Eq. (11), the only cases which have potentially divergent skeletons are the pseudoscalar vertex part itself, and the vacuum to two photon matrix element. For the pseudoscalar vertex part, the two constructions agree, by definition. For the vacuum to two photon matrix element, a possible difference $\Delta\tilde{G}$ between the two constructions must have the following properties: (i) $\Delta\tilde{G}$ must be a polynomial in the photon momentum variables k_1 and k_2. This restriction follows from generalized unitarity, which relates discontinuities in the nth order diagram to lower order matrix elements, for which the two constructions agree, by hypothesis. (ii) $\Delta\tilde{G}$ must satisfy the requirements of Weinberg's theorem, since both constructions do. This again means that if we set $k_1 = \xi Q$, $k_2 = -\xi Q + \xi R + S$ and let $\xi \rightarrow \infty$, $\Delta\tilde{G}$ must diverge at most as ξ times a power of $\ell n\ \xi$. Together with gauge invariance, the property (i) implies that $\Delta\tilde{G}$ has the form

$$\Delta\tilde{G} = k_1^{\xi} k_2^{\tau} \qquad \epsilon_2^{*\rho}\, \epsilon_{\xi\tau\sigma\rho} \times \text{ polynomial in } k_1, k_2, \tag{133}$$

but this diverges at least as ξ^2 in the Weinberg limit and violates property (ii). Thus we must have $\Delta\tilde{G} = 0$. We conclude that the two constructions agree in nth order, and by induction, in all orders. Consequently, the low energy theorem of Eq. (131) applies to the renormalized matrix element $\tilde{G}$ obtained from the usual skeleton expansion and makes the remarkable statement that all order α^2, α^3, ... contributions to $\tilde{G}(k_1 \cdot k_2)$ vanish at $k_1 \cdot k_2 = 0$.

4.2 Explicit Second Order Calculation

Let us now briefly outline a calculation which explicitly checks Eq. (131) to second order in perturbation theory. We wish to calculate the sum of the six radiative correction diagrams to the γ_5-γ_σ- γ_ρ triangle,



and to verify that they cancel to zero. The first step is to calculate the renormalized quantities $\tilde{\Gamma}_\mu(p,p')$, $\tilde{\Gamma}^5(p,p')$ and $\tilde{S}'_F(p)$. The most straightforward way of doing this is (a) to calculate the unrenormalized quantities Γ_μ, Γ^5 and S'_F using the cutoff Feynman rules, (b) to use Eqs. (12) and (13) to compute the renormalization constants m_0 and Z_2, and (c) to use the recipe of Eq. (132) to find the tilde quantities by taking the limit $\Lambda\to\infty$. This procedure gives, to second order,

$$\tilde{\Gamma}^{(2)}_\lambda(p,p') = \gamma_\lambda + \frac{e^2}{16\pi^2}\int_0^1 z\,dz \int_0^1 dy \left\{ 2\gamma_\lambda \ln\left[\frac{z^2m^2+(1-z)\mu^2}{D}\right] - \frac{N_\lambda}{D} - \frac{2m^2\gamma_\lambda P_1}{z^2m^2+(1-z)\mu^2}\right\},$$

$$\tilde{\Gamma}^{5(2)}(p,p') = \gamma_5 + \frac{e^2}{16\pi^2}\int_0^1 z\,dz \int_0^1 dy \left\{ 8\gamma_5 \ln\left[\frac{z^2m^2+(1-z)\mu^2}{D}\right] + \frac{\gamma_5 N}{D} + \frac{4m^2\gamma_5 P_2}{z^2m^2+(1-z)\mu^2}\right\},$$

$$\tilde{S}'^{(2)}_F(p) = [\,\not{p} - m - \tilde{\Sigma}^{(2)}(\not{p})\,]^{-1}, \tag{134}$$

$$\tilde{\Sigma}^{(2)}(\not{p}) = \frac{e^2}{16\pi^2}\int_0^1 z\,dz \left\{ 2g_1 \ln\left[\frac{z^2m^2+(1-z)\mu^2}{-p^2z(1-z)+zm^2+(1-z)\mu^2}\right] + g_2\,\frac{m^2-p^2(1-z)^2}{-p^2z(1-z)+zm^2+(1-z)\mu^2} + \frac{2m^2\not{p}\,P_1+4m^3P_2}{z^2m^2+(1-z)\mu^2}\right\},$$

$$D = (y^2z^2-yz)p^2 + [(1-y)^2z^2-(1-y)z]\,p'^2 + 2y(1-y)z^2\,p\cdot p' + zm^2 + (1-z)\mu^2,$$

with

$$N_\lambda = -2m^2\gamma_\lambda - 2[(1-z+yz)\not{p}' - yz\not{p}] \times \gamma_\lambda[(1-yz)\not{p} - (1-y)z\not{p}'] + 4m[(1-2yz)p_\lambda + (1-2z+2yz)p'_\lambda], \tag{135}$$

$$N = 4m^2 - 4[(1-yz)p - (1-y)zp']\cdot[(1-z+yz)p' - yzp] + 2m(\not{p}-\not{p}'),$$

$$P_1 = z^2+2z-2,\quad P_2 = 1-2z,$$

$$g_1 = 4m - \not{p},\ g_2 = 4m - 2\not{p}.$$

The quantity μ^2 is a fictitious virtual photon mass supplied to avoid logarithmic infrared singularities in the individual radiative correction diagrams. (The sum of the six radiative correction diagrams, however, has no infrared divergences because of cancellations of the troublesome terms.) As a check on our arithmetic, we note that Eqs. (134) and (135) satisfy the vector Ward identity

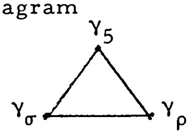
$$(p-p')^\lambda\ \tilde{\Gamma}_\lambda(p,p') = \tilde{S}'_F(p)^{-1} - \tilde{S}'_F(p')^{-1} \tag{136}$$

as well as the additional relation

$$0 = 2m\ \tilde{\Gamma}^5(p,p) + \tilde{S}'_F(p)^{-1}\gamma_5 + \gamma_5\ \tilde{S}'_F(p)^{-1}, \tag{137}$$

which follows from the $p=p'$ case of Eq. (70).

The next step is to substitute Eqs. (134) and (135) into the skeleton diagram



giving the lowest order triangle plus the six radiative correction diagrams illustrated above. The final step is to Taylor expand around the neighborhood $k_1=k_2=0$, since $\tilde{\mathcal{G}}(0)$ is the coefficient of the first nonvanishing term in this expansion. Although the integrals for general k_1 and k_2 are very formidible, the leading Taylor coefficient is not very complicated. Some straightforward algebra and integrations then show that the contributions of the radiative corrections to $\widetilde{G}(0)$ do indeed cancel, as required by Eq. (131).[15]

5. GENERALIZATIONS OF OUR RESULTS: π^0 DECAY; OTHER WARD IDENTITY ANOMALIES

Our discussion so far has dealt exclusively with the VVA triangle anomaly in QED. Let us now generalize our results in two directions. First, we will study the consequences of the VVA anomaly in other field theory models, especially in the so-called σ-models, which satisfy the partially-conserved axial-vector current (PCAC) condition as an exact operator identity. We will find that extension to this class of models of the low-energy theorem derived above leads to a prediction of the $\pi^0 \to 2\gamma$ decay rate. Comparison with experiment provides evidence <u>against</u> the quark model with fractional quark charges. Second, we will briefly examine other triangle, square and pentagon diagrams to see which have anomalous, and which have normal, Ward identities.

5.1 The σ-Models

As we have just noted, the σ-models are a special class of field theory models in which PCAC holds as an operator relation.[16] Since we are interested primarily in the neutral axial-vector current (which can couple to two photons through the triangle diagram), we consider a truncated version of the σ-model in which the charged axial-vector currents do not appear. This simplified model contains only a proton (ψ), a neutral pion (π) and a scalar meson (σ), with Lagrangian density

$$
\begin{aligned}
\mathcal{L} = \bar{\psi}[i\gamma\cdot\Box - G_0(g_0^{-1} + \sigma + i\pi\gamma_5)]\,\psi \\
+\lambda_0[\,4\sigma^2 + 4g_0\sigma(\sigma^2+\pi^2) + g_0^2(\sigma^2+\pi^2)^2\,] \\
+ \tfrac{1}{2}\mu_0^2[\,2\,g_0^{-1}\,\sigma + \sigma^2 + \pi^2\,] \\
+ \tfrac{1}{2}[\,(\partial\pi)^2 + (\partial\sigma)^2\,] - \tfrac{1}{2}\mu_1^2(\pi^2+\sigma^2).
\end{aligned}
\tag{138}
$$

In writing Eq. (138) we have chosen the fully translated form of the σ-model, with

$$
\langle 0 \mid \sigma \mid 0 \rangle = 0 \tag{139}
$$

to all orders of perturbation theory. The neutral axial-vector current is generated by making a chiral gauge transformation on the fields with position dependent gauge parameter $v(x)$,

$$
\begin{aligned}
\psi &\rightarrow (1 + \tfrac{1}{2}\,i\,\gamma_5 v)\psi \;, \\
\pi &\rightarrow \pi - v\,(g_0^{-1} + \sigma)\;, \\
g_0^{-1} + \sigma &\rightarrow g_0^{-1} + \sigma + v\pi\;.
\end{aligned}
\tag{140}
$$

Using the recipe of Eq. (90), we find

$$
j_\mu^5 = -\delta\mathcal{L}/\delta(\partial^\mu v) = \bar{\psi}\,\tfrac{1}{2}\gamma_\mu\gamma_5\psi + \sigma\,\partial_\mu\pi - \pi\partial_\mu\sigma + g_0^{-1}\partial_\mu\pi, \tag{141a}
$$

$$
\partial^\mu j_\mu^5 = -\delta\mathcal{L}/\delta v = -(\mu_1^2/g_0)\,\pi\;. \tag{141b}
$$

Thus, as claimed, the divergence of the axial-vector current is proportional to the canonical pion field. The various parameters appearing in Eq. (138) have the following significance:

(i) G_0 is the unrenormalized meson-nucleon coupling constant;

(ii) g_0 is related to the bare nucleon mass m_0 by $G_0/g_0 = m_0$;

(iii) μ_1^2 is the bare meson mass which appears in the bare σ and π propagators $(q^2-\mu_1^2+i\varepsilon)^{-1}$;

(iv) the term $\lambda_0[4\sigma^2+4g_0\sigma(\sigma^2+\pi^2)+g_0^2(\sigma^2+\pi^2)^2]$ is a chiral-invariant meson-meson scattering interaction;

(v) the term $\frac{1}{2}\mu_0^2(2g_0^{-1}\sigma+\sigma^2+\pi^2)$ is a chiral-invariant <u>counter term</u> which is necessary to guarantee that

$$\langle 0|\delta\mathcal{L}/\delta\sigma|0\rangle = \partial_\lambda \langle 0|\delta\mathcal{L}/\delta(\partial_\lambda\sigma)|0\rangle = 0\ , \qquad (142)$$

as is required by the Euler-Lagrange equations of motion and translation invariance. Eqs. (139) and (142) fix μ_0^2 to have the value

$$\mu_0^2 = \langle 0|G_0 g_0\bar{\psi}\psi - \lambda_0[4g_0^2(3\sigma^2+\pi^2)+4g_0^3\sigma(\sigma^2+\pi^2)]\,|0\rangle . \qquad (143)$$

The effect of μ_0^2, which is formally quadratically divergent, is to remove the "tadpole" diagrams of the type

so that the condition $\langle 0|\sigma|0\rangle = 0$ is maintained in each order of perturbation theory. It is easily seen that the μ_0^2 counter term simultaneously removes the quadratically divergent parts of the π- and σ-meson self-energies. Consequently, the remaining bare quantities appearing in the Lagrangian (G_0, g_0, μ_1), as well as the wave-function renormalizations, are at most logarithmically divergent, and the theory is no more singular than is QED.[17]

For our future work, it will prove convenient to rewrite the PCAC equation, Eq. (141b), as follows. First, we introduce the pion wave function renormalization constant Z_3^{π}, which enables us to express Eq. (141b) in terms of the renormalized pion field π^r,

$$\partial^{\mu} j_{\mu}^{5} = -(\mu_1^2/g_0)(Z_3^{\pi})^{\frac{1}{2}}\, \pi^r . \tag{144}$$

Next, we define the pion weak decay amplitude f_{π} by writing

$$<\pi(q)|\, j_{\lambda}^{5}\, |\, 0> = (2q_0)^{-\frac{1}{2}}(-iq_{\lambda}/\mu^2)\, f_{\pi}/\sqrt{2} , \tag{145}$$

with μ the physical pion mass. [In the full version of the σ-model, in which the neutral axial-vector current has charged isospin partners, f_{π} is just the amplitude for weak charged pion decay.] Taking the divergence of Eq. (145), substituting Eq. (144) and using $<\pi(q)|\pi^r|0> = (2q_0)^{-\frac{1}{2}}$, we find the relation

$$-(\mu_1^2/g_0)(Z_3^{\pi})^{\frac{1}{2}} = f_{\pi}/\sqrt{2} . \tag{146}$$

So we can eliminate the renormalization constants from Eq. (144) and rewrite the PCAC equation entirely in terms of physical quantities,

$$\partial^{\mu} j_{\mu}^{5} = (f_{\pi}/\sqrt{2})\pi^{r} \tag{147}$$

So far, we have only discussed the σ-model in the absence of electromagnetism. To include electromagnetism, we simply add to the Lagrangian density the terms

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - e_0\bar{\psi}\gamma_{\mu}\psi A^{\mu} \tag{148}$$

Because of the triangle diagram, the PCAC equation of Eq. (147) is modified to read

$$\partial^{\mu} j_{\mu}^{5} = (f_{\pi}/\sqrt{2})\,\pi^{r} + \tfrac{1}{2}\,\frac{\alpha_0}{4\pi}\,F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}, \tag{149}$$

with the factor $\frac{1}{2}$ in the anomaly term in Eq. (149) just a reflection of the factor $\frac{1}{2}$ appearing in the nucleon term in Eq. (141a). By introducing appropriate regulated Feynman rules and carrying out an analog of the argument of the previous section, one can show that Eq. (149) is exact to all orders of perturbation theory in both the electromagnetic and strong interactions. In other words, neither virtual photon nor virtual meson radiative corrections to the triangle diagram change the coefficient of the anomaly term.

All of the above considerations carry over directly to the isospin and full SU_3[18] generalizations of the sigma model. In the full SU_3 case, the proton ψ is replaced by a fermion triplet (ψ_1, ψ_2, ψ_3), the scalar and pseudoscalar mesons are replaced by nonets, and the axial-vector current j_μ^5 becomes the third component $\mathcal{F}_{3\mu}^5$ of the axial-vector current octet. The anomalous PCAC equation now becomes

$$\partial^\mu \mathcal{F}_{3\mu}^5 = (f_\pi/\sqrt{2})\, \pi^{0r} + S\, \frac{\alpha_0}{4\pi}\, F^{\xi\sigma} F^{\tau\rho} \epsilon_{\xi\sigma\tau\rho}, \tag{150}$$
$$S = \sum_j g_j Q_j^2 ,$$

with π^{0r} **the renormalized neutral pion field, with $Q_j e$ the** charge of the **jth fermion and with g_j the fermion coup**lings appearing in the expression for $\mathcal{F}_{3\mu}^5$ **in terms of** elementary fields,

$$\mathcal{F}_{3\mu}^5 = \sum_j g_j \bar{\psi}_j \gamma_\mu \gamma_5 \psi_j + \text{Meson Terms} . \tag{151}$$

Again, Eq. (150) is exact to all finite orders of perturbation theory. The interpretation of the expression for S is that the total coefficient of the anomaly is the sum of contributions from triangle graphs involving each of the individual elementary fermions.

Equation (150) generalizes even further, to models in which the naive divergence D_3^5 (i.e., the divergence of $\mathcal{F}_{3\mu}^5$ in the absence of electromagnetism) is not a canonical pion field. Because the argument of Section 4 depended primarily on the multiplicative renormalizability of the naive divergence, we would expect the equation

$$\partial^\mu \mathcal{F}_{3\mu}^5 = D_3^5 + S\, \frac{\alpha_0}{4\pi}\, F^{\xi\sigma} F^{\tau\rho} \epsilon_{\xi\sigma\tau\rho} \tag{152}$$

to be correct in any renormalizable field theory in which the naive axial divergence D_3^5 is multiplicatively renormalizable. Since, by the argument of Eqs. (76) - (79), multiplicative renormalizability of the naive divergence implies finite Z_A/Z_2 in the absence of electromagnetism, we may rephrase the above statement by saying that in any renormalizable field theory with finite axial-vector renormalizations g_A/g_V in the absence of electromagnetism, we expect Eq. (152) to be exact when electromagnetic effects are added. As long as D_3^5 is a smooth interpolating field for the pion, we may effectively make the replacement $D_3^5 \approx (f_\pi/\sqrt{2})\, \pi^{0r}$ in Eq. (152) for small extrapolations away from the pion mass shell. Thus, Eq. (150) is the correct PCAC equation, even in the more general class of models.

5.2 Low Energy Theorem for π^0 Decay

As we saw in Eqs. (125)-(131), the anomalous axial-vector Ward identity equation gives an exact low energy theorem for the vacuum to two photon matrix elements of the naive axial divergence. Since the naive axial divergence in Eq. (150) is the π^0 field, the low energy theorem in this case gives us a statement about the $\pi^0 \rightarrow 2\gamma$ decay amplitude, extrapolated off shell to zero pion mass.[19] The standard definition of the $\pi^0 \rightarrow 2\gamma$ amplitude $F^\pi(k_1 \cdot k_2)$ is

$$\langle \gamma(k_1, \epsilon_1)\ \gamma(k_2, \epsilon_2) | (\square^2 + \mu^2)\, \pi^{0r} | 0 \rangle \qquad (153)$$

$$= (4k_{10}k_{20})^{-\frac{1}{2}} k_1^{\xi} k_2^{\tau} \epsilon_1^{*\sigma} \epsilon_2^{*\rho} \epsilon_{\xi\tau\sigma\rho} F^{\pi}(k_1 \cdot k_2) .$$

Comparing with Eqs. (125)-(131), we see that the low energy theorem becomes

$$\widetilde{G}(0) = \mu^{-2} (f_\pi/\sqrt{2})\, F^{\pi}(0) = S(-2\alpha/\pi), \tag{154a}$$

that is

$$F^{\pi}(0) = (-\alpha/\pi)\,(2S)\,\sqrt{2}\,\mu^2/f_\pi . \tag{154b}$$

According to Eq. (154b), the off-shell $\pi^0 \rightarrow 2\gamma$ amplitude is directly proportional to the anomaly term in Eq. (150). If the anomaly term were omitted [i. e., if Eq. (147) were used to derive the low energy theorem], one would obtain instead the prediction

$$F^{\pi}(0) = 0 , \tag{155}$$

which states that the $\pi^0 \rightarrow 2\gamma$ decay is suppressed.[20] Let us briefly discuss some of the implications of Eq. (154b).

(i) The experimental π^0 decay rate predicted by Eq. (154b) depends on the parameter S, which in turn is determined by the charges and axial couplings of the elementary fermions. In the triplet-model, consisting of an SU_3-triplet of fermions $(\psi_1, \psi_2, \psi_3) \equiv (p, n, \lambda)$ interacting by meson exchange, the axial-vector couplings are $(g_1, g_2, g_3) = (\frac{1}{2}, -\frac{1}{2}, 0)$, and U-spin invariance of the electromagnetic current requires the respective charges of (ψ_1, ψ_2, ψ_3) to be $(Q, Q-1, Q-1)$. We immediately find

$$S = \tfrac{1}{2}Q^2 - \tfrac{1}{2}(Q-1)^2 = Q-\tfrac{1}{2} \equiv Q_{AV} = \tfrac{1}{2}[\,Q+(Q-1)] \ . \tag{156}$$

where Q_{AV} denotes the average charge of the fermions participating in the charged β-decay currents. We find, in the fractionally-charged quark model, that $Q = 2/3$ and $S = 1/6$, while in the integrally-charged quark models with $Q = 1$ and $Q = 0$, we find respectively $S = +\frac{1}{2}$ and $S = -\frac{1}{2}$. Using the formula

$$\tau^{-1} = (\mu^3/64\pi)\,|F^\pi(\mu^2)|^2, \tag{157}$$

taking the experimental value $f_\pi \approx 0.96\ \mu^3$ and approximatimating $F^\pi(\mu^2)$ by its off-shell value $F^\pi(0)$, we find for the π^0 decay rate

$$\tau^{-1} = 0.8 \text{ eV} \quad \text{for } S = \frac{1}{6} \tag{158a}$$

$$\tau^{-1} = 7.4 \text{ eV} \quad \text{for } S = \pm\frac{1}{2}\,. \tag{158b}$$

The experimental decay rate quoted by Rosenfeld[21] is

$$\tau^{-1}_{\text{expt}} = (1.12 \pm 0.22)\cdot 10^{16}\ \text{sec}^{-1} \approx (7.37 \pm 1.5)\,\text{eV}, \tag{159}$$

and may be as large as 11 eV if recent Primakoff effect experiments[22] turn out to be more reliable than earlier counter experiments included in Rosenfeld's average. In any case, we see that the <u>fractionally charged quark model is strongly excluded</u>, while the integrally charged models with triplet charges (1, 0, 0) or (0, -1, -1) are in satisfactory agreement with the experimental rate. Note that the apparent spectacular agreement between Eq. (158b) and Eq. (159)

is somewhat fortuitous, both because of the uncertainty in the experimental rate and because of the expected 10-20 percent extrapolation error involved in PCAC arguments. For example, if instead of using the experimental value of f_π in Eq.(154b) we use the Goldberger-Treiman relation

$$\frac{\sqrt{2}\,\mu^2}{f_\pi} \approx \frac{g_r}{M_N g_A}, \tag{160}$$

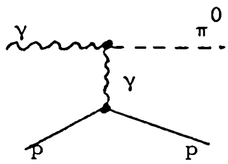
M_N = nucleon mass,

g_r = pion-nucleon coupling constant ≈ 13.6,

g_A = nucleon axial-vector coupling constant ≈ 1.22,

the theoretical prediction in the $S = \pm\frac{1}{2}$ cases is increased by 20 percent, to $\tau^{-1} = 9.1$ eV.

(ii) The comparison which we have made with the experimental π^0 decay rate tells us that $|S| \approx 0.5$, but does not determine the <u>sign</u> of S. However, there are a number of different ways of determining the sign of S, all of which, fortunately, seem to agree! The first method is to study $\pi^+ \to e^+\nu\gamma$ decay, the vector part of which is related by isospin rotation to F^π and the axial-vector part of which can be estimated by using hard pion techniques. The analysis,[23] using the experimentally-measured vector to axial-vector ratio for this process, gives a positive value of S. A second method is to make use of forward π^0 photoproduction, where one can observe the interference between the Primakoff amplitude

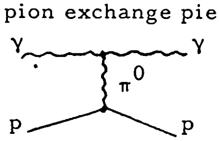


which is proportional to F^{π}, and the forward strong interaction amplitude. The sign of the latter can be determined by finite energy sum rules from the known sign of the pion photoproduction amplitude in the (3, 3) resonance region; the analysis[24] again indicates S positive. A third method consists of comparing Eq.(154b) with an approximate expression for the $\pi^0 \to 2\gamma$ amplitude derived[25] by applying a pole dominance argument to proton Compton scattering dispersion relations,

$$F^{\pi} \approx -4\pi\alpha \frac{\kappa_p}{g_r} \frac{1}{M_N}, \tag{161}$$

κ_p = proton anomalous magnetic moment = 1.79.

Eq. (161) gives a $\pi^0 \to 2\gamma$ rate of 2.0 eV, in fair agreement with experiment. The comparison again gives S positive. A fourth method which has been proposed[23] is to use Compton scattering data on protons to try to measure the interference of the pion exchange piece,



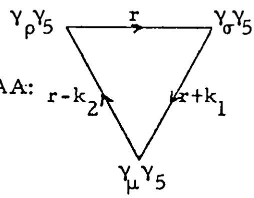
which is proportional to F^{π}, with the nucleon and nucleon isobar exchange pieces. The problem with this proposal[26] is that one does not know whether to take the pion exchange piece in its Born approximation form, $tF^{\pi}/(t-\mu^2)$, or in the polology form $\mu^2F^{\pi}/(t-\mu^2)$. Since t is negative in the physical region, this uncertainty leads to a sign ambiguity and renders the method dubious. In any case, with fair certainty one learns from the first three methods that S is positive. This means that the triplet model with $Q=1$ and triplet charges (1, 0, 0) is favored.

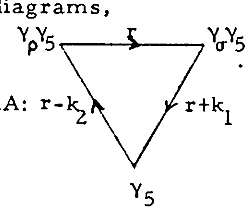
(iii) Although we have shown that Eq. (154b) is exact to all orders of perturbation theory in an interesting class of theoretical models, we have not dealt with the possibility that Eqs. (150) and (154b) are modified by nonperturbative effects. For example, should the coefficient S receive contributions from triangles involving bound states of the fundamental fields as well as from triangles involving the fundamental fields themselves, or would this be double counting? The answer to this question is not known. Our neglect of possible nonperturbative modifications in the analysis above is pure assumption.

(iv) Let us now backtrack from quantitative predictions to the more general question of how we <u>know</u> that π^0 decay is <u>not</u> really a suppressed decay, as would be suggested by PCAC with the triangle anomaly omitted. There is in fact an interesting experimental test,[27] which strongly suggests that π^0 decay is not suppressed. To see this, let us return to the suppression. argument in the case when one of the final photons is off mass shell, say $k_1^2 \neq 0$. As we have seen in Eqs. (106) and (107), the vacuum to two photon matrix element of $\partial^\mu \mathcal{F}^5_{3\mu}$ in this case is proportional to $k_1^\xi k_2^\tau \epsilon_1^{*\sigma} \epsilon_2^{*\rho} \epsilon_{\xi\tau\sigma\rho}[\mu^2 + \beta k_1^2]$, with β of order unity. We see that while the on shell part of the amplitude (in the absence of the anomaly) is suppressed by a factor μ^2, the photon off-shell dependence is not suppressed. Since the off-shell amplitude is measured in the reaction $\pi^0 \to e^+ e^- \gamma$, the suppression argument predicts that the k_1^2 dependence of this process will have the form $1 + (\beta/\mu^2)k_1^2$, which has a much larger slope than the form $1+(\beta/m_\rho^2)$ (with m_ρ the ρ-meson mass) expected in the absence of suppression of the $\pi^0 \to 2\gamma$ decay. A recent measurement[28] of this slope gives a matrix element $1 + \underline{a}\, k_1^2$, with $\underline{a} = (0.01 \pm 0.11)/\mu^2$. Clearly, ,this is strong evidence against $\pi^0 \to 2\gamma$ suppression, and therefore some mechanism, like the triangle anomalies which we have discussed in such great detail, is definitely needed to avoid the suppression prediction of Eq. (155).

5.3 Other Ward Identity Anomalies

So far, we have dealt exclusively with the VVA triangle diagram and its Ward identity anomaly. Let us now briefly examine the question of whether there are other diagrams with divergence anomalies. We begin with a study of the Ward identity relating the AAA and the PAA (P = pseudoscalar) fermion triangle diagrams,

AAA: 

PAA: 

Defining

$$\frac{-iR^5_{\sigma\rho\mu}}{(2\pi)^4} \equiv 2\int \frac{d^4r}{(2\pi)^4}(-1)\,\mathrm{Tr}\Big\{\frac{i}{\not r+\not k_1-m_0}(-i\gamma_\sigma\gamma_5) \times \frac{i}{\not r-m_0}(-i\gamma_\rho\gamma_5)\frac{i}{\not r-\not k_2-m_0}\gamma_\mu\gamma_5\Big\} \tag{162a}$$

$$\frac{-i2m_0R^5_{\sigma\rho}}{(2\pi)^4} \equiv 2\int \frac{d^4r}{(2\pi)^4}(-1)\,\mathrm{Tr}\Big\{\frac{i}{\not r+\not k_1-m_0}(-i\gamma_\sigma\gamma_5) \times \frac{i}{\not r-m_0}(-i\gamma_\rho\gamma_5)\frac{i}{\not r-\not k_2-m_0}2m_0\gamma_5\Big\}\ , \tag{162b}$$

we find, by manipulations under the r-integrals which ignore the linear divergence in $R^5_{\sigma\rho\mu}$, the naive Ward identity

$$-(k_1+k_2)^{\mu} R^5_{\sigma\rho\mu} = 2m_0 R^5_{\sigma\rho} \ . \qquad (163)$$

To search for possible corrections to Eq. (163), we must first make the definitions of $R^5_{\sigma\rho\mu}$ and $R^5_{\sigma\rho}$ precise. The latter quantity is given by a convergent integral and so is uniquely defined, but the integral in Eq. (162a) for $R^5_{\sigma\rho\mu}$ is linearly divergent, and hence its precise value depends on the choice of origin for symmetric integration. However, $R^5_{\sigma\rho\mu}$ is uniquely specified if we require that it be Bose symmetric under interchange of any pair of vertices. So our question becomes that of finding the extra terms (if any) which appear in Eq. (163) when the Bose-symmetric evaluation of the AAA triangle is used on the left-hand side. One way of doing this would be to explicitly calculate $R^5_{\sigma\rho\mu}$ and its divergence, as we did in our discussion of the VVA case in Eqs. (55)-(63). But this is not actually necessary: we can answer our question by comparison with our result for the VVA triangle. To see this, we note that arguments similar to those of Subsection 2.2 show that any anomaly term in Eq. (163) must be independent of the fermion mass m_0, so that it suffices to consider the case $m_0 = 0$. When $m_0 = 0$, the simple relation

$$(-i\gamma_{\sigma}\gamma_5) \ \frac{i}{\not{p}} \ (-i\gamma_{\rho}\gamma_5) = (-i\gamma_{\sigma}) \ \frac{i}{\not{p}} \ (-i\gamma_{\rho}) \qquad (164)$$

indicates that Eq. (162a) is formally identical to Eq. **(54)** for the VVA triangle $R_{\sigma\rho\mu}$ in the $m_0 = 0$ limit. **Remembering that $R^5_{\sigma\rho\mu}$** is obtained from Eq. **(162a) by <u>symmetrizing</u> with respect to the three vertices, this tells us that in the $m_0 = 0$ limit $R^5_{\sigma\rho\mu}(k_1, k_2)$ is related to $R_{\sigma\rho\mu}(k_1, k_2)$ by**

$$R^5_{\sigma\rho\mu}(k_1, k_2) = (\tfrac{1}{3})[R_{\sigma\rho\mu}(k_1, k_2) + R_{\rho\mu\sigma}(k_2, -(k_1+k_2)) + R_{\mu\sigma\rho}(-(k_1+k_2), k_1)]. \tag{165}$$

Taking the divergence of Eq. (165), and using the fact that $R_{\sigma\rho\mu}$ is divergenceless with respect to the first two indices (the vector indices) and has the third index divergence given by Eq. (63), we find

$$-(k_1+k_2)^\mu R^5_{\sigma\rho\mu}(k_1, k_2) = (\tfrac{1}{3})\, 8\pi^2 k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho} \tag{166}$$

when $m_0 = 0$. <u>Thus, the AAA diagram has a divergence anomaly which is 1/3 of the anomaly in the VVA diagram.</u> Comparing with Eq. (163), we see that for general m_0 Eq. (166) becomes

$$-(k_1+k_2)^\mu R^5_{\sigma\rho\mu}(k_1, k_2) = 2m_0 R^5_{\sigma\rho} + (8\pi^2/3) k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho}, \tag{167}$$

which is our final result for the AAA Ward identity. Note that Eq. (166) has the interesting implication that even in the $m_0 = 0$ limit, it is impossible to construct a quantum electrodynamics in which the photon is coupled to the axial vector current, since the AAA triangle diagram cannot be made to simultaneously satisfy the requirements of Bose symmetry and current conservation.

There is yet another way of obtaining Eq. (167) for the AAA diagram [and also our old results of Eq. (63) for the VVA case] without performing a full explicit calculation. This is to introduce a regulator mass M_0 by subtracting from Eqs. (54) and (162a) the corresponding expression with m_0 replaced by M_0:

$$
\begin{aligned}
\frac{-ie_0^2}{(2\pi)^4} R^{REG}_{\sigma\rho\mu} &\equiv 2\int \frac{d^4r}{(2\pi)^4} (-1)\ \mathrm{Tr}\{I_{VVA}\}, \\
\frac{-i}{(2\pi)^4} R^{5REG}_{\sigma\rho\mu} &\equiv 2\int \frac{d^4r}{(2\pi)^4} (-1)\ \mathrm{Tr}\{I_{AAA}\},
\end{aligned}
\tag{168}
$$

$$
\begin{aligned}
I_{VVA} = {} & \frac{i}{\not r + \not k_1 - m_0} (-ie_0\gamma_\sigma) \frac{i}{\not r - m_0} (-ie_0\gamma_\rho) \frac{i}{\not r - \not k_2 - m_0} \gamma_\mu\gamma_5 \\
& - \frac{i}{\not r + \not k_1 - M_0} (-ie_0\gamma_\sigma) \frac{i}{\not r - M_0} (-ie_0\gamma_\rho) \frac{i}{\not r - \not k_2 - M_0} \gamma_\mu\gamma_5 , \\
I_{AAA} = {} & \frac{i}{\not r + \not k_1 - m_0} (-i\gamma_\sigma\gamma_5) \frac{i}{\not r - m_0} (-i\gamma_\rho\gamma_5) \frac{i}{\not r - \not k_2 - m_0} \gamma_\mu\gamma_5 \\
& - \frac{i}{\not r + \not k_1 - M_0} (-i\gamma_\sigma\gamma_5) \frac{i}{\not r - M_0} (-i\gamma_\rho\gamma_5) \frac{i}{\not r - \not k_2 - M_0} \gamma_\mu\gamma_5 .
\end{aligned}
$$

Because the subtractions in Eq. (168) remove the linear divergences in both the VVA and AAA cases, $R^{REG}_{\sigma\rho\mu}$ will be automatically divergenceless with respect to the first two indices and $R^{5REG}_{\sigma\rho\mu}$ will be automatically Bose symmetric, with no need for additional subtractions. Thus, the quantities $R_{\sigma\rho\mu}$ and $R^5_{\sigma\rho\mu}$ are simply the limits of the corresponding regulated quantities as $M_0 \to \infty$,

$$R_{\sigma\rho\mu} = \lim_{M_0\to\infty} R^{REG}_{\sigma\rho\mu} \qquad (169)$$
$$R^5_{\sigma\rho\mu} = \lim_{M_0\to\infty} R^{5\,REG}_{\sigma\rho\mu}$$

To study the axial vertex divergences, we use the fact that since the regulated triangles have no linear divergences, they have no Ward identity anomalies, but rather satisfy the normal Ward identities

$$-(k_1+k_2)^\mu R^{REG}_{\sigma\rho\mu} = 2m_0 R_{\sigma\rho} - 2M_0 R_{\sigma\rho}\big|_{m_0\to M_0}, \qquad (170)$$
$$-(k_1+k_2)^\mu R^{5\,REG}_{\sigma\rho\mu} = 2m_0 R^5_{\sigma\rho} - 2M_0 R^5_{\sigma\rho}\big|_{m_0\to M_0}.$$

Taking the limit $M_0 \to \infty$, we find

$$-(k_1+k_2)^\mu R_{\sigma\rho\mu} = 2m_0 R_{\sigma\rho} - \lim_{M_0\to\infty} 2M_0 R_{\sigma\rho}\big|_{m_0\to M_0},$$
$$-(k_1+k_2)^\mu R^5_{\sigma\rho\mu} = 2m_0 R^5_{\sigma\rho} - \lim_{M_0\to\infty} 2M_0 R^5_{\sigma\rho}\big|_{m_0\to M_0}. \qquad (171)$$

As we saw in Eq. (61), $R_{\sigma\rho}$ is given by

$$R_{\sigma\rho} = k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho} 8\pi^2 m_0 I_{00}(k_1, k_2), \qquad (172a)$$

and a simple calculation shows that

$$R^5_{\sigma\rho} = k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho} 8\pi^2 m_0 [2(I_{10}+I_{01}) - I_{00}]. \qquad (172b)$$

Thus, it is easy to evaluate the limits in Eq. (171), giving

$$-\lim_{M_0\to\infty} 2M_0 R_{\sigma\rho}\big|_{m_0\to M_0} = -2M_0 k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho} 8\pi^2 M_0 \int_0^1 dx \int_0^{1-x} dy (-M_0^2)^{-1}$$
$$= 8\pi^2 k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho}, \qquad (173)$$
$$-\lim_{M_0\to\infty} 2M_0 R^5_{\sigma\rho}\big|_{m_0\to M_0} = -2M_0 k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho} 8\pi^2 M_0 \int_0^1 dx \int_0^{1-x} dy$$

$$\times [2(x+y)-1](-M_0^2)^{-1} = (8\pi^2/3)k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho},$$

in agreement with Eqs. (63) and (167). Thus we see that from the regulator point of view, the divergence anomalies result from the failure of the regulator term in the naive divergence to vanish in the limit of infinite regulator mass.

Let us now turn to the question of whether larger loops than triangle diagrams can have Ward identity anomalies. This question has been carefully analyzed, in the case of fermion loops, by both the ϵ-separation method[29] discussed in Subsection 2.4 and by the regulator method[30] discussed immediately above. In addition to considering vector and axial-vector vertices, the analyses allow scalar and pseudoscalar couplings as well, and internal degrees of freedom (such as SU_3 or isotopic spin) are also permitted. The results may be summarized as follows:

(i) No loops involving scalar or pseudoscalar couplings have Ward identity anomalies which cannot be removed by appropriately chosen subtractions. The only loops with true anomalies, which cannot be removed, are ones with only vector and axial-vector vertices, with the number of axial-vector vertices odd. If subtraction terms are chosen so that all vector index Ward identities are normal, the following loops have anomalous axial index Ward identities:

the VVA and AAA triangles, the VVVA and VAAA squares, and the VVVVA, VVAAA and AAAAA pentagons. The triangle anomalies are the ones which we have already discussed, and are the only anomalies when internal degrees of freedom are absent. The squares are anomalous because the naive Ward identity derivations for them involve a translation of integration variable in linearly divergent triangle diagrams. In the case of the pentagons, the naive derivations involve a translation of integration variable in logarithmically divergent square diagrams, which is allowed, but the Ward identities become anomalous as a result of the counter-terms which were added to the square diagrams to satisfy the vector current Ward identities. All diagrams larger than the pentagons have normal Ward identities.

(ii) A compact and explicit description of these anomalies[29] may be obtained by introducing external scalar (S), pseudoscalar (P), vector (V^μ) and axial-vector (A^μ) fields which couple to the respective currents, and which allow us to write a simple generating functional for all of the fermion loop diagrams. [Note the change in notation from our discussion of QED, where A^μ denoted the (vector) photon field.] We start with a Lagrangian density which we write as

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x), \tag{174}$$

$$\mathcal{L}_0(x) = \bar{\psi}(x)(i\gamma\cdot\Box - m_0)\psi(x),$$

$$\mathcal{L}_I(x) = \bar{\psi}(x)[S(x) + i\gamma_5 P(x) + \gamma_\mu V^\mu(x) + \gamma_\mu\gamma_5 A^\mu(x) + \Delta m_0]\,\psi(x),$$

with each of the fields $S(x)$, $P(x)$, $V^\mu(x)$, $A^\mu(x)$ a matrix in the internal space,

$$\begin{aligned} S(x) &= \sum_a \lambda_S^a S_a(x), & P(x) &= \sum_a \lambda_P^a P_a(x), \\ V^\mu(x) &= \sum_a \lambda_V^a V_a^\mu(x), & A^\mu(x) &= \sum_a \lambda_A^a A_a^\mu(x). \end{aligned} \tag{175}$$

The fields $S_a(x)$, ... are the external fields and the matrices λ_S^a, ... are their respective coupling matrices. The matrix Δm_0 supplies fermion mass splittings. Going over to the conventional interaction picture, the S-matrix is formally defined by

$$\mathcal{S} = T\,\exp\{i\int d^4x\,\mathcal{L}_I(x)\}, \tag{176}$$

with T the time ordering operator. If we take the fermion vacuum expectation $<0|\mathcal{S}|0>$ and drop disconnected diagrams, we clearly get a generating functional for all of the closed loops. Defining vector and axial vector currents by

$$\mathcal{F}_{a\mu}(x) = -i\frac{\delta}{\delta V_a^\mu(x)}\mathcal{S}, \quad \mathcal{F}_{a\mu}^5(x) = -i\frac{\delta}{\delta A_a^\mu(x)}\mathcal{S}, \tag{177}$$

the calculations described above lead to the divergence equations

$$\partial^\mu \mathcal{F}_{a\mu}(x) = D_a(x),$$

$$\partial^\mu \mathcal{F}^5_{a\mu}(x) = D^5_a(x) + \frac{1}{4\pi^2}\varepsilon_{\mu\nu\sigma\tau}\mathrm{Tr}_I\{\lambda^a_A[\tfrac{1}{4}F^{\mu\nu}_V(x)F^{\sigma\tau}_V(x)$$
$$+\frac{1}{12}F^{\mu\nu}_A(x)F^{\sigma\tau}_A(x) + \frac{2}{3}\,i\,A^\mu(x)\,A^\nu(x)\,F^{\sigma\tau}_V(x)$$
$$+\frac{2}{3}\,iF^{\mu\nu}_V(x)\,A^\sigma(x)\,A^\tau(x) + \frac{2}{3}iA^\mu(x)F^{\nu\sigma}_V(x)\,A^\tau(x) \tag{178}$$
$$-\frac{8}{3}A^\mu(x)\,A^\nu(x)\,A^\sigma(x)\,A^\tau(x)]\},$$

with Tr_I a trace over the internal degrees of freedom, with

$$F^{\mu\nu}_V(x) = \partial^\mu V^\nu(x) - \partial^\nu V^\mu(x) - i[V^\mu(x), V^\nu(x)]$$
$$-i[A^\mu(x), A^\nu(x)], \tag{179}$$
$$F^{\mu\nu}_A(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) - i[V^\mu(x), A^\nu(x)]$$
$$-i[A^\mu(x), V^\nu(x)],$$

and with D_a and D^5_a the naive vector and axial-vector divergences. From Eq. (178), all of the fermion loop Ward identities are easily generated by variation with respect to the external fields.

(iii) Finally, we note that a calculation similar to the one described above shows that there are no boson loop Ward identity anomalies.[31]

As a result of the fact that no loops involving scalar or pseudoscalar couplings have Ward identity anomalies, it is easy to see that $\pi^0 \to 2\gamma$, and the SU_3-related processes[32] $\eta \to 2\gamma$ and $X^0 \to 2\gamma$, are the only cases in which the anomalies alter the usual current-algebra-PCAC predictions.

In particular, the anomalies which we have discussed do not alter the predictions of current algebra in the troublesome $\eta \to 3\pi$ decays.

6. CONNECTION BETWEEN WARD IDENTITY ANOMALIES AND COMMUTATOR (BJORKEN-LIMIT) ANOMALIES

As we have seen, the VVA Ward identity anomaly implies other anomalies as well, such as in the renormalization of the axial-vector vertex and in the behavior of γ_5-transformations in massless electrodynamics. In the present section, we will see that the Ward identity anomaly, also implies that anomalous commutators are present: that is, the divergence anomaly requires that certain simple commutators, involving the electromagnetic potential and the currents, have values different from the canonical ones obtained by naive use of canonical commutation relations. The actual, correct expressions for the commutators will be deduced from our formula for the triangle diagram by a technique introduced by Bjorken[33] and Johnson and Low,[34] usually called the "Bjorken limit" method. Commutator anomalies are not a new phenomenon; in the usual QED without axial-vector currents, anomalies in potential-current commutators ("seagulls") and in current-current commutators ("Schwinger terms")[35] have been known for some time. The two anomalies in QED are related, and cancel exactly when the divergence of a covariant matrix element is taken, guaranteeing current conservation. The distinguishing feature of the commutator anomalies associated with the triangle diagram is that when the axial-vector divergence is taken, the seagull and Schwinger term do not cancel.[36] Rather, they combine to give the divergence anomaly found in Section 2, giving an alternative interpretation of the divergence anomaly as the result of non-cancellation of seagull and Schwinger term.

We begin our discussion by reviewing the seagull and Schwinger term, and their cancellation, in QED and also by outlining the Bjorken limit method. Then we will apply the concepts which we have developed to the VVA triangle diagram.

6.1 Schwinger Terms, Seagulls, the Reduction Formula and T and T* Products in QED

Let us consider the equal-time commutator of the time and space components of the ordinary electromagnetic current in QED,

$$[\, j_0(\underset{\sim}{x}, t),\ j_r(\underset{\sim}{y}, t)\,]\,. \tag{180}$$

By naive use of canonical commutation relations, we find for this commutator the value

$$\begin{aligned} &[\psi^{\dagger}(\underset{\sim}{x}, t)\ \psi(\underset{\sim}{x}, t), \psi^{\dagger}(\underset{\sim}{y}, t)\gamma_0\gamma_r\, \psi(\underset{\sim}{y}, t)\,] \\ &= \delta^3(\underset{\sim}{x}-\underset{\sim}{y})\ \psi^{\dagger}(\underset{\sim}{x}, t)[\,1, \gamma_0\gamma_r\,]\, \psi(\underset{\sim}{x}, t) = 0. \end{aligned} \tag{181}$$

It is easy to see, however, that Eq. (181) is false. To prove this we take the divergence $\partial/\partial y_r$ of Eq. (180) and use current conservation, which tells us that $\partial/\partial y_r\, j_r(\underset{\sim}{y}, t) = -(\partial/\partial t)\, j_0(\underset{\sim}{y}, t)$, giving

$$0=[\, j_0(\underset{\sim}{x}, t),\ \partial^r j_r(\underset{\sim}{y}, t)] = -[\, j_0(\underset{\sim}{x}, t), \frac{\partial}{\partial t}\ j_0(\underset{\sim}{y}, t)] \tag{182}$$

if Eq. (181) is valid. Taking the vacuum expectation value of Eq. (182), setting $\underset{\sim}{x} = \underset{\sim}{y}$ and inserting a complete set of intermediate states to evaluate the commutator, we find

$$\begin{aligned} 0 &= <0|[\, j_0(\underset{\sim}{x}, t), \frac{\partial}{\partial t}\ j_0(\underset{\sim}{x}, t)]\,|0> \\ &= \sum_n <0|j_0(\underset{\sim}{x}, t)|n><n|\frac{\partial}{\partial t} j_0(\underset{\sim}{x}, t)|0> - <0|\frac{\partial}{\partial t} j_0(\underset{\sim}{x}, t)|n><n|j_0(\underset{\sim}{x}, t)|0> \\ &= 2i \sum_n (E_n - E_0)\,|<0|j_0(\underset{\sim}{x}, t)|n>|^2 \end{aligned} \tag{183}$$

All the terms on the right-hand side of Eq. (183) are greater than equal to zero, so Eq. (183) can be satisfied only if $<0|j_0(\underset{\sim}{x}, t)|n>=0$ for all intermediate states n, which is manifestly untrue. Thus the commutator of Eq. (180) cannot vanish, as the naive use of canonical commutation relations suggests.

In order to understand this result, let us follow the procedure of Subsection 2.4 and define the space-component[37] of the vector current as the limit of a non-local current, with spacelike separation $\underset{\sim}{\varepsilon}$, averaged over the direction of $\underset{\sim}{\varepsilon}$,

$$j_r(\underset{\sim}{y}, t) = \lim_{\underset{\sim}{\varepsilon} \to 0} \underset{\underset{\sim}{\varepsilon}}{\mathrm{av}}\, j_r(\underset{\sim}{y}, t; \underset{\sim}{\varepsilon}), \tag{184}$$

$$j_r(\underset{\sim}{y}, t; \underset{\sim}{\varepsilon}) = \overline{\psi}(\underset{\sim}{y}+\tfrac{1}{2}\underset{\sim}{\varepsilon}, t)\gamma_r \psi(\underset{\sim}{y}-\tfrac{1}{2}\underset{\sim}{\varepsilon}, t)\exp[ie_0 \int_{\underset{\sim}{x}-\frac{1}{2}\underset{\sim}{\varepsilon}}^{\underset{\sim}{x}+\frac{1}{2}\underset{\sim}{\varepsilon}} d\underset{\sim}{\ell} \cdot \underset{\sim}{A}(\underset{\sim}{\ell}, t)] .$$

Substituting Eq. (184) into the commutator of Eq. (180) and evaluating by using canonical commutation relations, we find

$$\begin{aligned}[j_0(\underset{\sim}{x}, t), j_r(\underset{\sim}{y}, t; \underset{\sim}{\varepsilon})] &= [\delta^3(\underset{\sim}{x}-\underset{\sim}{y} - \tfrac{1}{2}\underset{\sim}{\varepsilon}) - \delta^3(\underset{\sim}{x}-\underset{\sim}{y}+\tfrac{1}{2}\underset{\sim}{\varepsilon})] \\ &\quad \times \overline{\psi}(\underset{\sim}{y}+\tfrac{1}{2}\underset{\sim}{\varepsilon}, t)\gamma_r \psi(\underset{\sim}{y}-\tfrac{1}{2}\underset{\sim}{\varepsilon}, t)\exp[\ \dots\ ] \\ &= -\underset{\sim}{\varepsilon} \cdot \underset{\sim}{\nabla}_{\underset{\sim}{x}}\, \delta^3(\underset{\sim}{x}-\underset{\sim}{y})\, j_r(\underset{\sim}{y}, t; \underset{\sim}{\varepsilon}) .\end{aligned} \tag{185}$$

Eq. (185) at first glance appears to vanish, because of the factor $\underset{\sim}{\varepsilon}$ in front, but an elementary calculation shows that

$$\langle 0 | j_r(\underset{\sim}{y}, t; \underset{\sim}{\varepsilon}) | 0 \rangle = \frac{n\, \varepsilon_r}{(\underset{\sim}{\varepsilon}^2)^2} \tag{186}$$

with n a numerical factor, so that Eqs. (184) and (185) give

$$[j_0(\underset{\sim}{x}, t), j_r(\underset{\sim}{y}, t)] = \partial_r\, \delta^3(\underset{\sim}{x}-\underset{\sim}{y}) \lim_{\underset{\sim}{\varepsilon} \to 0} n/(3\underset{\sim}{\varepsilon}^2) \tag{187}$$
$$+ \text{ possible operator term.}$$

The divergent c-number term on the right-hand side of Eq. (187), called the Schwinger term, eliminates the paradox of Eq. (183). In QED, it appears that no operator term is present, so that the Schwinger term is pure c-number, but this is not true in other field theory models.

Clearly, giving the space-component of the electromagnetic current an explicit dependence on the electromagnetic potential will alter the potential-current commutation relations. Thus, from Eq. (184) we find

$$
\begin{aligned}
&[\, j_r(\mathbf{y},t),\ \partial A_s(\mathbf{x},t)/\partial t] \\
&= \lim_{\boldsymbol{\varepsilon}\to 0} \operatorname*{av}_{\boldsymbol{\varepsilon}}\ j_r(\mathbf{y},t;\boldsymbol{\varepsilon})[\, ie_0 \boldsymbol{\varepsilon}\cdot \mathbf{A}(\mathbf{y},t),\ \partial A_s(\mathbf{x},t)/\partial t] \\
&= e_0\, g_{rs}\, \delta^3(\mathbf{x}-\mathbf{y}) \lim_{\boldsymbol{\varepsilon}\to 0} n/(3\boldsymbol{\varepsilon}^2),
\end{aligned}
\tag{188}
$$

whereas the naive value of this commutator, computed without the $\boldsymbol{\varepsilon}$-separation, would be zero. We note that the anomalous commutator in Eq. (188) has the same coefficient as the one in Eq. (187); this is not an accident, but rather is necessary to preserve the gauge-invariance of the theory. To understand this, let us study the matrix element for the photon scattering process $\gamma(k_1) + A \to \gamma(k_2) + B$, with A and B arbitrary states of the theory. Applying the LSZ reduction formula[38] to the initial photon, we find that this matrix element is proportional to $\langle B\gamma(k_2)|j_\mu(0)|A\rangle$. Applying the reduction formula a second time, to bring in the final photon, gives

$$
\begin{aligned}
&\langle B\gamma(k_2)|j_\mu(0)|A\rangle = \varepsilon^*_\lambda M^\lambda_\mu, \\
&M^\lambda_\mu = \frac{-i}{\sqrt{2k_{20}}\sqrt{Z_3}} \int d^4x\, e^{ik_2\cdot x}\, \Box_x^2 \langle B|\, T(A^\lambda(x) j_\mu(0))\,|A\rangle.
\end{aligned}
\tag{189}
$$

As usual, gauge invariance requires that $\varepsilon^*_\lambda M^\lambda$ be invariant under the gauge transformation $\varepsilon_\lambda \to \varepsilon_\lambda + v k_{2\lambda}$, which implies that

$$k_{2\lambda} M^{\lambda}_{\mu} = 0 \ . \tag{190}$$

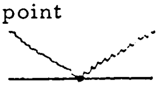
Let us now rewrite Eq. (189) by bringing the operator $\Box^2_x$ inside the time-ordered product, so that we can use the equation of motion $\Box^2_x A^{\lambda} = e_0 j^{\lambda}$. Keeping all equal-time commutators which arise from time derivatives of the time ordered product, we get

$$\begin{aligned} M^{\lambda}_{\mu} = \frac{-i}{\sqrt{2k_{20}}\sqrt{Z_3}} \int d^4x\, e^{ik_2 \cdot x} \{ & e_0 \langle B| T(j^{\lambda}(x) j_{\mu}(0)) |A\rangle \\ & + \delta(x_0) \langle B| [\partial A^{\lambda}(x)/\partial x_0, j_{\mu}(0)] |A\rangle \\ & + (\partial/\partial x_0)[\delta(x_0) \langle B| [A^{\lambda}(x), j_{\mu}(0)] |A\rangle] \}. \end{aligned} \tag{191}$$

Even with our $\underset{\sim}{\varepsilon}$-separation present, the third term on the right-hand side of Eq. (191) vanishes. The second term vanishes when either $\mu = 0$ or $\lambda = 0$, but when μ and λ are both spatial components it is just the anomalous potential-current commutator of Eq. (188),

$$\delta(x_0)[\partial A^{\lambda}(x)/\partial x_0, j_{\mu}(0)] = -e_0(g^{\lambda}_{\mu} - g^{\lambda 0} g_{\mu 0}) \delta^4(x) \lim_{\underset{\sim}{\varepsilon} \to 0} n/(3\underset{\sim}{\varepsilon}^2). \tag{192}$$

Eq. (192), which is colloquially called the "seagull" term, describes the coupling of two photons to an electron line at the same space-time point



which results from the potential-dependence of the electromagnetic current.

To see that the seagull term plays an essential role in maintaining gauge invariance, let us multiply Eq. (191) by $k_{2\lambda}$. By an integration by parts and by use of Eq. (3) (current-conservation), the contribution of the first term on the right-hand side of Eq. (191) becomes

$$\frac{e_0}{\sqrt{2k_{20}}\sqrt{Z_3}} \int d^4x \, e^{ik_2\cdot x} \frac{\partial}{\partial x^\lambda} \langle B| T(j^\lambda(x) j_\mu(0)) |A\rangle$$

$$= \frac{e_0}{\sqrt{2k_{20}}\sqrt{Z_3}} \int d^4x \, e^{ik_2\cdot x} \langle B|\delta(x_0)[\, j^0(x), j_\mu(0)]\, |A\rangle. \tag{193}$$

This expression is just the Schwinger term, and on substitution of Eq. (187) becomes

$$\frac{-ik_{2\lambda}e_0(g^\lambda_\mu - g^{\lambda 0} g_{\mu 0})}{\sqrt{2k_{20}}\sqrt{Z_3}} \langle B|A\rangle \lim_{\varepsilon \to 0} n/(3\varepsilon^2). \tag{194}$$

Thus, because of the presence of the Schwinger term, the divergence of the T-product term in Eq. (191) is not zero. But, combining Eqs. (191) and (192), we see that the divergence of the seagull term is

$$\frac{ik_{2\lambda}e_0(g^\lambda_\mu - g^{\lambda 0} g_{\mu 0})}{\sqrt{2k_{20}}\sqrt{Z_3}} \langle B|A\rangle \lim_{\varepsilon \to 0} n/(3\varepsilon^2), \tag{195}$$

which just cancels away the Schwinger term contributions and gives Eq. (190) for the total matrix element. So we see that gauge invariance is maintained by a cancellation between the divergence of the seagull term and the Schwinger term.

Equation (191) is frequently rewritten in the form

$$M_\mu^\lambda = \frac{-i}{\sqrt{2k_{20}}\sqrt{Z_3}} \int d^4x \, e^{ik_2 \cdot x} \langle B | T^*(j^\lambda(x) j_\mu(0)) | A \rangle, \quad (196)$$

with the T*-product defined as the sum of the T-product and the seagull term,

$$T^*(j^\lambda(x) j_\mu(0)) \equiv T(j^\lambda(x)\, j_\mu(0)) \qquad (197)$$
$$+ e_0(g_\mu^\lambda - g^{\lambda 0} g_{\mu 0}) \delta^4(x) \lim_{\underset{\sim}{\varepsilon} \to 0} \eta/(3\underset{\sim}{\varepsilon}^2).$$

As we have seen, because of the Schwinger term -seagull cancellation, the T* product satisfies the simple current conservation equation

$$\frac{\partial}{\partial x^\lambda} T^*(j^\lambda(x)\, j_\mu(0)) = 0 \,. \qquad (198)$$

Also, the T*-product transforms as a 2-index Lorentz tensor. (It is covariant because its matrix elements are the covariant Feynman amplitudes.) On the other hand, since the seagull term is not Lorentz covariant, it is clear that the T-product is not Lorentz covariant either. In other words, the properties which are naively attributed to the T-product are actually satisfied by the T*-product, and not by the T-product, when Schwinger terms and seagulls are present.

6.2 The Bjorken-Johnson-Low Method

Although we have found the $\underset{\sim}{\varepsilon}$-separation method to be useful in the above discussion, the method leads to inconsistencies when applied to more general types of commutator anomalies.[39] That this is so should not be too surprising since the averaging procedure of Eq. (184), which excludes timelike separations, is clearly noncovariant. If we include timelike separations to try to get a covariant definition of the current, we are no longer allowed to use canonical equal time commutation relations to evaluate the current-current commutators, but instead must use dynamics (the equations of motion) to follow the time evolution of the fields. Once we have to do this, however, we might just as well abandon our canonical procedure entirely, and instead calculate equal-time commutators as the limit as $t \to 0$ of unequal time commutators, with the latter calculated directly from Feynman diagrams. The Bjorken-Johnson-Low limit gives us a simple way of doing this.

We consider the T-product of two operators $J_{(1)}$ and $J_{(2)}$ and take the matrix element between arbitrary states A and B,

$$< B \,|\int d^4x \; e^{iq\cdot x} T(J_{(1)}(x)\, J_{(2)}(0))\,| A>. \qquad (199)$$

This matrix element has an analytic continuation into the upper half q_0 plane given by the retarded commutator

$$<B|\int d^4x\, e^{iq\cdot x}\theta(x_0)[J_{(1)}(x), J_{(2)}(0)]|A> \quad (200)$$
$$= \int_0^\infty dt\, e^{iq_0 t}\phi(t),$$
$$\phi(t) \equiv <B|\int d^3x\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}[J_{(1)}(\underset{\sim}{x}, t), J_{(2)}(0)]|A>.$$

Let us now let q_0 approach infinity in the upper half plane, that is, we set $q_0 = iR$, $R \to \infty$. To find the behavior of Eq. (200), we Taylor expand $\phi(t)$ about $t = 0$,

$$\phi(t) = \phi(t)|_{t\to 0} + t\,\phi'(t)|_{t\to 0} + \ldots, \qquad (201)$$

which on substitution into Eq. (200) gives

$$\int_0^\infty dt\, e^{-Rt}\phi(t) = \frac{1}{R}\phi(t)|_{t\to 0} + \frac{1}{R^2}\phi'(t)|_{t\to 0} + \cdots \qquad (202)$$

Thus we have learned that the matrix elements of the equal time commutators $[J_{(1)}, J_{(2)}]$, $[\partial J_{(1)}/\partial t, J_{(2)}]$, ... are just the coefficients of q_0^{-1}, q_0^{-2}, ... as we take $q_0 \to i\infty$ in Eq. (199):

$$\lim_{q_0\to i\infty} <B|\int d^4x\, e^{iq\cdot x}T(J_{(1)}(x)\, J_{(2)}(0))|A> \qquad (203)$$
$$= (-iq_0)^{-1}<B|\int d^3x\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}[J_{(1)}(\underset{\sim}{x}, t), J_{(2)}(0)]|_{t\to 0}|A>$$
$$+ (-iq_0)^{-2}<B|\int d^3x\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}[\partial J_{(1)}(\underset{\sim}{x}, t)/\partial t, J_{(2)}(0)]|_{t\to 0}|A> + \ldots$$

This formula is the recipe of Bjorken and of Johnson and Low. Clearly, the series in Eq. (203) cannot be extended arbitrarily far; a necessary condition for it to be valid out to power q_0^{-n} is that the Taylor coefficients $\phi^j(t)|_{t\to 0}$, $0 \le j \le n-1$ must exist. Although Eq. (203) has been formulated in terms of the T-product, it can be applied to the T*-

product as well. To see this, we note that the x-dependence of the seagull term consists of $\delta^4(x)$ and possibly, a finite number of derivatives of $\delta^4(x)$, and therefore the Fourier transform $\int d^4x\, e^{iq\cdot x}$(seagull) is purely a polynomial in q_0. Hence from a Feynman amplitude, or T*-product, we obtain a T-product to which Eq. (203) can be applied by dropping polynomial terms in q_0 which do not vanish as $q_0 \to i\infty$.

The equal-time commutator defined by Eq. (203) has the nice property that, barring pathological oscillatory behavior,[40] it agrees with the usual definition of the commutator as a sum over intermediate states. To show this, we write the Low equation for the left-hand side of Eq. (203),

$$
\begin{aligned}
&\langle B| \int d^4x\, e^{iq\cdot x} T(J_{(1)}(x)\, J_{(2)}(0))|A\rangle \\
&= i\int_{-\infty}^{\infty} dq_0' \,\frac{\rho_{AB}(\underset{\sim}{q}, q_0')}{q_0 - q_0'} \quad ,
\end{aligned}
\tag{204}
$$

with ρ the spectral function

$$
\begin{aligned}
\rho_{AB}(\underset{\sim}{q}, q_0) &= (2\pi)^3 \sum_n \langle B|J_{(1)}|n\rangle\langle n|J_{(2)}|A)\, \delta^4(q+p_B-p_n) \\
&-(2\pi)^3 \sum_n \langle B|J_{(2)}|n\rangle\langle n|J_{(1)}|A\rangle\, \delta^4(q+p_n-p_A).
\end{aligned}
\tag{205}
$$

Provided that the spectral function does not oscillate an infinite number of times [and it cannot in perturbation theory, where we will be applying Eq. (203)] , when the coefficient of q_0^{-1} exists it is equal to the integral

$$\int_{-\infty}^{\infty} dq_0' \, \rho_{AB}(\underset{\sim}{q}, q_0'), \tag{206}$$

which is just the usual sum-over-intermediate-states definition of the commutator.

In conclusion, we note that the results of Eqs. (187) and (188) for the Schwinger term and seagull in QED, which were obtained above by the $\underset{\sim}{\varepsilon}$-separation method, can equally well[33] be obtained by applying the Bjorken-limit technique to the vacuum polarization tensor $\Pi(q)^{\mu\nu}$. Nothing will be lost, then, by abandoning the $\underset{\sim}{\varepsilon}$-separation method of determining commutators in favor of the recipe of Eq. (203).

6.3 Anomalous Commutators Associated with the VVA Triangle Anomaly[41]

Let us now apply what we have learned to the lowest order VVA triangle diagram. We start with the two photon to vacuum matrix element of the axial-vector current, $\langle 0 | j_\mu^5(0) | \gamma(k_1, \varepsilon_1)\gamma(k_2, \varepsilon_2)\rangle$, and apply the reduction formula once to pull in one of the initial photons. This gives

$$\begin{aligned} &\langle 0 | j_\mu^5(0) | \gamma(k_1, \varepsilon_1)\gamma(k_2, \varepsilon_2)\rangle [4k_{10}k_{20}]^{\frac{1}{2}} \\ &= -i\varepsilon_1^\sigma \int d^4x \, e^{-ik_1\cdot x} \Box_x^2 \langle 0 | T(j_\mu^5(0) A_\sigma(x)) | \gamma(k_2, \varepsilon_2)\rangle [2k_{20}]^{\frac{1}{2}} \\ &= -i \, \varepsilon_1^\sigma \varepsilon_2^\rho [e_0^2/(2\pi)^4] \, R_{\sigma\rho\mu}(k_1, k_2), \end{aligned} \tag{207}$$

with $R_{\sigma\rho\mu}$ the explicit expression for the lowest order triangle diagram given in Eqs. (55)-(59). [Since, in Eqs. (207)-

(214), we work to lowest order only, we omit the usual wave-function renormalization factor $Z_3^{\frac{1}{2}}$ from Eq. (207).] Bringing $\Box_x^2$ inside the time-ordered product, as in Eq. (191), we get

$$\int d^4x\, e^{-ik_1\cdot x}\Box_x^2 <0|\, T(j_\mu^5(0)A_\sigma(x))\,|\gamma(k_2,\varepsilon_2)>$$
$$= A_{\mu\sigma}k_{10} + B_{\mu\sigma} + C_{\mu\sigma}(k_{10}), \qquad (208)$$

with $A_{\mu\sigma}$ and $B_{\mu\sigma}$ the seagull terms

$$A_{\mu\sigma} = i\int d^4x\, e^{i\underset{\sim}{k_1}\cdot\underset{\sim}{x}}\delta(x_0)<0|\,[\,A_\sigma(x), j_\mu^5(0)]\,|\gamma(k_2,\varepsilon_2)>,$$
$$B_{\mu\sigma} = \int d^4x\, e^{i\underset{\sim}{k_1}\cdot\underset{\sim}{x}}\delta(x_0)<0|\,[\,\dot{A}_\sigma, j_\mu^5(0)]\,|\gamma(k_2,\varepsilon_2)>, \qquad (209)$$
$$\dot{A}_\sigma(x) \equiv \partial A_\sigma(x)/\partial x_0,$$

and with $C_{\mu\sigma}(k_{10})$ the T-product

$$C_{\mu\sigma}(k_{10}) = e_0\int d^4x\, e^{-ik_1\cdot x}<0|\, T(j_\mu^5(0)\, j_\sigma(x))\,|\gamma(k_2,\varepsilon_2)> \qquad (210)$$
$$= e_0\int d^4x\, e^{i(k_1+k_2)\cdot x}<0|\, T(j_\mu^5(x)j_\sigma(0))\,|\gamma(k_2,\varepsilon_2)>.$$

Let us first show that the assumption that no commutator anomalies are present leads to a contradiction. If all commutators are given by their naive values, $A_{\mu\sigma}$ and $B_{\mu\sigma}$ vanish and no Schwinger term appears when we take the axial-index divergence of $C_{\mu\sigma}$, so that

$$< 0|\partial^\mu j_\mu^5\,|\gamma(k_1,\varepsilon_1)\gamma(k_2,\varepsilon_2)>[\,2k_{10}]^{\frac{1}{2}} \qquad (\mathrm{A})$$
$$= -i(k_1+k_2)^\mu < 0|\, j_\mu^5\,|\gamma(k_1,\varepsilon_1)\gamma(k_2,\varepsilon_2)>[\,2k_{10}]^{\frac{1}{2}} \qquad (211)$$
$$= -i\,\varepsilon_1^\sigma\,[\,-i(k_1+k_2)^\mu\; C_{\mu\sigma}(k_{10})]$$
$$= -i\,\varepsilon_1^\sigma\; e_0\int d^4x\, e^{i(k_1+k_2)\cdot x}<0|\, T(\partial^\mu j_\mu^5(x)j_\sigma(0))\,|\gamma(k_2,\varepsilon_2)> \qquad (\mathrm{B})$$

Substituting Eq. (68) into Eq. (211), we get from term (A) the matrix element of $2im_0 j^5$, plus a contribution from the anomaly of order e_0^2. On the other hand, from the term (B) in Eq. (211) we get the reduction formula for the matrix element of $2im_0 j^5$, plus a contribution from the anomaly which is of order e_0^3 at least, but the order e_0^2 contribution of the anomaly is missing. Thus, our assumption that all commutators have their naive values is in contradiction with the anomalous axial-vector divergence equation of Eq. (68).

To determine the required values of the anomalous commutators, we use the $k_{10} \to i\infty$ limit discussed above. The seagull terms in Eq. (208) have polynomial dependence on k_{10}, while Eq. (203) tells us that the large k_{10} limit of $C_{\mu\sigma}$ is

$$C_{\mu\sigma}(k_{10}) = \frac{-ie_0}{k_{10}} \int d^4x \, e^{i\underset{\sim}{k}_1 \cdot \underset{\sim}{x}} \delta(x_0) \langle 0 | [j_\sigma(\underset{\sim}{x}, 0), j_\mu^5(0)] | \gamma(k_2, \varepsilon_2) \rangle$$
$$+ \text{ higher order.} \qquad (212)$$

Thus, the equal-time commutators $[A_\sigma(x), j_\mu^5(0)]$, $[\dot{A}_\sigma(x), j_\mu^5(0)]$ and $[j_\sigma(x), j_\mu^5(0)]$ are to be identified, respectively, with the parts of $R_{\sigma\rho\mu}$ behaving like k_{10}, 1 and k_{10}^{-1} as k_{10} becomes infinite. From Eqs. (55)-(59), we find

$$\begin{aligned}
&\epsilon_2^{\ \rho} R_{\sigma\rho\mu}(k_1, k_2) \qquad (213)\\
&= 4\pi^2 (k_2^\tau \epsilon_2^{\ \rho} - k_2^\rho \epsilon_2^{\ \tau})\{\epsilon_{\tau\sigma\rho\mu} + g_{\sigma 0}\epsilon_{0\tau\rho\mu} - g_{\rho 0}\epsilon_{0\tau\sigma\mu}\\
&\quad + k_{10}^{-1}[\tfrac{1}{2}(1-g_{\sigma 0})(k_{2\sigma}\epsilon_{0\tau\rho\mu} + k_{20}\epsilon_{\sigma\tau\rho\mu})\\
&\quad + g_{\sigma 0}(1-g_{\eta 0})k_1^\eta \epsilon_{\eta\tau\rho\mu} - g_{\rho 0}(1-g_{\eta 0})k_1^\eta \epsilon_{\eta\tau\sigma\mu}\\
&\quad + (\text{terms which vanish when } \sigma=0 \text{ or}\\
&\qquad \mu = 0)]\} + O(k_{10}^{-2} \ln k_{10})
\end{aligned}$$

Comparing Eq. (213) with Eqs. (209) and (212), we find the equal-time commutation relations

$$\begin{aligned}
[A_\sigma(x), j_\mu^5(y)] &= [\dot{A}_0(x), j_\mu^5(y)] = 0, \qquad (214)\\
[\dot{A}_r(x), j_0^5(y)] &= (-2i\alpha_0/\pi)\delta^3(\underset{\sim}{x}-\underset{\sim}{y})B^r(y),\\
[\dot{A}_r(x), j_s^5(y)] &= (i\alpha_0/\pi)\delta^3(\underset{\sim}{x}-\underset{\sim}{y})\epsilon^{rst}E^t(y),\\
[j_0(x), j_0^5(y)] &= (-ie_0/2\pi^2)\underset{\sim}{B}(y)\cdot \underset{\sim}{\nabla}_{\underset{\sim}{x}}\delta^3(\underset{\sim}{x}-\underset{\sim}{y}),\\
[j_r(x), j_0^5(y)] &= (-ie_0/4\pi^2)[\underset{\sim}{E}(x)\times \underset{\sim}{\nabla}_{\underset{\sim}{y}}\delta^3(\underset{\sim}{x}-\underset{\sim}{y})]^r,\\
[j_0(x), j_s^5(y)] &= (ie_0/4\pi^2)[\underset{\sim}{E}(y)\times \underset{\sim}{\nabla}_{\underset{\sim}{x}}\delta^3(\underset{\sim}{x}-\underset{\sim}{y})]^s,
\end{aligned}$$

with

$$\begin{aligned}
B^t(x) &= [\underset{\sim}{\nabla}\times \underset{\sim}{A}(x)]^t = \epsilon^{rst}\frac{\partial}{\partial x^r}A^s(x),\\
E^t(x) &= -\dot{A}^t(x) - \frac{\partial}{\partial x^t}A^0(x), \qquad (215)\\
\epsilon^{123} &= 1.
\end{aligned}$$

We have only listed the current-current commutators containing at least one time component, since these are the only ones which appear when divergences with respect to the vector or axial-vector indices (σ or μ) are brought inside the time-ordered product in Eq. (210). All of the nonvanishing commutators in Eq. (214) are anomalous in the sense that if they are calculated by naive use of canonical commutation relations they vanish.

It is easy to check that the anomalous commutation relations of Eq. (214), together with the reduction formula of Eqs. (208)-(210), correctly reproduce the known divergence properties of the lowest-order triangle diagram. To check gauge invariance for the photon which has been reduced in, we multiply Eq. (208) by k_1^σ and use vector current conservation, giving

$$\begin{aligned} &k_1^\sigma \int d^4x\, e^{-ik_1\cdot x} \Box_x^2 <0|\, T(j_\mu^5(0)\; A_\sigma(x))\, |\gamma(k_2, \epsilon_2)> \\ &= k_1^\sigma \int d^4x\, e^{i\underset{\sim}{k_1}\cdot \underset{\sim}{x}} \delta(x_0)\; <0|\, [\dot{A}_\sigma(x), j_\mu^5(0)]\; |\gamma(k_2, \epsilon_2)> \qquad (216) \\ &\quad -ie_0 \int d^4x\, e^{i\underset{\sim}{k_1}\cdot \underset{\sim}{x}} \delta(x_0)\; <0|\, [\, j_0(x), j_\mu^5(0)]\; |\gamma(k_2, \epsilon_2)> . \end{aligned}$$

Substituting the commutators of Eq. (214), we easily see that the seagull and Schwinger terms on the right-hand side of Eq. (216) cancel, as expected. To check the axial-vector divergence of the triangle, we multiply by $-i(k_1+k_2)^\mu$, as we did in Eq. (211). When seagulls and Schwinger terms are kept, term (B) of Eq. (211) becomes

$$\begin{aligned} &-i\epsilon_1^\sigma \{ e_0 \int d^4x e^{i(k_1+k_2)\cdot x} <0|\, T(\partial^\mu j_\mu^5(x)\; j_\sigma(0))\, |\gamma(k_2, \epsilon_2)> \\ &+i \int d^4x\, e^{i\underset{\sim}{k_1}\cdot \underset{\sim}{x}} \delta(x_0) \Big[-(k_1+k_2)^\mu <0|\, [\dot{A}_\sigma(x), j_\mu^5(0)]\; |\gamma(k_2, \epsilon_2)> \\ &\quad +ie_0 <0|\, [\, j_\sigma(x), j_0^5(0)]\; |\gamma(k_2, \epsilon_2)> \Big] \} . \qquad (217) \end{aligned}$$

Using Eq.(214) to evaluate the seagull and Schwinger terms in the heavy square brackets, we find

$$\int d^4x\, e^{i\underset{\sim}{k_1}\cdot\underset{\sim}{x}}\, \delta(x_0)\Big[-(k_1+k_2)^{\mu} <0|[\dot{A}_\sigma(x), j^5_\mu(0)]\,|\gamma(k_2,\varepsilon_2)> + ie_0<0|[\,j_\sigma(x), j^5_0(0)]\,|\gamma(k_2,\varepsilon_2)>\Big][\,(2\pi)^3 2k_{20}]^{\frac{1}{2}}$$
$$= -\varepsilon_2^{\rho}(e_0^2/2\pi^2)\, k_1^{\xi} k_2^{\tau}\, \varepsilon_{\xi\sigma\tau\rho}, \qquad (218)$$

which is just the axial-divergence anomaly obtained by substituting Eq. (68) into term (A) of Eq. (211). We see that <u>the anomalous axial divergence arises from a failure of the Schwinger term</u> $[j_\sigma, j_0^5]$ <u>and the seagull</u> $[\,\dot{A}_\sigma, j^5_\mu\,]$ <u>to cancel.</u> As a point of consistency, we note that the pseudoscalar-two-photon triangle $R_{\sigma\rho}$[Eq. (60)] has the asymptotic behavior $R_{\sigma\rho}(k_1,k_2)\to 0$ as $k_{10}\to i\infty$. Thus the naive equal-time commutation relations

$$[\,A_\sigma(x),\ j^5(y)\,] = [\,\dot{A}_\sigma(x), j^5(y)\,] = 0 \qquad (219)$$

remain valid, and no extra seagull terms are picked up when the one photon reduction formula is applied to the matrix element $<0|\,2im_0 j^5\,|\gamma(k_1,\varepsilon_1)\gamma(k_2,\varepsilon_2)>$.

We proceed next to check whether the commutation relations of Eqs. (214) and (219) are formally consistent with each other, with the equations of motion, and with the electromagnetic-field canonical commutation relations of Eq. (5). We start with the equation of motion

$$\Box^2 A_\mu = \ddot{A}_\mu - \nabla^2 A_\mu = e_0 j_\mu , \tag{220}$$

and the divergence equations satisfied by the currents $j_\mu(\underset{\sim}{x}, t)$ and $j^5_\mu(\underset{\sim}{x}, t)$,

$$\frac{\partial}{\partial t} j_0 + \underset{\sim}{\nabla} \cdot \underset{\sim}{j} = 0, \tag{221}$$

$$\frac{\partial}{\partial t} j_0^5 + \underset{\sim}{\nabla} \cdot \underset{\sim}{j}^5 = 2im_0 j^5 + (2\alpha_0/\pi)\underset{\sim}{E} \cdot \underset{\sim}{B} .$$

We proceed to combine Eqs. (5), (220) and (221) with Eqs. (214) and (219). All the commutators which we write down are at equal time, with $x_0 = y_0 = t$.

(i) From $[A_\sigma(x), j_0^5(y)] = 0$, we deduce

$$[\dot{A}_\sigma(x), j_0^5(y)] + [A_\sigma(x), (\partial/\partial t) j_0^5(y)] = 0. \tag{222}$$

On substituting Eq. (221) for $(\partial/\partial t) j_0^5(y)$ and using $[A_\sigma(x), j^5(y)] = [A_\sigma(x), \underset{\sim}{j}^5(y)] = 0$, we find

$$[\dot{A}_\sigma(x), j_0^5(y)] = -[A_\sigma(x), 2\alpha_0/\pi)\underset{\sim}{E}(y) \cdot \underset{\sim}{B}(y)] . \tag{223}$$

Using the canonical commutation relations we then get

$$[\dot{A}_0(x), j_0^5(y)] = 0, \tag{224}$$

$$[\dot{A}_r(x), j_0^5(y)] = (-2i\alpha_0/\pi)\, \delta^3(\underset{\sim}{x} - \underset{\sim}{y}) B^r(y) ,$$

in agreement with Eq. (214).

(ii) From $[\dot{A}_0(x), j_0^5(y)] = 0$, we deduce

$$[\ddot{A}_0(x), j_0^5(y)] + [\dot{A}_0(x), (\partial/\partial t) j_0^5(y)] = 0. \tag{225}$$

Substituting Eq. (221) for $(\partial/\partial t) j_0^5(y)$ and Eq. (220) for $\ddot{A}_0(x)$, and using the commutators $[A_0(x), j_0^5(y)] = [\dot{A}_0(x), j^5(y)] = [\dot{A}_0(x), \underset{\sim}{j}^5(y)] = 0$, we find

$$[e_0 j_0(x), j_0^5(y)] = -[\dot{A}_0(x), (2\alpha_0/\pi)\mathbf{E}(y)\cdot\mathbf{B}(y)]$$
$$= (-2i\alpha_0/\pi)\mathbf{B}(y)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}), \tag{226}$$

that is,

$$[j_0(x), j_0^5(y)] = (-ie_0/2\pi^2)\mathbf{B}(y)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}), \tag{227}$$

in accord with Eq. (214).

(iii) In a similar manner, the relations obtained by time differentiation of $[\dot{A}_r(x), j_0^5(y)] = -(2i\alpha_0/\pi)\delta^3(\mathbf{x}-\mathbf{y})B^r(y)$ and $[j_0(x), j_0^5(y)] = -(ie_0/2\pi^2)\mathbf{B}(y)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y})$ are found to be consistent with Eqs. (214), (219), (220) and (221).

(iv) Finally, to check the consistency of quantization in the Feynman gauge, we must verify that

$$L \equiv \dot{A}_0 + \boldsymbol{\nabla}\cdot\mathbf{A} \tag{228}$$

and $\dot{L}$ remain dynamically independent of the axial-vector current. That is, we must verify that

$$[L(x), j_\mu^5(y)] = 0 \tag{229}$$

and that

$$[\dot{L}(x), j_\mu^5(y)] = 0. \tag{230}$$

Equation (229) follows immediately from the first line of Eq. (214). To check Eq. (230), we substitute Eq. (220) for $\ddot{A}_0$ and use $[A_0(x), j_\mu^5(y)] = 0$, giving

$$[\dot{L}(x), j_\mu^5(y)] = [e_0 j_0(x), j_\mu^5(y)] + [\boldsymbol{\nabla}_{\mathbf{x}}\cdot\dot{\mathbf{A}}(x), j_\mu^5(y)]. \tag{231}$$

Substituting commutators from Eq. (214) then shows that right-hand side of Eq. (231) vanishes.

We conclude that the commutation relations of Eq. (214), which were obtained from the triangle graph in lowest-order perturbation theory, are consistent with the equations of motion and canonical commutation relations. Moreover, the fact that Eq. (224) for $[\dot{A}_r(x), j_0^5(y)]$ and Eq. (227) for $[j_0(x), j_0^5(y)]$ were deduced from simpler, exact commutators and equations of motion suggests that Eqs.(224) and (227) are themselves exact to all orders of perturbation theory. The values given in Eq. (214) for $[\dot{A}_r(x), j_s^5(y)]$, $[j_r(x), j_0^5(y)]$ and $[j_0(x), j_s^5(y)]$ cannot, on the other hand, be deduced from the consistency argument. To see this, we note that the consistency checks of items (iii) and (iv) above are unchanged if we modify these commutators to read

$$
\begin{aligned}
[\dot{A}_r(x), j_s^5(y)] &= \frac{i\alpha_0}{\pi}\delta^3(\underset{\sim}{x}-\underset{\sim}{y})\epsilon^{rst}E^t(y) - ie_0\delta^3(\underset{\sim}{x}-\underset{\sim}{y})S^{rs}(y),\\
[j_r(x), j_0^5(y)] &= \frac{-ie_0}{4\pi^2}[\underset{\sim}{E}(x)\times\underset{\sim}{\nabla}_y\delta^3(\underset{\sim}{x}-\underset{\sim}{y})]^r + i\frac{\partial}{\partial y^s}[\delta^3(\underset{\sim}{x}-\underset{\sim}{y})S^{rs}(y)], \quad (232)\\
[j_0(x), j_s^5(y)] &= \frac{ie_0}{4\pi^2}[\underset{\sim}{E}(y)\times\underset{\sim}{\nabla}_x\delta^3(\underset{\sim}{x}-\underset{\sim}{y})]^s - i\frac{\partial}{\partial x^r}[\delta^3(\underset{\sim}{x}-\underset{\sim}{y})S^{rs}(y)],
\end{aligned}
$$

with $S^{rs}(y)$ a pseudotensor operator. In other words, the consistency check does not rule out the possibility that higher orders of perturbation theory may modify Eq. (214) by adding Schwinger terms and seagulls of the usual type, which cancel against each other when vector or axial-vector divergences are taken. It is expected[42] on general grounds that the seagull commutator $[\dot{A}_r(x), j_s^5(y)]$ does not involve derivatives of the δ function and the Schwinger term commutators $[j_r(x), j_0^5(y)]$ and $[j_0(x), j_s^5(y)]$ do not involve derivatives of the delta function higher than the first. [This structure for seagulls and Schwinger terms was found in the pure QED example discussed above in Subsection 6.1.] Under this assumption, Eq. (232) represents the most general form for these commutators consistent with Eqs. (220) and (221).

Using Eq. (224), we can easily complete the argument, sketched in Subsection 3.3, that the operator

$$\bar{Q}^5 = \int d^3x [\, j_0^5(x) + (\alpha_0/\pi) \underset{\sim}{A}(x) \cdot \underset{\sim}{\nabla}_x \times \underset{\sim}{A}(x)] \tag{233}$$

is the conserved generator of the γ_5 transformations in massless electrodynamics. We have already shown that $\bar{Q}^5$ is conserved and that it satisfies the correct commutation relations with the fermion fields. We now show that $\bar{Q}^5$ commutes with the photon field variables. From the first line of Eq. (214) we find

$$[\bar{Q}^5, A_\sigma(y)] = [\bar{Q}^5, \dot{A}_0(y)] = 0, \tag{234a}$$

while from Eq. (224) we find

$$[\bar{Q}^5, \dot{A}_r(y)] = \left[\int d^3x\, j_0^5(x), \dot{A}_r(y)\right] + \left[\int d^3x (\frac{\alpha_0}{\pi}) \underset{\sim}{A}(x) \cdot \underset{\sim}{\nabla}_x \times \underset{\sim}{A}(x), \dot{A}_r(y)\right]$$

$$= \frac{2i\alpha_0}{\pi} B^r(y) - \frac{2i\alpha_0}{\pi} B^r(y) = 0, \tag{234b}$$

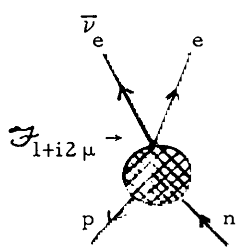
as required.

7. APPLICATIONS OF THE BJORKEN LIMIT

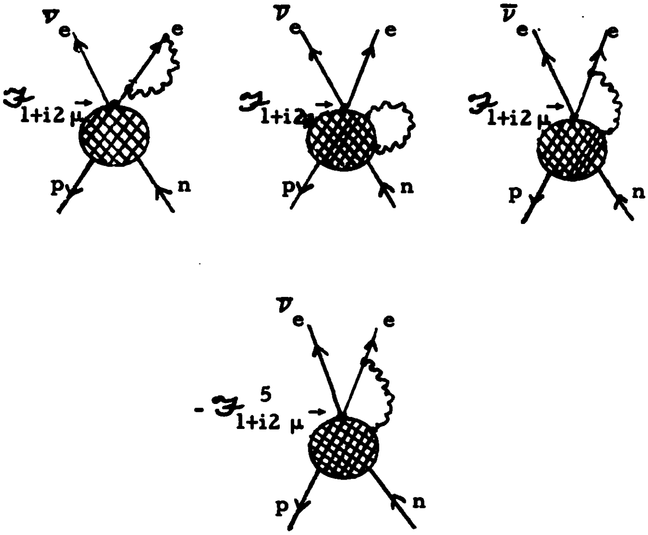
The Bjorken limit formula of Eq. (203) has been extensively applied over the past several years to the study of radiative corrections to the hadronic β decay[43] and to the derivation of asymptotic sum rules[44] and asymptotic cross section relations[45] for high energy inelastic electron and neutrino scattering. In all of these applications, it is _assumed_ that the equal-time commutators appearing on the right-hand side of Eq. (203) are the same as the "naive" commutators obtained by straightforward use of canonical commutation relations and equations of motion. As we have seen in the previous section, in the case of vacuum polarization and triangle diagrams in QED, this assumption is not borne out, and we find cases in which the Bjorken-limit and the naive commutator do not agree. Because of special features of the diagrams which lead to these counter-examples, they do not directly invalidate the applications of Eq. (203) mentioned above. However, when detailed perturbation theory calculations are made on a wider class of diagrams,[46] one does find anomalous commutators which invalidate all of the above-mentioned applications. In the present section, we will briefly derive various consequences of the Bjorken-limit formula in the case when anomalous behavior is neglected. Then, in Section 8, we will discuss the changes that result from the presence of perturbation theory anomalies.

7.1 Radiative Corrections to Hadronic β Decay

We begin by considering the theory of second order radiative corrections to hadronic β decay in the local current-current theory of weak interactions. For definiteness, we will discuss only the vector amplitude for the specific process of neutron decay, $n \to p + e^- + \bar{\nu}_e$. The lowest order matrix element M for this process is represented by the diagram



with $\mathcal{J}_{1+i2\mu}$ the component of the hadronic current to which the leptons couple. The radiative corrections to this process come from the following four diagrams (As before, the wavy line denotes a virtual photon)



Although the fourth diagram involves the axial-vector current $\mathcal{J}^{5}_{1+i2\,\mu}$, it has a piece which contains the pseudotensor $\epsilon_{\mu\nu\lambda\sigma}$ and therefore transforms as a vector coupling and contributes to the radiative-corrected vector amplitude.

The three radiative correction diagrams on the first line can be analyzed by the standard methods of <u>time component</u> current algebra, without the use of Bjorken limits. In the approximation of zero momentum transfer to the leptons (an eminently reasonable approximation, since the ratio of the leptonic momentum transfer to the nucleon mass is of the same order as higher order electromagnetic corrections) one finds[47] the remarkable result that these three diagrams sum to a universal, structure-independent, divergent correction to the vector amplitude,

$$\delta M^{\text{first line}} = \frac{3\alpha}{8\pi}(\ell n\, \Lambda^2) M,$$
$$M = (\tilde{G}/\sqrt{2})\, \bar{u}_e \gamma^\mu (1-\gamma_5) v_\nu < p | \mathcal{F}_{1+i2\;\mu}(0) | n > . \tag{235}$$

Here $\tilde{G} = G \cos\theta_C$ is the effective Fermi constant, and Λ is the mass of a regulator photon which has been introduced to permit evaluation of the integrals. The fourth diagram cannot be treated by using solely the techniques of time-component current algebra, but it can be evaluated by use of the Bjorken limit. The contribution of this diagram may be written as

$$\delta M^{\text{second line}} = \frac{ie^2}{(2\pi)^4}\int \frac{d^4k}{k^2}\frac{\tilde{G}}{\sqrt{2}}\, \bar{u}_e \gamma^\sigma \frac{1}{\not{\ell} - \not{k} - m_e} \gamma^\mu (1-\gamma_5) v_\nu$$
$$\times [\, T_{\sigma\mu}(k) + \text{seagull}] \tag{236}$$

with

$$T_{\sigma\mu}(k) = -i\int d^4x\, e^{ik\cdot x} < p | T(j_\sigma^{EM}(x)\, \mathcal{F}^5_{1+i2\mu}(0)) | n> \tag{237}$$

and with ℓ the electron four-momentum, m_e the electron mass and j_σ^{EM} the hadronic electromagnetic current. Making our approximation of zero momentum transfer to the leptons, we drop ℓ and m_e. Since the factor multiplying the seagull is odd in k and since we expect the seagull to be proportional to $\int d^4x\, e^{ik\cdot x}\, \delta^4(x)$, which is k-independent, the seagull drops out and we get

$$\delta M^{\text{second line}} = \frac{-ie^2}{(2\pi)^4}\int \frac{d^4k}{k^2}\frac{\tilde{G}}{\sqrt{2}}\,\bar{u}_e\gamma^\sigma \frac{\not{k}}{k^2}\gamma^\mu(1-\gamma_5)v_\nu T_{\sigma\mu}(k). \quad (238)$$

In order to isolate the divergent part of Eq. (238), we need only calculate the large-k behavior of $T_{\sigma\mu}(k)$. According to the Bjorken limit formula of Eq. (203), the large-k_0 behavior of $T_{\sigma\mu}(k)$ is given by

$$T_{\sigma\mu}(k)_{k_0 \to i\infty} \to k_0^{-1}\int d^4x\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\delta(x_0) < p|[\, j_\sigma^{EM}(x), \mathcal{F}^5_{1+i2\mu}(0)]\,|n> \quad (239)$$

In order to evaluate Eq. (239), we will adopt a specific model of the strong interactions, in which the basic fields are a fermion SU_3-triplet $\psi \equiv (\psi_1, \psi_2, \psi_3)$ with electric charges (Q, Q-1, Q-1), bound by the exchange of SU_3-singlet vector, scalar and pseudoscalar bosons. The electromagnetic and weak axial-vector currents in this model are given by the simple expressions

$$j_\sigma^{EM}(x) = \bar{\psi}(x)\lambda_Q\gamma_\sigma\psi(x),\quad \mathcal{F}^5_{1+i2\mu}(x) = \bar{\psi}(x)\lambda_+\gamma_\mu\gamma_5\psi(x),$$

$$\lambda_Q = \begin{bmatrix} Q & 0 & 0 \\ 0 & Q-1 & 0 \\ 0 & 0 & Q-1 \end{bmatrix},\quad \lambda_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (240)$$

Let us further make the assumption that the equal-time commutator appearing in Eq. (239) is the same as the naive canonical equal time commutator, giving

$$\delta(x_0)[j_\sigma^{EM}(x), \mathcal{F}^5_{1+i2\mu}(0)] = \delta^4(x)\bar{\psi}(x)\gamma_0[\lambda_Q\gamma_0\gamma_\sigma, \lambda_+\gamma_0\gamma_\mu\gamma_5]\psi(x)$$

$$= \delta^4(x)\,\bar{\psi}(x)\gamma_0(\tfrac{1}{2}\{\lambda_Q,\lambda_+\}[\gamma_0\gamma_\sigma,\gamma_0\gamma_\mu\gamma_5] + \tfrac{1}{2}[\lambda_Q,\lambda_+]\{\gamma_0\gamma_\sigma,\gamma_0\gamma_\mu\gamma_5\})\psi(x) \quad . \tag{241}$$

The second term on the right-hand side of Eq. (241) is pure axial-vector in character, and so can be dropped. Using the elementary relations

$$\begin{aligned} &\tfrac{1}{2}\{\lambda_Q,\lambda_+\} = \tfrac{1}{2}(Q+Q-1)\,\lambda_+ = Q_{AV}\,\lambda_+ \,, \\ &\gamma_\sigma\gamma_0\gamma_\mu - \gamma_\mu\gamma_0\gamma_\sigma = -2i\,\epsilon_{\sigma 0\mu\eta}\,\gamma^\eta\gamma_5, \end{aligned} \tag{242}$$

the first term can be rewritten as

$$-2i\delta^4(x)\,Q_{AV}\,\epsilon_{\sigma 0\mu\eta}\,\mathcal{F}^{\eta}_{1+i2}(x), \tag{243a}$$

which, on substitution into Eq. (239) gives

$$T_{\sigma\mu}(k)_{\overrightarrow{k_0\to i\infty}}\ \frac{-2iQ_{AV}}{k_0}\,\epsilon_{\sigma 0\mu\eta}\,\langle p|\,\mathcal{F}^{\eta}_{1+i2}\,|n\rangle. \tag{243b}$$

Let us now define[48] the auxiliary tensor $t_{\sigma\mu}(k)$ by

$$t_{\sigma\mu}(k) \equiv \frac{-2iQ_{AV}}{k^2}\,\epsilon_{\sigma\lambda\mu\eta}\,k^\lambda\langle p|\,\mathcal{F}^{\eta}_{1+i2}\,|n\rangle, \tag{244}$$

so that, by construction, $\Delta_{\sigma\mu}(k) \equiv T_{\sigma\mu}(k) - t_{\sigma\mu}(k)$ approaches zero faster than k^{-1} as $k_0 \to i\infty$. Arguments based on dispersion theory[49] can then be used to show that $\Delta_{\sigma\mu}(k)$ decreases faster than k^{-1} as k approaches infinity in an arbitrary direction, which means that to extract the ultraviolet divergent part of Eq. (238), we need only evaluate the k-integrals with $T_{\sigma\mu}(k)$ replaced by $t_{\sigma\mu}(k)$. This is a completely straightforward calculation, which gives

$$\delta M^{\text{second line}} = \frac{3\alpha}{8\pi}\,(\ln\Lambda^2)\,2Q_{AV}\,M. \tag{245}$$

Adding Eqs. (245) and (235), we get for the total second order radiative correction to the vector part

$$\delta M = \frac{3\alpha}{8\pi}\,(1 + 2Q_{AV})\,\ell n\,\Lambda^2 M\,. \qquad (246)$$

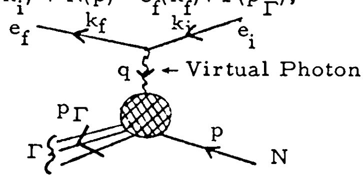
Thus, if $Q_{AV} = -\frac{1}{2}$, which corresponds to the choice of triplet charges (0, -1, -1), the radiative corrections are finite; for other choices of Q_{AV}, such as the value $+\frac{1}{2}$ favored by our analysis of $\pi^0 \to 2\gamma$ decay, the radiative corrections diverge. A similar calculation can be done for the axial-vector part of the amplitude and for more general β decay processes, with again the conclusion that the radiative corrections are finite only for $Q_{AV} = -\frac{1}{2}$. _If_ the analysis leading to Eq. (246) were correct, one could adopt one of two points of view (just as in our discussion of purely leptonic processes in Subsection 3.2): (i) The radiative corrections should be finite in the local current-current theory, requiring the choice $Q_{AV} = -\frac{1}{2}$; (ii) The radiative corrections need not be finite, since the divergence in Eq. (246) is a weak one which becomes a significant correction to the weak decay amplitude only for large values of Λ, where the local current-current theory must fail in any case. In actual fact, we will see that when interactions of the fermion triplet with the mesons are taken into account, the assumption of identity of the Bjorken-limit and naive commutators used to derive Eq. (246) breaks down. As a result, even the choice $Q_{AV} = -\frac{1}{2}$ leaves the radiative corrections infinite, and so the second point of view seems to be the more reasonable one.

We note, in conclusion, that the analysis given above also applies to the second order radiative corrections to μ-meson decay. Since the average charge of the μ-meson and the μ-neutrino is $-\frac{1}{2}$, Eq. (246) predicts that the radiative corrections in this case are finite, as is indeed found both by explicit calculation and by the Ward identity arguments of Subsection 3.1.

7.2 Asymptotic Sum Rules and Asymptotic Cross Section Relations

We consider next the use of the Bjorken limit formula to derive asymptotic sum rules and asymptotic cross section relations for lepton-nucleon scattering. In inelastic electron-nucleon scattering, one observes the reaction

$$e_i(k_i) + N(p) \rightarrow e_f(k_f) + \Gamma(p_\Gamma), \tag{247}$$



with e_i and e_f the initial and final electrons, with N the nucleon (typically, a free proton or a neutron bound in a deuteron) and with Γ any inelastic hadron final state. Four-momenta of the particles are indicated in parentheses. In the experiments done at SLAC and other laboratories,[50] one measures the incident and final electron energies E_i and E_f and the laboratory scattering angle θ between the electron directions, but obtains no detailed information about the conposition of the state Γ. Thus, the experimentally measured differential cross section is $d^2\sigma/d\Omega_f\, dE_f$, with Ω_f the final electron solid angle. Since the matrix element for Eq. (247) is proportional to $<\Gamma(p_\Gamma)|j_\lambda^{EM}|N(p)>$, after squaring, averaging over nucleon spin and summing over final states Γ we are clearly measuring the quantity

$$\frac{1}{2}\sum_{\mathrm{spin}(N),\,\Gamma} <N(p)|j_\mu^{EM}|\Gamma(p_\Gamma)><\Gamma(p_\Gamma)|j_\lambda^{EM}|N(p)>(2\pi)^3\delta^4(p_\Gamma-p-q), \quad (248)$$

$$q = k_i - k_f,$$

which is essentially the imaginary part of the amplitude for forward Compton scattering of virtual photons of $(\mathrm{mass})^2 = q^2$ on the target nucleon. Using Lorentz invariance and gauge invariance, Eq. (248) can be rewritten in the form

$$W_1(\nu, q^2)\left(-g_{\mu\lambda} + \frac{q_\mu q_\lambda}{q^2}\right) + M_N^{-2} W_2(\nu, q^2)\left(p_\mu - \frac{\nu}{q^2}q_\mu\right)\left(p_\lambda - \frac{\nu}{q^2}q_\lambda\right), \quad (249)$$

$$\nu = q\cdot p, \quad M_N = \text{nucleon mass},$$

and in terms of W_1 and W_2 the experimentally measured differential cross section is

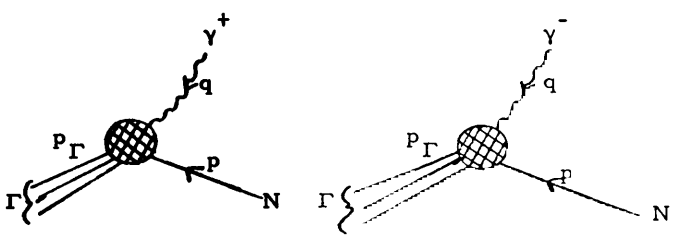
$$\frac{d^2\sigma}{d\Omega_f dE_f} = \frac{\alpha}{\pi}\frac{E_f^2}{(q^2)^2}[2W_1(\nu, q^2)\sin^2(\frac{\theta}{2})+W_2(\nu,q^2)\cos^2(\frac{\theta}{2})] . \tag{250}$$

Roughly speaking, at small scattering angles one measures W_2 and at large scattering angles one measures W_1. In the neutrino scattering reaction

$$\nu_\mu(k_i) + N(p) \rightarrow \mu(k_f) + \Gamma(p_\Gamma), \tag{251}$$

the doubly differential cross section is given by a formula similar in form to Eq. (250), but containing a third term, proportional to $(E_i+E_f)\sin^2(\frac{\theta}{2})$, arising from vector-axial-vector interference.[51]

Rather than considering the physically realistic electron-nucleon and neutrino nucleon processes themselves, we will illustrate the application of Bjorken-limit techniques to these reactions by studying the cross sections for the absorption of fictitious charged, isovector virtual photons by nucleons,



These cross sections differ only in isospin structure from the corresponding virtual photon cross sections appearing in the electron scattering case, and differ from the cross sections appearing in νN and $\bar{\nu}$N scattering in that the axial-vector current terms have been omitted. The results which we will obtain for the simpler, fictitious reactions are readily extended to the realistic cases.

To proceed, we first develop some properties of the nucleon-spin averaged amplitude for the forward scattering of isovector photons, from initial isotopic state b to final isotopic state a, on a nucleon target. This is given by

$$\begin{aligned}(M_N/p_0)\, T^*_{a\mu\, b\lambda}(p, q) &= \tfrac{1}{2}\sum_{\mathrm{spin}(N)} -i\int d^4x\, e^{iq\cdot x} \\ &\quad \times \langle N(p)|\, T^*(\mathcal{F}_{a\mu}(x)\mathcal{F}_{b\lambda}(0))\,|N(p)\rangle \qquad (252)\\ &= \text{seagull} + \tfrac{1}{2}\sum_{\mathrm{spin}(N)} -i\int d^4x\, e^{iq\cdot x}\langle N(p)|\, T(\mathcal{F}_{a\mu}(x)\mathcal{F}_{b\lambda}(0))\,|N(p)\rangle,\\ &\qquad a, b = 1, 2, 3,\end{aligned}$$

where, as before, the seagull term is a polynomial in q_0. Since we have seen that amplitudes involving only vector currents have normal Ward identities, by using both isospin current conservation and the ordinary time component current algebra[52] we find that the divergence of Eq. (252) is given by

$$\begin{aligned}q^\mu T^*_{a\mu\, b\lambda} &= (p_0/M_N)\,\tfrac{1}{2}\sum_{\mathrm{spin}(N)}\int d^4x\, e^{iq\cdot x}\\ &\quad \times \langle N(p)|\,\delta(x_0)[\mathcal{F}_{a0}(x), \mathcal{F}_{b\lambda}(0)]\,|N(p)\rangle \qquad (253)\\ &= (p_0/M_N)\,\tfrac{1}{2}\sum_{\mathrm{spin}(N)} \langle N(p)|\mathcal{F}_{[a,b]\lambda}|N(p)\rangle\\ &\doteq \tfrac{1}{2}\,\mathrm{tr}\left\{\left(\frac{\not{p}+M_N}{2M_N}\right)\gamma_\lambda\left[\tfrac{1}{2}\lambda a, \tfrac{1}{2}\lambda b\right]\right\}\end{aligned}$$

[For notational convenience, we have omitted the initial and final nucleon isospinors, and so Eqs. (252) and (253) are really matrix equations in isospin space.] If we examine the Born approximation to $T^{*}_{a\mu\, b\lambda}(p,q)$,

$$T^{*BORN}_{a\mu\, b\lambda}(p,q)=\tfrac{1}{2}\mathrm{tr}\{(\frac{\not p+M_N}{2M_N})(\gamma_\mu\tfrac{1}{2}\lambda_a\frac{1}{\not p+\not q-M_N}\gamma_\lambda\tfrac{1}{2}\lambda_b+\gamma_\lambda\tfrac{1}{2}\lambda_b \times\frac{1}{\not p-\not q-M_N}\gamma_\mu\tfrac{1}{2}\lambda_a)\} \tag{254}$$

we easily see that $q^\mu T^{*BORN}_{a\mu\, b\lambda}(p,q)$ agrees with Eq. (253), and that $q^\lambda T^{*BORN}_{a\mu\, b\lambda}(p,q)$ agrees with the λ-index analog of Eq. (253). This means that the non-Born part of Eq. (252) is divergenceless, and therefore Eq. (252) has the general structure

$$T^{*}_{a\mu\, b\lambda}(p,q)=T^{*\,BORN}_{a\mu\, b\lambda}(p,q)+T_{1\,ab}(q^2,\omega)(-g_{\mu\lambda}+\frac{q_\mu q_\lambda}{q^2}) + M_N^{-2}T_{2ab}(q^2,\omega)(p_\mu-\frac{\nu}{q^2}q_\mu)(p_\lambda-\frac{\nu}{q^2}q_\lambda). \tag{255}$$

In writing Eq. (255) we have eliminated ν in terms of the dimensionless variable

$$\omega=-q^2/\nu\,. \tag{256}$$

To proceed further, we separate off the isospin dependence of the non-Born amplitudes,

$$T_{1,2\,ab}(q^2,\omega)=T^{(+)}_{1,2}(q^2,\omega)\ \{\tfrac{1}{2}\lambda_a,\tfrac{1}{2}\lambda_b\}+T^{(-)}_{1,2}(q^2,\omega)\ [\tfrac{1}{2}\lambda_a,\tfrac{1}{2}\lambda_b]\,; \tag{257}$$

the crossing symmetry relation $T^{*}_{a\mu\, b\lambda}(p,q)=T^{*}_{b\lambda\, a\mu}(p,-q)$ implies that the (+) amplitudes are even and the (-) amplitudes are odd functions of ω. A standard forward dispersion relations analysis, very similar to that for the familiar case[53] of pion-nucleon scattering, shows that the amplitudes $T_{1,2}^{(\pm)}$ satisfy the following dispersion relations,

$$
\begin{aligned}
T_1^{(+)}(q^2,\omega) &= T_1^{(+)}(q^2,0) - \int_0^2 d\omega' [W_1^-(q^2,\omega') + W_1^+(q^2,\omega')]\left(\frac{1}{\omega'-\omega} + \frac{1}{\omega'+\omega}\right), \\
T_1^{(-)}(q^2,\omega) &= -\int_0^2 d\omega' \, [W_1^-(q^2,\omega') - W_1^+(q^2,\omega')]\left(\frac{1}{\omega'-\omega} - \frac{1}{\omega'+\omega}\right), \\
T_2^{(+)}(q^2,\omega) &= -\omega \int_0^2 \frac{d\omega'}{\omega'} \, [W_2^-(q^2,\omega') + W_2^+(q^2,\omega')]\left(\frac{1}{\omega'-\omega} - \frac{1}{\omega'+\omega}\right), \\
T_2^{(-)}(q^2,\omega) &= -\omega \int_0^2 \frac{d\omega'}{\omega'} \, [W_2^-(q^2,\omega') - W_2^+(q^2,\omega')]\left(\frac{1}{\omega'-\omega} + \frac{1}{\omega'+\omega}\right),
\end{aligned}
\tag{258}
$$

with absorptive parts given by

$$
\begin{aligned}
&-(2\pi)^3 (p_0/M_N)\tfrac{1}{2}\sum_{\text{spin}(N)} \sum_\Gamma \langle N(p)|\,2^{-\frac{1}{2}}(\mathcal{F}_{1\mu} \mp i\mathcal{F}_{2\mu})\,|\Gamma(p_\Gamma)\rangle \\
&\quad \times \langle \Gamma(p_\Gamma)|\,2^{-\frac{1}{2}}(\mathcal{F}_{1\lambda} \pm i\,\mathcal{F}_{2\lambda})\,|N(p)\rangle\, \delta^4(p_\Gamma - p - q). \qquad (259) \\
&= W_1^{\pm}(q^2,\omega)\left(-g_{\mu\lambda} + \frac{q_\mu q_\lambda}{q^2}\right) + M_N^{-2} W_2^{\pm}(q^2,\omega)\left(p_\mu - \frac{\nu}{q^2}q_\mu\right)\left(p_\lambda - \frac{\nu}{q^2}q_\lambda\right).
\end{aligned}
$$

The structure functions $W_{1,2}^{\pm}$ appearing here are the charged-photon analogs of the electron scattering structure functions defined in Eqs. (248) - (249). In writing Eq. (258), we have assumed one subtraction each for $T_1^{(\pm)}$ [the subtraction constant $T_1^{(-)}(q^2,0)$ vanishes by crossing symmetry] and no subtraction for $T_2^{(\pm)}$, as is suggested both by perturbation theory calculations and by a simple Regge model[54] for the high energy (large ν, small ω) behavior of the amplitudes.

Having finished our preliminaries, we can now derive a so-called asymptotic sum rule satisfied by the structure functions $W_1^{\pm}(q^2, \omega)$ in the asymptotic limit as $q^2 \to -\infty$. Referring back to Eq. (255), we set $\underset{\sim}{q} = \underset{\sim}{p} = 0$, $\mu = \lambda = 1$ and take the Bjorken limit $q_0 \to i\infty$. Using Eqs. (254) - (258), we find that

$$T^*_{a1b1}(p, q)_{q_0 \to i\infty} q_0^{-1} [\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b]\{1 - \lim_{q^2 \to -\infty} 2M_N \int_0^2 d\omega' [W_1^-(q^2, \omega') - W_1^+(q^2, \omega')]\} + \text{symmetric in } a, b + O(q_0^{-2} \ell n q_0), \tag{260}$$

while applying the Bjorken-Johnson-Low recipe of Eq. (203) to Eq. (252) gives the alternative evaluation

$$T^*_{a1b1}(p, q)_{q_0 \to i\infty} \text{ polynomial} + q_0^{-1}(p_0/M_N)\tfrac{1}{2}\sum_{\text{spin}(N)} \int d^4x\, e^{iq\cdot x} \times \delta(x_0)\langle N(p)| [\mathcal{F}_{a1}(\underset{\sim}{x}, 0), \mathcal{F}_{b1}(0)] |N(p)\rangle + O(q_0^{-2} \ell n q_0). \tag{261}$$

To calculate the equal-time commutator appearing in Eq. (261), we will again adopt the SU_3-triplet model of the strong interactions described in the preceding subsection, and again we assume the identity of the Bjorken-limit and the naive canonical commutator. We thus get

$$\begin{aligned} &\delta(x_0)[\mathcal{F}_{a1}(\underset{\sim}{x}, 0), \mathcal{F}_{b1}(0)] \\ &= \delta(x_0)[\overline{\psi}(\underset{\sim}{x}, 0)\tfrac{1}{2}\lambda_a\gamma_1\psi(\underset{\sim}{x}, 0), \overline{\psi}(0)\tfrac{1}{2}\lambda_b\gamma_1\psi(0)] \\ &= \delta^4(x)\, \overline{\psi}\, \gamma_0[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b]\,\psi\,, \end{aligned} \tag{262}$$

and the right-hand side of Eq. (261) becomes

$$\text{polynomial} + q_0^{-1}[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b] + O(q_0^{-2} \ell n\, q_0). \tag{263}$$

Comparing with Eq. (260), we thus get the asymptotic sum rule

$$0 = \lim_{q^2 \to -\infty} 2M_N \int_0^2 d\omega' \, [W_1^-(q^2,\omega') - W_1^+(q^2,\omega')] \ . \qquad (264)$$

The value 0 appearing on the left-hand side of Eq. (264) is peculiar to the SU_3-triplet model; other field theoretic models, which also satisfy the Gell-Mann time component current algebra, have different values for the space-space commutator of Eq. (262) and therefore lead to modified sum rules. For example, in the algebra of fields model,[55] the commutator in Eq. (262) vanishes and the left-hand side of Eq. (264) is 1. Eq. (264) is readily generalized to the physically realistic cases, where it yields a similar sum rule[49] for the W_1 structure functions in ν and $\bar{\nu}$-nucleon scattering, and an inequality for the W_1 structure functions in electron-nucleon scattering. We note finally that in the usual form in which Eq. (264) appears in the literature, the Born terms are not separated out of W_1; if the Born terms are included in W_1, the left-hand side of Eq. (264) becomes, respectively, -1 in the quark-model and 0 in the field algebra cases.

A useful alternative form of Eq. (264) is obtained by recalling that the usual fixed-q^2 sum rule, following from the local time component algebra alone,[56] is in our present notation

$$0 = \int_0^2 \frac{d\omega'}{\omega'^2}[\, W_2^-(q^2,\omega') - W_2^+(q^2,\omega')\,] \tag{265}$$

Multiplying Eq. (265) by $2q^2/M_N$ and adding to Eq. (264), we get the modified sum rule[57]

$$0 = \lim_{q^2 \to -\infty} 2\int_0^2 d\omega' \,[\, L^-(q^2,\omega') - L^+(q^2,\omega')\,] \tag{266}$$

with

$$L^{\mp}(q^2,\omega) = M_N[\, W_1^{\mp}(q^2,\omega) + \frac{q^2}{\omega^2}\frac{1}{M_N^2} W_2^{\mp}(q^2,\omega)\,] \tag{267}$$

the total longitudinal cross section[58] for inelastic charged-photon-nucleon scattering. Eq. (266) has a very interesting analog in the case of electron scattering, obtained by Callan and Gross[45] by a more complicated derivation involving the commutator $\delta(x_0)[\partial j_r^{EM}/\partial x_0, j_s^{EM}]$, which appears as the coefficient of the q_0^{-2} term when the recipe of Eq. (203) is applied to the T-product of two electromagnetic currents. In the quark model, this analog reads

$$0 = \lim_{q^2 \to -\infty} \int_0^2 d\omega\, \omega\, L^{EM}(q^2,\omega) \,, \tag{268}$$

and since L^{EM} is positive definite Eq.(268) has the very strong implication that

$$\lim_{q^2 \to -\infty} L^{EM}(q^2,\omega) = 0 \tag{269}$$

for all ω in the range $0 \le \omega \le 2$. Just as in our previous examples, the derivation of Eq. (269) involves in a crucial way the assumption that the Bjorken-limit commutator and the naive canonical commutator are the same.

8. BREAKDOWN OF THE BJORKEN LIMIT IN PERTURBATION THEORY

As we have greatly emphasized, all of the applications of the Bjorken limit method discussed in the preceding section involve the assumption that the commutators appearing in the Bjorken-Johnson-Low recipe can be evaluated by naive application of canonical commutation relations. We have also seen, in Section 6, that this assumption is not generally true in perturbation theory, since it fails, for example, in the simple case of the triangle diagram. Consequently, it is natural to ask whether the assumption is valid for the Compton-like perturbation theory diagrams involved in the applications of Section 7; we will find, upon detailed examination, that it is not, and that all of the applications fail in perturbation theory. For definiteness, we will consider the SU_3-triplet model described above, consisting of a fermion triplet bound by the exchange of SU_3-singlet boson "gluons" which can be spatially scalar, pseudoscalar or vector in character. It will not be possible to test the applications of the Bjorken limit in precisely the formulations given in Section 7, where we always assumed involvement of a nucleon, which appears (conjecturally) in the gluon model only as a complicated multiparticle bound state. However, it is easy to see that the derivations of Section 7 are equally valid if the nucleons p and n are replaced by the first two fermion triplets ψ_1 and ψ_2, with the nucleon mass M_N replaced in the kinematic formulas by the triplet mass m. Eq. (246) then becomes a statement about radiative corrections to the vector amplitude for the triplet β-decay $\psi_2 \to \psi_1 + e + \overline{\nu}_e$, while Eqs. (264), (266) and (269) become asymptotic sum rules and cross section relations for charged photon-triplet and electron-triplet scattering. It is the validity of these relations which we will directly test in perturbation theory.

8.1 Computational Results

We begin by summarizing the results of some perturbation theory calculations in the triplet-gluon model. In order to treat simultaneously commutators involving scalar, pseudoscalar and tensor currents as well as the usual vector and axial-vector currents, we introduce the abbreviated notation

$$\begin{gathered} J_{(1)} = \overline{\psi}\,\gamma_{(1)}\psi\,, \quad J_{(2)} = \overline{\psi}\,\gamma_{(2)}\psi\,, \\ \gamma_{(1)} = \gamma_\mu \tfrac{1}{2}\lambda_a (\gamma_\mu\gamma_5 \tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_a, \ldots) \\ \gamma_{(2)} = \gamma_\lambda \tfrac{1}{2}\lambda_b (\gamma_\lambda\gamma_5 \tfrac{1}{2}\lambda_b, \tfrac{1}{2}\lambda_b, \ldots) \end{gathered} \tag{270}$$

according to whether the first or second current is a vector (axial-vector, scalar, ...) current. The naive equal time commutator of the two currents is

$$\delta(x_0)[J_{(1)}(x), J_{(2)}(0)] = \delta^4(x)\,\overline{\psi}(x)\,C\,\psi(x), \tag{271}$$

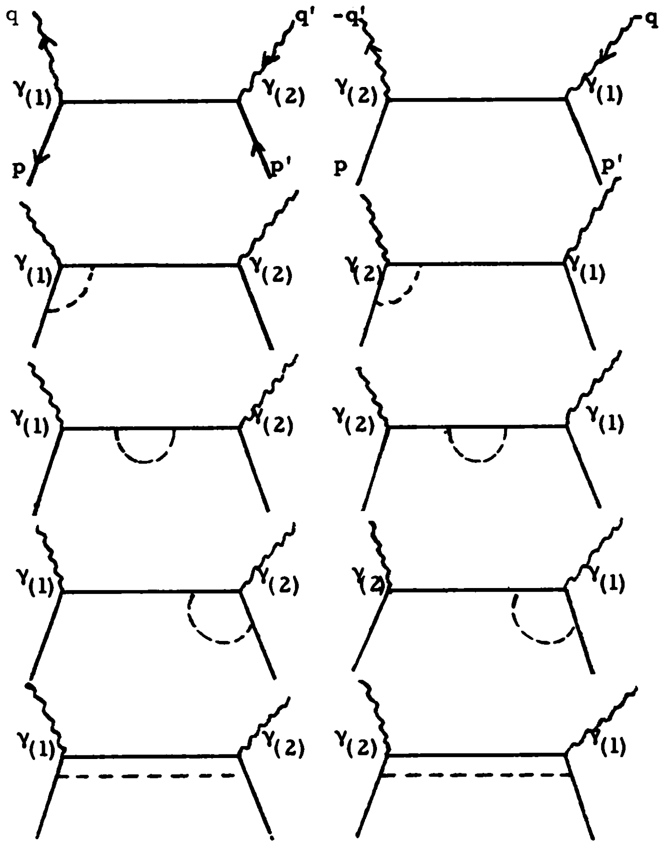
$$C = \gamma_0[\gamma_0\gamma_{(1)}, \gamma_0\gamma_{(2)}] = \gamma_{(1)}\gamma_0\,\gamma_{(2)} - \gamma_{(2)}\gamma_0\gamma_{(1)}.$$

We wish to compare the Bjorken-limit commutator with the naive commutator in the special case in which Eqs. (270) and (271) are sandwiched between triplet states. To do this, we calculate the renormalized current-fermion scattering amplitude $\tilde{T}^*_{(1)(2)}(p, p', q)$ in the limit $q_0 \to i\infty$, and compare the coefficient of the q_0^{-1} term with the renormalized vertex $\tilde{\Gamma}(C;p,p')$ of the naive commutator. Identity of the Bjorken-limit and the naive commutators would mean that

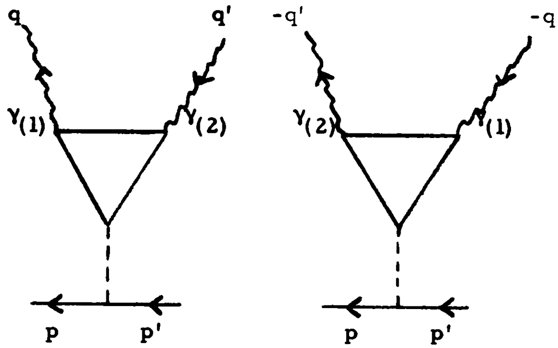
$$\lim_{\substack{q_0 \to i\infty \\ \underset{\sim}{q}, p, p' \text{ fixed}}} \tilde{T}^*_{(1)(2)}(p, p', q) = \underset{\text{in } q_0}{\text{polynomial}} + q_0^{-1}\tilde{\Gamma}(C;p,p') + O(q_0^{-2}\,\ell n\, q_0), \tag{272}$$

with the polynomial, as usual, coming from the seagull term. In the calculations which follow, we test the validity of Eq. (272) in perturbation theory.

(i) Second order. To second order in the gluon-fermion coupling constant g_r, there are two classes of diagrams which contribute to $\tilde{T}^*_{(1)(2)}$. The diagrams of the first class



consist of the lowest order current-fermion diagrams and the second order diagrams obtained from the lowest order ones by insertion of a single virtual gluon (denoted by the dashed line). The diagrams of the second class

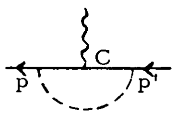


involve a fermion triangle graph. We denote the contributions of these two classes to $\tilde{T}^*_{(1)(2)}$ by $\tilde{T}^{*\ \mathrm{Compt}}_{(1)(2)}$ and $\tilde{T}^{*\ \mathrm{Triang}}_{(1)(2)}$, respectively.

The first-class diagrams are evaluated by the standard technique of regulating the gluon propagator with a regulator of mass Λ, which defines an <u>unrenormalized</u> amplitude $T^{*\ \mathrm{Compt}}_{(1)(2)}$. To get the renormalized amplitude we multiply by the fermion wave function renormalization constant Z_2 (the Feynman rules supply us with a factor $\sqrt{Z_2}$ for each of the two external fermion legs) and take the limit $\Lambda \to \infty$,

$$\tilde{T}^{*\ \mathrm{Compt}}_{(1)(2)} = \lim_{\Lambda \to \infty} Z_2\, T^{*\ \mathrm{Compt}}_{(1)(2)} . \tag{273}$$

In certain cases, as discussed below, this limit diverges logarithmically; in these cases we take Λ to be finite but very large, dropping terms which vanish as $\Lambda \to \infty$ but retaining all terms which are proportional to $\ell n\, \Lambda^2$. The renormalized vertex $\widetilde{\Gamma}(C;p,p') = \lim(\Lambda\to\infty)\, Z_2\Gamma(C;p,p')$ is calculated by the same techniques from the diagram



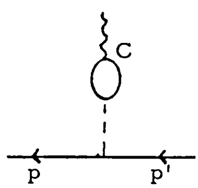
Finally, we take the limit $q_0 \to i\infty$ in our expression for $\widetilde{T}^{*\,\mathrm{Compt}}_{(1)(2)}$ and compare with $\widetilde{\Gamma}(C;p,p')$, giving the results[59]

$$\lim_{\substack{q_0\to i\infty \\ \underset{\sim}{q},\, p,\, p'\ \mathrm{fixed}}} \widetilde{T}^{*\,\mathrm{Compt}}_{(1)(2)}(p,p',q) = q_0^{-1}[\widetilde{\Gamma}(C;p,p') + \Delta^{\mathrm{Compt}}] + O(q_0^{-2}\,\ell n\, q_0), \tag{274a}$$

$$\begin{aligned}\Delta^{\mathrm{Compt}} = \frac{g_r^2}{32\pi^2}\Big\{\ell n\Big(\frac{\Lambda^2}{|q_0|^2}\Big)\Big[&-\gamma_{(1)}\gamma_0\underset{\sim}{\gamma}\gamma_0\underset{\sim}{\gamma}\gamma_0\gamma_{(2)} + \tfrac{1}{2}\underset{\sim}{\gamma}\gamma_\tau\gamma_{(1)}\gamma_0\gamma_{(2)}\gamma^\tau\underset{\sim}{\gamma} \\ &-\tfrac{1}{2}\gamma_{(1)}\gamma_0\underset{\sim}{\gamma}\gamma_\tau\gamma_{(2)}\gamma^\tau\underset{\sim}{\gamma} - \tfrac{1}{2}\underset{\sim}{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\underset{\sim}{\gamma}\gamma_0\gamma_{(2)}\Big] \\ &-\tfrac{3}{2}\gamma_{(1)}\gamma_0\underset{\sim}{\gamma}\gamma_0\underset{\sim}{\gamma}\gamma_0\gamma_{(2)} - \tfrac{1}{2}\underset{\sim}{\gamma}\gamma_0\gamma_{(1)}\gamma_0\gamma_{(2)}\gamma_0\underset{\sim}{\gamma} \\ &+\gamma_{(1)}\gamma_0\underset{\sim}{\gamma}\gamma_0\gamma_{(2)}\gamma_0\underset{\sim}{\gamma} + \underset{\sim}{\gamma}\gamma_0\gamma_{(1)}\gamma_0\underset{\sim}{\gamma}\gamma_0\gamma_{(2)} \\ &-\tfrac{1}{4}\gamma_{(1)}\gamma_0\underset{\sim}{\gamma}\gamma_\tau\gamma_{(2)}\gamma^\tau\underset{\sim}{\gamma} - \tfrac{1}{4}\underset{\sim}{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\underset{\sim}{\gamma}\gamma_0\gamma_{(2)} \\ &+\tfrac{1}{4}\underset{\sim}{\gamma}[\gamma_\tau\gamma_{(1)}\gamma^\tau\gamma_{(2)}\gamma_0 + \gamma_0\gamma_{(1)}\gamma_\tau\gamma_{(2)}\gamma^\tau]\underset{\sim}{\gamma} - (1)\longleftrightarrow(2)\Big\}.\end{aligned} \tag{274b}$$

In Eq. (274), the notation $\underset{\sim}{\gamma}\ldots\underset{\sim}{\gamma}$ is a shorthand for $1\ldots1$ in the scalar gluon case, $i\gamma_5\ldots i\gamma_5$ in the pseudoscalar gluon case and $(-\gamma_\rho)\ldots\gamma^\rho$ in the vector gluon case. If more than one type of gluon is present in the theory, the quantity Δ^{Compt} appearing in Eq. (274a) is simply the sum of contributions as in Eq. (274b) for each gluon.

Because our model contains only SU_3-singlet gluons, the second class diagrams contribute only to the SU_3-singlet part of the commutator. Taking the Bjorken limit, and comparing with the bubble diagram contribution to $\tilde{\Gamma}(C;p,p')$



one finds[34]

$$\lim_{\substack{q_0 \to i\infty \\ \underset{\sim}{q},\, p,\, p' \text{ fixed}}} \tilde{T}^{*\,\mathrm{Triang}}_{(1)(2)}(p,p',q) = \text{constant} + q_0^{-1}[\,\tilde{\Gamma}(C;p,p')^{\mathrm{Bubble}} + \Delta^{\mathrm{Triang}}] + O(q_0^{-2}\ln q_0). \tag{275a}$$

We will not exhibit the detailed form of Δ^{Triang}, but only remark that in all cases Δ^{Triang} vanishes when the three-momenta $\underset{\sim}{q}$ and $\underset{\sim}{q}' = \underset{\sim}{q} + \underset{\sim}{p} - \underset{\sim}{p}'$ associated with the currents $J_{(1)}$ and $J_{(2)}$ vanish,

$$\Delta^{\mathrm{Triang}}\Big|_{\underset{\sim}{q} = \underset{\sim}{q}' = 0} = 0\ . \tag{275b}$$

[Eq. (275) holds when the triplet of fermions are degenerate in mass. The effect of mass splittings is discussed in Ref. 34.] Thus, for the physically-interesting case of the commutator of spatially integrated currents, the entire answer is given by Eq. (274). No cancellation between the SU_3-singlet part of Δ^{Compt} and Δ^{Triang} is possible, and we conclude that the Bjorken-limit and the naive commutator in our models differ in second order perturbation theory.

For future reference it will be useful to write out in detail some special cases of Eq. (274). We consider first the commutator of two vector currents, with $\gamma_{(1)}=\gamma_\mu \frac{1}{2}\lambda_a$, $\gamma_{(2)}= \gamma_\lambda \frac{1}{2}\lambda_b$ and with naive commutator $\widetilde{\Gamma}(C;p,p')$,

$$C =\tfrac{1}{2}\{ \tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b\}(\gamma_\mu\gamma_0\gamma_\lambda-\gamma_\lambda\gamma_0\gamma_\mu)+\tfrac{1}{2}[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b](\gamma_\mu\gamma_0\gamma_\lambda+\gamma_\lambda\gamma_0\gamma_\mu). \tag{276}$$

In the vector gluon case we find

$$\Delta^{\mathrm{Compt}}=\frac{g_r^2}{16\pi^2}\{2(g_{\mu\lambda}-g_{\mu0}g_{\lambda0})\gamma_0[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b] + \tfrac{3}{2}(\gamma_\lambda\gamma_0\gamma_\mu -\gamma_\mu\gamma_0\gamma_\lambda)\{ \tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b\}\}, \tag{277}$$

while in the scalar and pseudoscalar gluon cases we find

$$\Delta^{\mathrm{Compt}}=\frac{g_r^2}{16\pi^2}\{(g_{\mu\lambda}-g_{\mu0}g_{\lambda0})\gamma_0[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b] -\tfrac{1}{2}(\gamma_\lambda\gamma_0\gamma_\mu -\gamma_\mu\gamma_0\gamma_\lambda)\{ \tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b\}[\ln (\frac{\Lambda^2}{|q_0|^2})-1]\}. \tag{278}$$

When one or both of the currents in the vector gluon case is an axial-vector current, Δ^{Compt} is obtained from the expression in Eq. (277) by the following simple substitutions:

Current $J_{(1)}$	Current $J_{(2)}$	Change in Eq. (277)
V	V	none
A	V	$\Delta^{\mathrm{Compt}} \to -\gamma_5\, \Delta^{\mathrm{Compt}}$
V	A	$\Delta^{\mathrm{Compt}} \to \Delta^{\mathrm{Compt}}\, \gamma_5$
A	A	$\Delta^{\mathrm{Compt}} \to -\gamma_5 \Delta^{\mathrm{Compt}} \gamma_5$

The results for axial-vector currents in the scalar and pseudoscalar gluon cases are not so simple.

(ii) Fourth order.

To fourth order in g_r, the number of diagrams contributing to $\tilde{T}^*_{(1)(2)}$ is so large that a direct calculation of the Bjorken limit, in analogy with our treatment of the second order case, is prohibitively complicated. However, as we have seen in Eqs. (260), (265) and (266), dispersion relations and unitarity provide a connection between the Bjorken limit for two vector currents and an integral over the longitudinal current-fermion inelastic cross section,

$$\lim_{q_0 \to i\infty} T^*_{a1\,b1}(p,q) = \lim_{q_0\to i\infty} \tfrac{1}{2}\mathrm{tr}\{(\tfrac{\not p+m}{2m})\tilde{T}_{(1)(2)}(p,p,q)\}\Big|_{\substack{\gamma_{(1)}=\gamma_1\frac{1}{2}\lambda_a \\ \gamma_{(2)}=\gamma_1\frac{1}{2}\lambda_b}}$$

$$= q_0^{-1}[\tfrac{1}{2}\lambda^a, \tfrac{1}{2}\lambda^b]\{1 - \lim_{q^2\to-\infty} 2\int_0^2 d\omega' \, [\, L^-(q^2,\omega') - L^+(q^2,\omega')\,]\,\}$$

$$+ \text{ symmetric in } a, b \;+ O(q_0^{-2}\, \ln\, q_0). \tag{279}$$

From this equation we can calculate the part of Δ which is proportional to $[\,\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b]$ if the longitudinal cross sections are known. The longitudinal cross sections themselves are, in general, extremely complicated. But multiplying Eq. (259) by $p^\lambda p^\mu$ and comparing with Eq. (267) (with M_N replaced by m) shows that, in the limit as the triplet mass m approaches zero, the longitudinal cross sections are given by the simple expression (280)

$$L^{\mp} = \lim_{m\to 0} \frac{-p_0\omega^2}{q^2}(2\pi)^3 \sum_{\text{spin}(\psi)} \sum_{\Gamma} |\langle\psi(p)|\tfrac{1}{2}p^\mu(\mathcal{F}_{1\mu}\pm\mathcal{F}_{2\mu})|\Gamma(p_\Gamma)\rangle|^2 \times \delta^4(p+q-p_\Gamma).$$

The factor p_0 in front of this equation is of purely kinematic origin, and cancels against a factor p_0^{-1} arising from our choice of normalization. The important point is that the factor p^μ in the matrix element in Eq. (280) leads to a considerable simplification in the calculation of $L^{\mp}$ in the zero triplet mass limit. Eqs. (279) and (280) have been applied,[59] in the scalar and pseudoscalar gluon cases, to the calculation of the part of the vector-vector commutator which is proportional to $[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b]$, independent of m, $\underset{\sim}{q}$, $\underset{\sim}{q}'$, $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ and which is logarithmically divergent as $|q_0|^2$ becomes infinite, with the result that

$$\Delta = (g_{\mu\lambda}-g_{\mu 0}g_{\lambda 0})\gamma_0[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b]\left[\frac{g_r^2}{16\pi^2} + 7\left(\frac{g_r^2}{16\pi^2}\right)^2 \ln|q_0|^2 + g_r^4 \times \text{constant}\right] \tag{281}$$

$$+ \text{ symmetric in } a,b + \text{ terms proportional to } m, \underset{\sim}{q}, \underset{\sim}{q}', \underset{\sim}{p}, \underset{\sim}{p}'.$$

Even the calculation of this one special case is very complicated, and involves the consideration of three-body final states containing either a triplet plus two gluons or a triplet plus a triplet-antitriplet pair.

8.2 Discussion

We proceed next to discuss a number of features of the results of Eqs. (274), (276-279) and (281), and in particular, to indicate their effect on the applications of the Bjorken limit developed above.

(i) We begin by noting that to second order in g_r^2, Δ^{Compt} contains terms $\ell n(\Lambda^2/|q_0|^2)$ which diverge logarithmically both in the Bjorken limit $q_0 \rightarrow i\infty$ and in the infinite cutoff limit $\Lambda \rightarrow \infty$. It is easy to see that the $\ell n\Lambda^2$ divergences result from a mismatch between the multiplicative factors needed to make $T^{*\,\mathrm{Compt}}_{(1)(2)}(p, p', q)$ and $\Gamma(C;p,p')$ finite (i. e., $\ell n\Lambda^2$-independent) as $\Lambda \rightarrow \infty$. As we recall, the <u>renormalized</u> quantities $\widetilde{T}^{*\,\mathrm{Compt}}_{(1)(2)}(p, p', q)$ and $\widetilde{\Gamma}(C;p,p')$ are obtained from $T^{*\,\mathrm{Compt}}_{(1)(2)}(p, p', q)$ and $\Gamma(C;p,p')$ by multiplying by the wave function renormalization Z_2 and taking the limit $\Lambda \rightarrow \infty$, keeping any residual $\ell n\Lambda^2$ dependence. On the other hand, the <u>finite</u> quantities $T^{*\,\mathrm{Compt}}_{(1)(2)}(p, p', q)^{\mathrm{finite}}$ and $\Gamma(C;p,p')^{\mathrm{finite}}$ are obtained by multiplying by appropriate vertex and propagator renormalization factors which completely remove the $\ln \Lambda^2$ dependence,

$$\Gamma(C;p,p')^{\mathrm{finite}} = Z(C)\Gamma(C;p,p'), \qquad (282)$$

$$T^{*\,\mathrm{Compt}}_{(1)(2)}(p,p',q)^{\mathrm{finite}} = Z(\gamma_{(1)})Z(\gamma_{(2)})Z_2^{-1}T^{*\,\mathrm{Compt}}_{(1)(2)}(p,p',q).$$

In general, the vertex renormalizations $Z(C)$, $Z(\gamma_{(1)})$ and $Z(\gamma_{(2)})$ are not equal to each other, or to Z_2. For example, in the case of the vector gluon model, we have seen in Subsection 3.1 that the pseudoscalar vertex renormalization factor is $Z(\gamma_5) = Z_2 m_0$, with m_0 the divergent fermion bare mass. If we write

$$Z(C) = 1 + \Lambda(C), \qquad (283)$$

$$Z_2 = 1 + \Lambda(\gamma_\mu) = 1 + \Lambda_2,$$

then we find to second order that

$$\tilde{T}^{*\,\mathrm{Compt}}_{(1)(2)}(p,p',q) = T^{*\,\mathrm{Compt}}_{(1)(2)}(p,p',q)^{\mathrm{finite}} \qquad (284)$$

$$+[2\Lambda_2 - \Lambda(\gamma_{(1)}) - \Lambda(\gamma_{(2)})][\gamma_{(1)} \frac{1}{\not p + \not q - m} \gamma_{(2)} + \gamma_{(2)} \frac{1}{\not p - \not q' - m} \gamma_{(1)}],$$

$$\tilde{\Gamma}(C;p,p') = \Gamma(C;p,p')^{\mathrm{finite}} + [\Lambda_2 - \Lambda(C)]C.$$

Since finite quantities on the left and right hand sides of Eq. (284) must match up, we see that

$$\Delta^{\mathrm{Compt}} = [\Lambda_2 + \Lambda(C) - \Lambda(\gamma_{(1)}) - \Lambda(\gamma_{(2)})]C + \text{finite}, \quad (285)$$

confirming that the $\ln \Lambda^2$ dependence in Δ^{Compt} results from a mismatch between the renormalization factors on the left and right hand sides of Eq. (274a). A simple calculation shows that

$$\Lambda(C)C = \frac{g_r^2}{32\pi^2} \tfrac{1}{2} \underset{\sim}{\gamma} \gamma_\tau C \gamma^\tau \underset{\sim}{\gamma} \ln \Lambda^2, \tag{286}$$

which on substitution into Eq. (285) does indeed give the $\ln \Lambda^2$ terms in Eq. (274).

The presence of terms which diverge as $\ln|q_0|^2$ in Eq. (274) indicates that, <u>in the general case, the Bjorken limit does not exist in perturbation theory.</u> The fact that the $\ln|q_0|^2$ and $\ln \Lambda^2$ terms occur in the combination $\ln(\Lambda^2/|q_0|^2)$ means that, <u>to second order, the existence of the Bjorken limit is directly connected with the matching of renormalization factors on the left and right hand sides of Eq. (274a)</u>: When the renormalization factors match, the Bjorken limit exists; when the factors do not match, the Bjorken limit diverges.[60] Unfortunately, we shall see that this simple result does not hold in higher orders in perturbation theory.

(ii) There are a number of interesting cases in which the renormalization factors do match, and hence the Bjorken limit exists in second order. In the vector gluon model, Eq. (277) and the table following Eq. (278) show that this is true for all commutators involving vector and axial-vector currents. In the scalar and pseudoscalar gluon models, it is true for the vector piece of the V-V commutator [Eq. (278)]

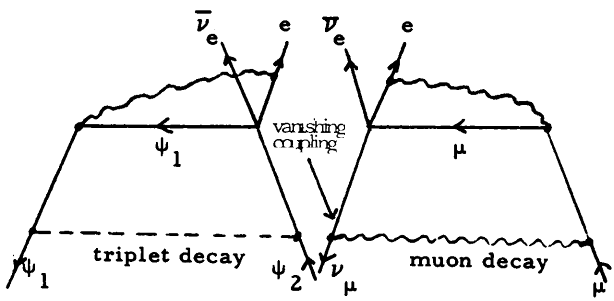
and for the axial-vector piece of the V-A and A-V commutators. The remarkable result that emerges from these examples is that, even when the Bjorken limit exists in second order, it does not agree with the naive commutator. According to the discussion of Eqs. (204)-(206) above, this means that the Bjorken limit agrees with the spectral function integral of Eq. (206), but the naive commutator does not. So it is really somewhat of a misnomer to talk about Bjorken limit breakdown; it is the naive commutator, and not the Bjorken limit, which breaks down.

Armed with the explicit formulas of Eqs. (277) and (278), we can now go back to see what happens to the various Bjorken limit applications developed above. First we consider the discussion of radiative corrections to β-decay given in Subsection 7.1.[61] As we have seen, the term proportional to Q_{AV} in Eq. (246) comes from the spatially vector, isospin symmetric part of a V-A commutator [the $\{\lambda_Q, \lambda_+\}$ $[\gamma_0\gamma_\sigma, \gamma_0\gamma_\mu\gamma_5]$ term in Eq. (241)]. In the vector gluon model, this corresponds to the $\{\frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b\}$ term in the V-A version of Eq. (277). Comparing Eq. (277) with Eq. (276), we see that to second order the coefficient of $\frac{1}{2}\{\frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b\}$ $(\gamma_\mu\gamma_0\gamma_\lambda - \gamma_\lambda\gamma_0\gamma_\mu)$ is changed from 1 to $1 - 3g_r^2/(16\pi^2)$, and hence Eq. (246) is modified in the vector gluon model to read

$$\delta M = \frac{3\alpha}{8\pi} \left[1 + \left(1 - \frac{3g_r^2}{16\pi^2}\right) 2Q_{AV} \right] \ln \Lambda^2 M, \qquad (287)$$

which diverges even in the special case $Q_{AV} = -\frac{1}{2}$. In the scalar and pseudoscalar gluon models, the situation is even worse, since the vector part of the V-A commutator is divergent in these models, as a result of mismatch of renormalization factors, and consequently the coefficient of $\ln \Lambda^2$ in Eq. (287) is itself logarithmically divergent. In fact, detailed field theoretic analyses[11] show that the vector gluon model with $Q_{AV} = -\frac{1}{2}$ is the only renormalizable, SU_3-symmetric model of the strong interactions which has the possibility of having finite radiative corrections to β-decay, so our result of Eq. (287) shows that there are in fact <u>no</u> renormalizable, SU_3-symmetric models with this property.

It is important to note that Eq. (287) does not contradict the result, mentioned above, that the radiative corrections to μ-meson decay are finite to all orders in QED. The point is that while the vector gluon is an SU_3-singlet, and hence couples symmetrically to all of the triplet fermions, the photon couples to the muon and electron but <u>not</u> to the neutrino. As a result, certain diagrams which are present in the triplet decay process are absent in muon decay, e.g.



It turns out to be precisely these missing diagrams which cause the disagreement between the Bjorken limit and naive commutators.

Next, we consider the asymptotic sum rules and cross section relations of Eqs. (260) and (266). We have actually already seen what happens in these cases: according to Eq. (279), the integrals no longer vanish, but instead are proportional to the coefficient of the $[\frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b]$ term in Eq. (277) or Eq. (278),

$$\begin{aligned}
&\lim_{q^2\to-\infty} 2M_N^2 \int_0^2 d\omega' \left[W_1^-(q^2,\omega') - W_1^+(q^2,\omega') \right] \\
&= \lim_{q^2\to-\infty} 2\int_0 d\omega' \left[L^-(q^2,\omega') - L^+(q^2,\omega') \right] \qquad (288) \\
&= \begin{cases} g_r^2/(8\pi^2) & \text{vector gluon} \\ g_r^2/(16\pi^2) & \text{scalar or pseudoscalar gluon} \end{cases}
\end{aligned}$$

In other words, the backward neutrino sum rule and the backward electron-scattering inequality depend on the dynamics of the triplet-gluon interaction, not just on the kinematic structure of the weak and electromagnetic currents. If, for definiteness, we take the fermion state ψ in Eq. (280) to be ψ_1, we find that the entire contribution to the longitudinal cross section to order g_r^2 comes from the intermediate state $\Gamma = \psi_2$ + gluon. The resulting two-body phase space integral is easily evaluated in the center of mass frame, giving

$$L^+ = 0 \tag{289}$$

$$\lim_{q^2 \to -\infty} L^-(q^2,\omega) = \begin{cases} g_r^2\omega/32\pi^2 & \text{vector gluon} \\ g_r^2\omega/64\pi^2 & \text{scalar or pseudoscalar gluon,} \end{cases}$$

in agreement with Eq. (288). The corresponding electromagnetic longitudinal cross section is obtained by replacing $\mathcal{F}_{1\mu} - i\,\mathcal{F}_{2\mu}$ in Eq. (280) by j_μ^{EM}, and receives its only contribution from the intermediate state $\Gamma = \psi_1$ + gluon. Clearly, the ratio $L^{EM}(q^2,\omega)/L^-(q^2,\omega)$ is just the squared charge Q^2 of ψ_1, indicating that the Callan-Gross relation of Eq. (269) also fails in our perturbation theory models.[61,62] We conclude that none of the principal applications of the Bjorken limit method are valid in perturbation theory.

(iii) From an inspection of Eq. (277) we see that in the vector gluon case, for all commutators involving vector and axial-vector currents, Δ^{Compt} vanishes when either $\mu=0$ or $\nu=0$. In other words, only the space-component-space-component commutators are anomalous. When $J_{(1)}$ and $J_{(2)}$ are both vector currents, this result can be deduced directly from the Ward identity of Eq. (253), which in our present notation reads, on the mass shell,

$$\tilde{T}^{*\,\mathrm{Compt}}_{(1)(2)}(p,p',q)\Big|_{\substack{\gamma_{(1)}=q^{\mu}\gamma_{\mu}\frac{1}{2}\lambda_a \\ \gamma_{(2)}=\gamma_{\lambda}\frac{1}{2}\lambda_b}} = \tilde{\Gamma}([\tfrac{1}{2}\lambda_a,\tfrac{1}{2}\lambda_b]\,\gamma_{\lambda};p,p'). \quad (290)$$

Multiplying by q_0^{-1} and taking the limit $q_0 \to i\infty$ gives immediately

$$\lim_{q_0\to i\infty} \tilde{T}^{*\,\mathrm{Compt}}_{(1)(2)}(p,p',q)\Big|_{\substack{\gamma_{(1)}=\gamma_0\frac{1}{2}\lambda_a \\ \gamma_{(2)}=\gamma_{\lambda}\frac{1}{2}\lambda_b}} = q_0^{-1}\tilde{\Gamma}([\tfrac{1}{2}\lambda_a,\tfrac{1}{2}\lambda_b]\,\gamma_{\lambda};p,p') + O(q_0^{-2}\ln q_0), \quad (291)$$

confirming our explicit calculation. A similar derivation holds in the cases involving axial-vector currents, provided that the divergence of the axial-vector current is "soft",[63] as it is in the vector gluon model. We thus see that the breakdown of the Bjorken limit which we have found in Compton-like graphs is consistent with the constraints imposed by Ward identities, just as we found a similar consistency in the triangle graph case in Subsection 6.3. This means that except in cases such as $\pi^0 \to 2\gamma$, where Ward identity anomalies occur, the standard results of the Gell-Mann time component algebra, which are derived directly from Ward identities, remain valid. From the point of view of trying to distinguish between different models of the hadronic current, it is unfortunate that the Bjorken-limit fails precisely in the case of space-space commutators, where the usual current algebra "infinite momentum limit" and "low energy theorem" methods also do not work.

(iv) Basically, the origin of Bjorken limit breakdown is the very singular nature of perturbation expansions in field theory.[64] To see this, we note that if we take the Bjorken limit $q_0 \to i\infty$ before letting the regulator mass Λ approach infinity, so that we are dealing with a convergent cutoff field theory, the Bjorken limits and naive commutators agree.[59;65] The order of limits is thus of crucial importance here, unlike the situation in the low energy theorem discussion of Section 4. The reason is that when Λ is held finite while the Bjorken limit is taken, the spectral function integral of Eq. (206) contains contributions from the regulator particle, which cancel away the anomalous terms which we have found. On the other hand, if the limit $\Lambda \to \infty$ is taken first, so that we are dealing with renormalized perturbation theory, and then the Bjorken limit is taken, these regulator contributions are absent. Although the regulator theory has no Bjorken limit anomalies, it is not a very satisfactory physical model, since cross sections for regulator particle production can be negative. For instance, the Callan-Gross limit of Eq. (269) is satisfied in the regulator theory, but only by virtue of a cancellation between the cross section for fermion + gluon production, which is positive, and a negative cross section for fermion + regulator particle production.

(v) We turn next to the order g_r^4 result of Eq. (281), which gives the V-V→V commutator in the scalar and pseudoscalar gluon models. We see that <u>even though the renormalization factors match, the Bjorken limit in this case diverges in fourth order.</u> We note, however, that the divergence behaves as $g_r^4 \ell n|q_0|^2$, whereas in fourth order terms behaving like $g_r^4(\ell n|q_0|^2)^2$ could in principle be present. On the basis of this behavior and our second order results, the following <u>conjecture</u> seems reasonable: When the renormalization factors needed to make $T^{*\mathrm{Compt}}_{(1)(2)}(p,p',q)$ and $\Gamma(C;p,p')$ finite are the same, the Bjorken limit in order 2n of perturbation theory contains no terms $g_r^{2n}(\log|q_0|^2)^n$, but begins in general with terms

$g_r^{2n}(\log|q_0|^2)^{n-1}$.

(vi) Finally, we must emphasize that all of our results have been obtained in perturbation theory,[66] whereas strong interactions are notoriously non-perturbative in behavior. Thus, one is always free to postulate that non-perturbative effects somehow conspire to "damp out" the anomalous terms when all orders of perturbation theory are summed, although the need for this assumption would mean that asymptotic sum rules would not give a test of the space-space current algebra alone, but would involve deep dynamical considerations as well. There is an alternative point of view which has been analyzed in detail recently.[67] This is that Bjorken limits and naive commutators may well have little relation to each other, but still Bjorken limits may be interesting because, via asymptotic sum rules and Eq. (206), they furnish an experimental means for measuring equal-time commutators and other singular behavior of time-ordered products. In order for this point of view to bear fruit, it will be necessary to find new ways, not involving naive commutators, of correlating this singular behavior with the underlying structure of the theory or of relating to one another the singular behavior measured in different types of experiments.

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