I

Introduction: Purpose and plan of the inquiry

The creation of a formal mathematical language was of decisive significance for the constitution of modern mathematical physics. If the mathematical presentation is regarded as a mere device, preferred only because the insights of natural science can be expressed by "symbols" in the simplest and most exact manner possible, the meaning of the symbolism as well as of the special methods of the physical disciplines in general will be misunderstood. True, in the seventeenth and eighteenth century it was still possible to express and communicate discoveries concerning the "natural" relations of objects in nonmathematical terms, yet even then — or, rather, particularly then — it was precisely the mathematical form, the *mos geometricus*, which secured their dependability and trustworthiness. After three centuries of intensive development, it has finally become impossible to separate the content of mathematical physics from its form. The fact that elementary presentations of physical science which are to a certain degree nonmathematical and appear quite free of presuppositions in their derivations of fundamental concepts (having recourse, throughout, to immediate "intuition") are still in vogue should not deceive us about the fact that it is impossible, and has always been impossible, to grasp the meaning of what we nowadays call physics independently of its mathematical form. Thence arise the insurmountable difficulties in which discussions of modern physical theories become entangled as soon as physicist or nonphysicists attempt to disregard the mathematical apparatus and to present the results of scientific research in popular form. The intimate connection of the formal mathematical language with the content of mathematical physics stems from the special kind of conceptualization which is a concomitant of modern science and which was of fundamental importance in its formation.

Before entering upon a discussion of the problems which mathematical physics faces today, we must therefore set ourselves the task of inquiring into the origin and the conceptual structure of this formal language. For this reason the fundamental question concerning the inner relations between mathematics and physics, of "theory" and "experiment," of "systematic" and "empirical" procedure within mathematical physics, will be wholly bypassed in this study, which will confine itself to the limited task of recovering to some degree the sources, today almost completely hidden from view, of our modern symbolic *mathematics*. Nevertheless, the inquiry will never lose sight of the fundamental question, directly related as it is to the conceptual difficulties arising within mathematical *physics* today. However far afield it may run, its formulation will throughout be determined by this as its ultimate theme.

The creation of the formal language of mathematics is identical with the foundation of modern algebra. From the thirteenth until the middle of the sixteenth century, the West absorbed the Arabic science of "algebra" (*al-g'abr wa'l-muqābala*) in the form of a theory of equations, probably itself derived from Indian as well as from Greek sources.[1] As far as the Greek sources are concerned, the special influence of the *Arithmetic* of Diophantus on the content, but even more so on the form, of this Arabic science is unmistakable[2] — if not in the *Liber Algorismi* of al-Khowarizmi himself, at any rate from the tenth century on.[3] Now concurrently with the elaboration, particularly in Italy, of the theory of equations which the Arabs had passed on to the West, the original text of Diophantus began, as early as the fifteenth century, to become well known and influential. But it was not until the last quarter of the sixteenth century that Vieta undertook to broaden and to modify Diophantus' technique in a really crucial way. He thereby became the true founder of modern mathematics.

The conventional presentations of the history of this development do not, indeed, fail to see the significance of the revival and assimilation of Greek mathematics in the sixteenth century. But they always take for granted, and far too much as a matter of course, the *fact* of symbolic mathematics. They are not sufficiently aware of the *character* of the conceptual transformation which occurred in the course of this assimilation and which constitutes the indispensable condition of modern mathematical symbolism. Moreover, most of the standard histories attempt to grasp Greek mathematics itself with the aid of modern symbolism, as if the latter were an altogether external "form" which may be tailored to any desirable "content." And even in the case of investigations intent upon a genuine understanding of Greek science, one finds that the inquiry starts out from a conceptual level which is, from the very beginning, and precisely with respect to the fundamental concepts, determined by modern modes of thought. To disengage ourselves as far as possible from these modes must be the first concern of our enterprise.

Hence our object is not to evaluate the revival of Greek mathematics in the sixteenth century in terms of its results retrospectively, but to rehearse the actual course of its genesis prospectively. Now in Vieta's assimilation and transformation of the Diophantine technique, we have, as it were, a piece of the seam whereby the "new" science is attached to the old. But in order to be able to throw light on the essential features of this assimilation and transformation, we must first of all see the work of Diophantus *from the point of view of its own presuppositions*. Only then can we begin to distinguish Vieta's "Ars analytice" from its Greek foundations so as to reveal the conceptual transformation which is expressed in it.

The *Arithmetic* of Diophantus must, then, be given its proper place within the general framework of Greco-Hellenistic science, whatever one may imagine its prehistory to have been. This, however, immediately leads to a comparison of the foundations of the *Arithmetic* with those of the Neoplatonic "arithmetical" literature which forms its background, although the Neoplatonic categories were such as to prevent the integration of the *Arithmetic* into this literature. Sections 2–4 of Part I are devoted to the investigation of the classification of mathematical sciences in the Neoplatonic writers; these classifications go back to corresponding formulations in Plato, without, however, being identical with them. It will be shown that the Neoplatonic division of the science of numbers into "theoretical arithmetic" and "practical logistic" (the art of calculation) cannot assign an unambiguous position to the "theory of ratios and proportions." The latter does, on the other hand, seem identical with the "theoretical logistic" postulated by Plato. For Plato, this "theoretical logistic" bears a relation to "practical logistic" similar to that which "theoretical arithmetic" has to "practical arithmetic." "Theoretical logistic" and "theoretical arithmetic" both have as objects — in contrast to the corresponding practical arts — not things experienced through the senses but indivisible "pure" units which are completely uniform among themselves and which can be grasped as such only in thought. Both theoretical disciplines arise directly, on the one hand from actual *counting*, and on the other from *calculating*, i.e., from the act of relating numbers to one another; and the task of the theoretical disciplines is to reduce these "practical" activities to their true presuppositions. The Neoplatonic commentaries on the Platonic definitions of arithmetic and logistic in the *Charmides* and in the *Gorgias* show that in this "reduction" arithmetic is concerned with the "kinds" (εἴδη) of numbers while logistic is concerned with their "material" (ὕλη).

The Platonic postulation of a theoretical logistic as a noetic analogue for, and as the presupposition of, any art of calculation was ignored, as Section 5 will show, by the Neoplatonists, essentially because of the property of indivisibility of the noetic monads; the use of fractional parts of the unit of calculation, which is unavoidable in calculations, cannot be justified on the basis of such monads. An additional reason was the elaboration of the theory of ratios into a *general* theory of proportion, which depended on the discovery of incommensurable magnitudes and which led altogether beyond the realm of counted collections.

However, the difficulties which arise from the Platonic postulation of a theoretical logistic can be fully understood only if the ontological foundations which determine this conception are called to mind. And this requires, in turn, a thoroughgoing clarification of the *arithmos* concept which forms the basis of all Greek arithmetic and logistic. It can be shown (Section 6) that *arithmos never* means anything other than "a definite number of definite objects." Theoretical arithmetic grows initially out of the understanding that in the process of "counting off" any objects whatever we make use of a prior knowledge of "counting-numbers" which are already in our possession and which, as such, can only be collections of "undifferentiated" objects, namely assemblages of "pure" units. The problem of the possibility of such assemblages, i.e., the question how it is possible that *many* "ones" should ever form *one* collection of "ones," leads to the search for *eide* with definite "specific properties" such as will give unity to, and permit a classification of, all counted collections. Greek arithmetic is therefore originally nothing but the theory of the *eide* of numbers, while in the art of "calculating," and therefore in theoretical logistic as well, these counted collections are considered only with reference to their "material," their *hyle*, that is, with reference to the units as such. The possibility of theoretical logistic is therefore totally dependent on the mode of being which the pure units are conceived to have.

For this reason Pythagorean and Platonic philosophy in their relation to the fundamental problems of Greek mathematics are considered next (Section 7). In the first part (7A), the general point of view of Pythagorean cosmological "mathematics" and its connection with the *arithmos* concept as such is presented. In the second part (7B), the significance which is attached to "the ability to count and calculate" in Platonic philosophy is discussed: In "pure" arithmetic and "pure" logistic, human thinking (διάνοια) succeeds in becoming conscious of the true object and the true presuppositions of its activity, an activity which always remains tied to sense perception (αἴσθησις). A third part (7C) follows through the consequences which arise for Plato from the privileged position he assigns to the theory of number: In the structure of the *arithmos* concept he discovers the possibility of a fundamental solution of the problem of participation (μέθεξις) to which his "dialectic" necessarily leads, without, however, being of itself able to provide a solution. Thus the Pythagorean attempt at an "arithmological" ordering of all being is repeated by Plato within the realm of the ideas themselves; this amounts to a decisive correction of the Eleatic thesis of the "One."

This conception of numbers, eidetic as well as mathematical, as assemblages whose being is self-subsistent and originally "separate" from sense perception, a conception which is basic in Platonic teaching, is then criticized by Aristotle (Section 8). He shows that the "pure" units are merely the product of a "reduction" performed in thought, which turns everything countable into "neutral" material.

The "pure" units have, therefore, no being of their own. Their indivisibility is only an expression of the fact that counting and calculating always presuppose a last, irreducible "unit," which is to be understood as the given "measure." It follows that there is nothing to prevent the introduction of a new and "smaller" measure; in other words, we may operate with fractional parts of the former unit. Only on the basis of this Aristotelian conception can the Platonic demand for a "scientific" logistic be realized.

In Part II of this study we turn to the relation of symbolic algebra to the *Arithmetic* of Diophantus. After a general consideration of the difference between ancient and modern concept formation, the work of Diophantus is, on the basis of the results of Part I, interpreted as a "theoretical logistic" (Section 9). In the formulation and solution of problems, this theoretical logistic always retains a dependence on the Greek *arithmos* concept, although it apparently incorporates a more general, pre-Greek "algebraic" tradition as well (Section 10).

Finally, in Sections 11 and 12 the transformation of the Diophantine technique at the hands of Vieta and Stevin is described. In these concluding sections we show that the revival and assimilation of Greek logistic in the sixteenth century are themselves prompted by an already current *symbolic* understanding of number, and we attempt to clarify the conceptual structure of the algebraic symbolism which is its product. At the same time we trace out the general transformation, closely connected with the symbolic understanding of number, of the "scientific" consciousness of later centuries. This transformation will be shown to appear characteristically in Stevin, Descartes, and Wallis.