CHAPTER I

Geometry—A Phenomenological Discussion

1. The axiomatic geometer demands nothing of his reader except the ability to draw a logical conclusion. He sets up a number of axioms, containing words that sound like geometry, and then from these axioms he undertakes to derive theorems of many different kinds. On the other hand, the analytic (or better, the algebraic) geometer attaches geometric names to certain algebraic objects and then proves by algebraic methods that they have certain properties. But in both cases some sort of groundwork should be laid; there should be some discussion of the particular choice of axioms and of the geometric names for the given algebraic objects.

Since the concepts of geometry have been taken from the space of our everyday experience and visualization, and since conversely they often find applications there, we can proceed a surprisingly long way with a purely phenomenological analysis of this empirical space before making any start on a more or less clean-cut axiomatic or analytical treatment. In school the intuitive approach is never entirely abandoned, and Euclid himself, in spite of all his rigor, did not set up an unobjectionable system of axioms. Thus, in dealing with any particular part of geometry, the teacher must clearly realize why and how far he is willing, or compelled, to base his instruction on the intuitive powers of his students; he must know what further steps, and what choice of axioms, would be necessary to make his instruction entirely independent of intuition. In short, both for his own knowledge of the subject and for his instruction of others, he must undertake an analysis of our intuition of space. Only then can he teach with a good conscience; only then will he be able to lead his pupils, who at the beginning of the journey are at the mercy of their intuition, across its treacherous shoals onto higher ground.

The discussion in the present chapter is entirely phenomenological, although we assume that the reader knows his geometry. In our analysis of space we unhesitatingly make use of concepts analyzed in later chapters and do not give any proofs, often assuming that the reader can easily prove certain simple statements for himself.

Order

2. In our intuition of space the concept of a segment precedes that of a straight line. In fact, we arrive at the concept of a straight line by continually extending a segment in both directions. The straight line contains many segments, each of which is determined by its two endpoints. Every segment is an *infinite* set but can be determined by two data, namely, its endpoints.

By the segment AB we obviously mean the points of the straight line AB lying *between* A and B (exclusive). The *relation of betweenness,* which underlies the concept of a segment, is a relation "C lies between A and B" among three (arbitrary) points of a (fixed) line. Euclid, giving free rein to intuition, paid no attention to a relation of this sort, and Pasch was the first to recognize its importance. In the early stages of geometry, recognizing the similarity of two figures such as la and lb, which differ only in their order properties, represented a difficult feat of abstraction, so difficult indeed that even today many beginners are confused by it.

In the time of Pasch, on the other hand, it was a bold deed to free oneself from the Euclidean tradition and recognize the mathematical importance of these neglected questions of *order.*

Betweenness is one of the concepts of order. With its help, for example, we can describe the intuitive order which is imposed on the set of points in a straight line when we traverse the line in one direction; in a passage in the direction $A \rightarrow B$ the points *between* A and B are those which come *before* B.

But this relation of betweenness is quite inconvenient, since it is a relation among three things (a three-place relation), so that any nontrivial statement about it must take at least four things into account; for example, one of Hilbert's axioms runs as follows: "If four points are given on a line, they can

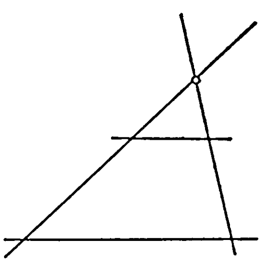

Fig. 1a

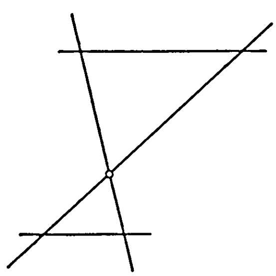

Fig. 1b

always be denoted by A, B, C, D in such a way that B lies between A and C and also between A and D, and C lies between A and D and also between B and D."[1]

It is much more satisfactory if we proceed not from our intuition of betweenness but from the idea of passage along a straight line. Here the relation "A before B," which we shall also write as $A < B$, is meaningful and, fortunately, is a two-place relation, so that we can make nontrivial statements about it by considering only three things.

A set on which the relation "before" is defined is thereby made into a *totally ordered set.* More precisely, a set is said to be ordered if for every pair of distinct elements A and B exactly one of the relations

$$A < B, \quad B < A$$

is satisfied in such way that for every three elements A, B, C it follows from

$$A < B, \quad B < C$$

that

$$A < C.$$

Instead of $A < B$ we also write $B > A$.

Of course, a set can be ordered in many different ways. But on a straight line we intuitively distinguish two special orders, one of them being the *opposite* of the other; i.e., if $A < B$ in one of them, then $B < A$ in the other. Instead of the axioms of betweenness, as they are to be found in Hilbert, we can postulate: on every straight line (i.e., *oriented, or directed straight line*) two (opposite) orders are distinguished.

Every point A on a line determines two *halflines,* the set of points $B < A$ and the set of points $B > A$, and it does not matter which of the two orders is adopted. Two points A and B on a line determine four halflines and then, if $A < B$, the segment AB is defined as the intersection of the sets $C < B$ and $C > A$.

If a halfline is distinguished, the line is thereby oriented; for if A is the point determining the halfline and B is any point belonging to it, we may distinguish the order in which $A < B$.

3. There is not much more of importance to say about order on a line. But there is also a certain natural order in the *plane.*

Every line divides the plane into two parts, namely, two *halfplanes;* every point of the plane that does not lie on the line lies in exactly one of its two halfplanes. A halfplane has the property that two arbitrary points in it can be joined by a segment lying entirely in the halfplane. On the other hand, two points in different halfplanes determined by the same line l cannot be joined by a line segment that does not cross l.

[1]This axiom was subsequently derived from axioms of order in the *plane.*

This situation can be described in another way, in terms of convexity. A set is said to be *convex* if with every pair of points A, B in it the whole segment AB belongs to the set. Thus a line, a halfline, a segment, a disk, and the surface of a triangle are convex sets.

Then the above property of the two halfplanes of a line l can be described by saying that each of the two halfplanes is convex, but if to either of them we add a single point not on l from the other halfplane, the resulting set is no longer convex.

4. Like the line, a plane α can also be oriented. For let us choose an oriented line l in α and decide which of the two resulting halfplanes is to be called the *left side* of l (in α). Then we shall say that the plane α has been *oriented,* or *directed,* since we have now distinguished between the two sides of it as a plane in space; for when we are looking along the directed line l, our choice for its left-hand side will obviously depend on which side of the plane we are on in space. A plane in space has exactly two sides.

But the concept of an oriented plane can also be understood intuitively without any reference to space. For we need only consider, in addition to the oriented line l, a second line m, crossing l from right to left; i.e., the orientation of m is such that on it the points of the right halfplane of l come *before* those of the left halfplane. Or conversely, if we begin with the pair l,m of oriented lines in α, we know which side of l is to be regarded as the left; it is the side into which the line m points. (Automobiles on m have the right-of-way over those on l.)

Thus the choice of two intersecting oriented lines l,m in α orients the plane α. Let us note the importance here of the order in which lines l,m are taken. If the order is reversed, the plane α is given the opposite orientation; for if m crosses l from right to left, then l crosses m from left to right. Thus an ordered pair of intersecting oriented lines l,m in a plane α, or alternatively an ordered pair of intersecting halflines (such a pair will be called a *bilateral*), orients the plane α. This orientation is reversed if l and m are interchanged, or if either l or m is reversed in direction. The orientation of a plane can also be described by means of an *oriented triangle* ABC, where B is the intersection of l and m and $B < A$, $B < C$ on l and m respectively. The same orientation is determined by the triangles ABC, BCA, CAB and the opposite one by the triangles ACB, CBA, BAC, so that in an oriented triangle we are interested only in the *sense* in which the triangle is traversed. In an oriented plane the area of a triangle can be given a sign, which is positive or negative according to whether or not the triangle determines the given orientation of the plane.

If for an oriented line l in a plane α we have determined which is its left side, then from the above discussion we also know which is the left side of any oriented line m intersecting l (Fig. 2); for if m crosses l from right to left, then l will cross m from left to right. The manner in which the left

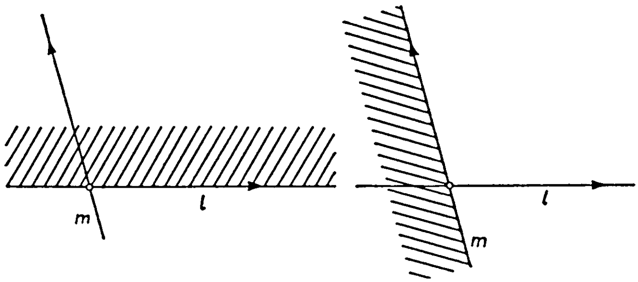

Fig. 2

side of m is determined by the left side of l is clear from the two sketches in Fig. 2. Also, it is intuitively clear that if the oriented line l moves continuously into the position of the oriented line m, its left side is "carried along with it"; i.e., its left side remains its left side in any continuous motion. (If m is parallel to l, we can determine the left side of m either by means of a third line cutting both l and m or by a continuous motion.)

Instead of an oriented line l and its left side, we may continuously transport a pair of oriented lines l,m (a bilateral), which will then constantly determine the same orientation of the plane.

Thus an affine transformation with which the identical transformation is continuously connected within the set of (nondegenerate) affine transformations takes a bilateral into another bilateral determining the same orientation of the plane. But there also exist affine transformations of the plane into itself (for example, reflections) that reverse its orientation. A given bilateral cannot be transported continuously in the plane into a bilateral determining the opposite orientation; at some stage the two lines of the moving bilateral must coincide, but then it ceases to be a bilateral.

5. From the algebraic point of view the situation is as follows; in the oriented plane let us choose an orienting bilateral, whose oriented lines can now be taken as the x-axis and the y-axis. The equation of a straight line is

$$l \equiv ax + by + c = 0.$$

Taking l as a symbol for the oriented line, we let ρl denote the same oriented line, for all $\rho > 0$, or the oppositely directed line, for all $\rho < 0$. We then take the left side of l to be the set of points (x, y) with $l > 0$, and note that under multiplication with $\rho > 0$ or $\rho < 0$ the sides are in fact preserved or interchanged.

The reader may verify that the line $l \equiv -x + y$ points from the lower halfplane into the upper, and the line $l \equiv x - y$ from the upper, and the line $l \equiv x - y$ from the upper into the lower.

Instead of operating with ordered pairs of real numbers we can also coordinatize the plane by means of the complex numbers. After choice of an oriented line as the "real axis" and assignment of its points in the usual way to the real numbers, we choose another line, perpendicular to the first, as the "imaginary axis," whose points correspond to the pure imaginary numbers. If the points 0 and 1 have been chosen, the real axis and its co-ordinatization are thereby determined. But now we must fix the position of i. Here there are the two possibilities that i may lie on the left or right of the real axis. If the given plane is already oriented, we take i on the *left* side of the real axis, traversed in the sense of increasing numbers; or conversely, we orient the plane in such a way that i lies on the left side of the real axis.

A circle centered on the origin consists of the set of points $re^{i\varphi}$ ($r > 0$ fixed, φ a real variable). If φ traverses the real axis in the positive sense (i.e., if φ increases), then $re^{i\varphi}$ traverses the circle in the sense $1, i, -1, -i$, which we agree to call the *positive sense*, where it is to be noted that the positive sense depends on the orientation of the plane. Or conversely, we may orient the plane by stating which is the positive sense of traversal on the circumference of the circle.

If the circle is traversed in the positive sense, the origin (together with the whole interior of the circle) lies to the left of the direction of traversal, i.e., to the left of the tangent directed at each point in the sense of the traversal. We have already spoken about the sense of traversal of a triangle. Here again the interior of the triangle lies in each case to the left of the positive direction of traversal. More generally, we can define a positive traversal for arbitrary convex curves; the interior must always lie to the left of the direction of traversal.

6. The situation in space is analogous. A plane divides the space into two halfspaces. Each of these two halfspaces is convex and becomes nonconvex when a single point (not on the plane) of the other halfspace is added to it. The space becomes *oriented* (left- and right-handed screws are distinguished) if for an oriented plane α we state which is its left side. Or we may choose an oriented plane α and an intersecting oriented line. Or again we may orient the space by means of a *trilateral,* i.e., an ordered triple of distinct oriented lines (for example, all of them through the same point) or of halflines. Interchange of two elements of the trilateral produces the opposite orientation, but cyclic permutation of its three elements leaves the orientation unchanged. Again, in place of all these methods, we may take an *oriented tetrahedron ABCD* (where A is the intersection of the three lines and in each case $A < B$, $A < C$, $A < D$). An even permutation of the vertices preserves the orientation of the space, and an odd permutation reverses it.

It is a remarkable fact that the space can be oriented by means of an ordered pair of oriented lines l,m, *provided* l,m *are skew.* For we have only to draw a third oriented line n intersecting l and m and pointing from l to m.

Of course, the orientation of space obtained in this way is independent of the choice of n.

A continuous rotation about an oriented line l in an oriented space can take place in either the positive or the negative sense; if we construct a plane α perpendicular to l in such a way that l passes through α from right to left, the given rotation will take place in the positive sense if it moves a point of α in the positive sense (see 5 above).

If we combine a rotation about l with a steady motion along l, we obtain a *screw*, which will be positive if the rotation about l takes place in the same sense (for example, in the positive sense) as the motion along l. The points of the space then describe helical lines like the thread on a screw. The ordinary screws of everyday life are right-handed. In the space of physics the right-handed sense is called positive.

The above discussion for the plane can be repeated here, and we can proceed analogously in higher dimensions. The n-dimensional space is oriented by an ordered set of n-oriented lines (an n-lateral), the even or odd permutations of which preserve or reverse the orientation of the space.

Cyclic Order

7. In the oriented plane it is obvious that there also exists an order among the halflines issuing from a given point (*pencil of halflines*) and that this order is different in character from the order of the points on a straight line (see 2 above). The order among the halflines is said to be *cyclic*, and the same sort of order is to be found on the face of a clock or in the cycle of months in a year. On the oriented line we were able to ask whether A comes before B or not, but we cannot ask whether noon comes before midnight or summer before winter. Of course, we can say that the sequence "morning, noon, evening," or "summer, autumn, winter" is correct and the reverse sequence is wrong; but the sequence "noon, evening, morning," for example, or "winter, summer, autumn" is also correct.

The n objects $a_1, a_2, \ldots, a_n$ can be arranged in $n!$ ways. Two arrangements such as

$$a_{i_1}, a_{i_2}, \ldots, a_{i_p}, a_{i_{p+1}}, \ldots, a_{i_n}$$

and

$$a_{i_p}, a_{i_{p+1}}, \ldots, a_{i_n}, a_{i_1}, a_{i_2}, \ldots, a_{i_{p-1}}$$

are said to be cyclically equivalent and are assigned to the same (cyclic) equivalence class.[2] To provide a cyclic order for a finite set means simply to

[2]In the symmetric group S_n this equivalence class is a left coset with respect to the cyclic subgroup generated by $\{i \to i + 1\}$.

distinguish one equivalence class among all its (cyclic) equivalence classes. If V and W are arbitrary sets with $W \subset V$, each cyclic order in V generates the cyclic order in W obtained by simply discarding all the elements not in W. To provide a cyclic order for an arbitrary set Z means providing a cyclic order for all its finite subsets in such a way that for $W \subset V \subset Z$ (where V is finite) the cyclic order in W is the one determined by V (in the subset W).

A triple a, b, c admits two cyclic orders: $abc = bca = cab$ and $acb = cba = bac$, and it can be shown that the cyclic order of any set is already determined by the cyclic order of each of its triples.

By omitting a fixed element a we can interpret a cyclically ordered set Z as an ordered set Z'; we have only to write $x < y$ if axy is a triple in the cyclic order of Z. If we do this, the transitive law does in fact hold; for if $x < y$, $y < z$, then the triples axy and ayz correspond to the cyclic order of Z, and this result admits only the cyclic order $axyz$ for the quadruple, so that $x < z$ as desired.

A cyclically ordered set Z admits an n-fold "covering," as follows. For every $z \in Z$ we define a set of elements z_i (where i is an integer mod n) and agree, for example, that for $x < y$, $z \neq a$ the order

$$a_i x_i y_i a_{i+1} z_{i+1} a_{i+2}$$

is to be cyclic, where $x < y$ is defined as just above by means of a fixed element a.

The set Z can also be ∞-times covered, but then the result is an ordered set (i.e., not cyclically ordered). To do this we define, for every $z \in Z$, a sequence of elements z_i (where i is an integer) and agree that (for $x < y$, $z \neq a$)

$$a_i < x_i < y_i < a_{i+1} < z_{i+1}.$$

These "coverings" are essentially independent of the choice of a.

8. The lines through a point in the oriented plane can be so ordered as to form a cyclically ordered set (a cyclically ordered pencil of lines); for let us orient one of these lines a arbitrarily and then orient the others in such a way that they cross a from right to left. For two such lines x, y let us set $x < y$ if and only if x is crossed by y from right to left, and then regard $axyz \ldots$ as a cyclic order if $x < y < z < \ldots$. This order is independent of the choice of the line a and of the orientation given to it, but is reversed by a reversal of the orientation of the plane.

The oriented lines or halflines through a point in the oriented plane can also be cyclically ordered, and in fact as a double covering of the cyclic order of the pencil of (unoriented) lines described above. It is easy to see how this is done.

The cyclic order of the pencil of lines or of halflines can also be called a *sense of rotation*. Orienting a plane is thus equivalent to determining a sense of rotation.

Magnitude

9. The basic statements in Euclid fall into two classes: postulates (αἰτήματα) and axioms (κοιναὶ ἔννοιαι, common notions). The postulates are geometric in nature, whereas the axioms refer to magnitudes in general.[3] The first of these statements is: "Things that are equal to the same thing are equal to each other." Nowadays we would say: equality is a two-place relation $a = b$ with the property of comparativity; namely, from $a = c$ and $b = c$ it follows that $a = b$. The words "equal to each other" imply that this relation is also symmetric; i.e., from $a = b$ it follows that $b = a$. We also assume that the relation is reflexive; i.e., every magnitude is equal to itself. (The axiom of symmetry is then superfluous.)

A relation with these properties is nowadays called an *equivalence*. Examples of such relations are: equally long, equally heavy, equally old. An equivalence relation in a set generates a partition into classes. A definite length, weight, or age is an example of equivalence class (a class of equally long, equally heavy, equally old things). But in this respect present-day language is usually somewhat careless. Concerning a segment AB, for example, people say that $AB = 3$ cm. But "3 cm" is not a segment; it is an equivalence class of segments (which are 3 cm long). A segment is not equal to an equivalence class of segments but is at most contained in it. When AB denotes a segment, we should say something like $AB \in 3$ cm.

Things can be compared not only with respect to equality but also with respect to "greater and smaller," whereupon the equivalence classes become an ordered set. But we arrive at the concept of magnitude only when we are able to add and subtract (the smaller from the greater). In general, we cannot add segments but only their lengths, i.e., we can only add equivalence classes. A system of magnitudes is thus an ordered set with an addition that has certain properties (such as commutativity). The exact definition is rather complicated, and it is easier to begin in the first place with an ordered Abelian group (IB1, §§2.5 and 2.3). Its positive elements constitute exactly what is meant by a system of magnitudes.

10. We can also take multiples of magnitudes: if x is a magnitude and n is a natural number, then $nx = x + \cdots + x$ (with n summands). Given two magnitudes, it may happen that neither of them is a multiple of the other; in fact, they do not even need to have a common multiple; for example, the diagonal and side of a square are *incommensurable*, i.e., they have no common measure and thus no common multiple.

This situation becomes quite unpleasant when we wish, for example, to prove that the areas of the rectangles $ABB'A'$ and $ACC'A'$ (with equal altitudes) are to each other as their bases AB and AC (Fig. 3); or again (Fig. 4)

[3]But the tradition on this division into postulates and axioms is by no means consistent.

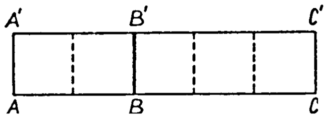

Fig. 3

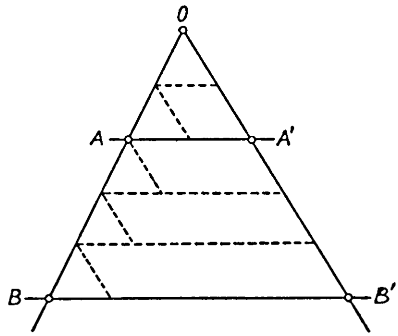

Fig. 4

that $OA:OB = OA':OB'$. If the segments are proportional to integers (i.e., if they are commensurable), it is easy to give a proof by subdividing the two rectangles and using the theorems on congruence. But how are we to proceed in general?

Eudoxus (in the Fifth Book of Euclid's Elements) avoids this difficulty in a very ingenious way. He simply states that by definition

$$a:b = a':b'$$

means that for all positive integers m and n the two relations in each of the three pairs of relations

$$ma > nb \quad \text{and} \quad ma' > nb',$$

$$ma = nb \quad \text{and} \quad ma' = nb',$$

$$ma < nb \quad \text{and} \quad ma' < nb'$$

are either both correct or both incorrect, whereupon the proof of the desired proportions follows at once.

Eudoxus continues in the natural way by defining

$$a:b > a':b'$$

to mean the existence of a pair m, n such that

$$ma > nb, \quad \text{but} \quad ma' \leqq nb'.$$

However, we are now involved in a new difficulty. If we wish to show, for example, that for

$$a < b$$

we have

$$a:a > a:b,$$

we must find m, n such that

$$ma > na, \quad \text{but} \quad ma \leqq nb.$$

If we try to do this by setting $m = n + 1$, we have

$$(n + 1)a \leqq nb$$

or

$$a \leqq n(b - a).$$

In other words, for the magnitudes a and d (with $d = b - a$) we must find a positive integer n such that

$$nd \geqq a.$$

The requirement that "for two magnitudes a and d there exists an n such that $nd \geqq a$" is called the *Axiom of Archimedes*,[4] although, disguised as a definition, it was already formulated by Eudoxus. A system of magnitudes satisfying this axiom is called an *Archimedean system.*

The concepts of Eudoxus are closely related to those of Dedekind. The ratio $a{:}b$ of two magnitudes determines two sets of rational numbers m/n such that $ma > nb$ if m/n is in the first set and $ma \leq nb$ if m/n is in the second set. These sets have the properties that Dedekind requires for the upper and lower classes of a cut. The definition given by Eudoxus for the equality of two ratios means that a cut determines *at most* one (real) number. For Dedekind a cut must also, by definition, determine at least one number. Dedekind is seeking to define the real numbers in terms of the rational numbers. On the other hand, for Eudoxus, magnitudes are already given geometrically. Unlike Dedekind, he has no need to provide a definition for $\sqrt{2}$, for example; for him this magnitude already exists as the ratio of the diagonal to the side of a square.

An Archimedean system of magnitudes is isomorphic to a subset of the system of real numbers. An Archimedean system that satisfies Dedekind's postulate is isomorphic to the system of real numbers.

11. In one respect the concept of a magnitude, formulated in this way, is still too restrictive. Angles are an example of a cyclic magnitude, at least if we count up to 360° and identify 360° with 0° (i.e., if we calculate mod 360°), so they do not form an ordered set but, like the halflines in a pencil, a cyclically ordered set (and furthermore an Abelian group). Let us look at what we mean by an angle.

Angle

12. In elementary geometry the concept of an angle is ambiguous and hazy. Euclid defines it as an inclination of lines (including curved lines) to

[4]Compare here also IB1, §3.4.

each other, where he is obviously thinking of halflines, since otherwise he could not distinguish an angle from its adjacent angle. But in the next definition he goes on to speak of the lines (straight lines) enclosing an angle, where it is clear that he is (also) thinking of a part of the plane.

Euclid does not recognize zero angles, straight angles, or reflex angles. But this procedure is often inconvenient; for example, given an obtuse angle at the circumference of a circle, what becomes of the theorem that it is half as great as the angle at the center; or what about the sum of a set of angles that add up to more than 180°?

In the theory of the measurement of angles (i.e., in goniometry) angles are considered as being at the center of a circle, say with unit diameter, and are related to the corresponding arcs (Fig. 5); in fact, the angle is even measured by the arc, with the result that, unlike line segments, angles have a natural unit of measure (the complete circumference, corresponding to 360° or 2π). Thus angle magnitudes are dimensionless.

In goniometry angles are measured, not up to 180°, but up to 360°. Then we can either go on or else neglect multiples of 360°; in other words, we can calculate mod 360° (mod 2π). But even this latter procedure, though it is the most satisfactory one from a logical point of view, does not get us out of all our difficulties; in the statement that the sum of the angles in a quadrilateral is equal to 360° it is *not* convenient to replace 360° by 0°.

Moreover, the goniometric definition of an angle deals with arcs of a circumference and not with angles between halflines. For if we are given only two halflines (with common endpoint) we cannot say which of the two circular arcs α and $2\pi - \alpha$ should be regarded as measuring the angle between them; nothing in the appearance of the halflines themselves will settle this question. Of course, we mean the arc that lies "between" the sides of the angle. But what is meant by the word "between"? Again the answer depends on which of the two sides is taken as the first. The goniometric angle is a function of the *ordered* pair of halflines, and the corresponding arc is the one that begins on the left of the first halfline and ends on the right of the second.

But again we must be cautious. The left side and the right side of a straight line are meaningful only in an oriented plane, and it is only in such a plane that angles are defined at all in goniometry.

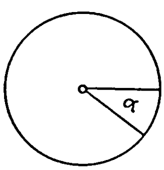

Fig. 5

The concept of an ordered pair of halflines is reminiscent of the bilateral but is more inclusive. For we now allow the halflines to coincide or to point in opposite directions, with corresponding angles 0 and π whereas in §4 matters were so arranged that a bilaterial agreeing with the given orientation of the plane corresponds to an angle between 0 and π. But if we reverse the orientation, the angle α becomes the angle $2\pi - \alpha$, which means that in an unoriented plane we cannot distinguish between these two angles. In this case it is better to deal with angles only from 0 to π, so that Euclid was quite right in not admitting greater angles. For then he would have had to begin by orienting the plane, a procedure quite foreign to his way of thought, since the choice of orientation is arbitrary. This inability to distinguish between α and $2\pi - \alpha$ can also be interpreted as meaning that an angle is no longer a function of an ordered pair of halflines but of an unordered pair, since interchange of the order of the halflines takes α into $2\pi - \alpha$.

The formula for calculating the angle α between two unit vectors x and y is

$$\cos \alpha = \mathfrak{x}\mathfrak{y}. \tag{1}$$

This formula is symmetric in $\mathfrak{x}$ and $\mathfrak{y}$. So we are dealing here with the angle between an unordered pair of vectors, in agreement with the fact that the value of the cosine does not indicate whether it comes from α or from $2\pi - \alpha$. To be sure, we have another formula

$$\sin \alpha = x_1 y_2 - x_2 y_1, \tag{2}$$

which seems to help us if we are trying to decide between α and $2\pi - \alpha$. But this help is only apparent. For the right side of (1) does not depend on our choice of (rectangular) coordinate system, whereas the right side of (2) changes sign if we replace one of the axes by the oppositely oriented line (say $x_2 \to -x_2, y_2 \to -y_2$). The choice of axes has oriented the plane, and the angle α calculated from (2) depends on this orientation. Thus formulas (1) and (2) together determine the goniometric angle in the oriented plane.

Confusion about the concept of an angle is particularly troublesome in plane analytic geometry, where it is customary to talk about the angle between two lines (instead of two halflines), so that apparently we cannot even distinguish between an angle and its adjacent angle. But again things are not so bad as they seem. In analytic geometry an angle between two lines l and m is determined by its (trigonometric) tangent, which is of period π, so the angle is determined only mod π. Let us look at this more closely.

If in the x_1x_2-plane we choose the one line l as the x_1-axis and describe the other line m by the equation $x_2 = \mu x_1$, the angle α between m and the x_1-axis is given by the formula

$$\operatorname{tg} \alpha = \mu.$$

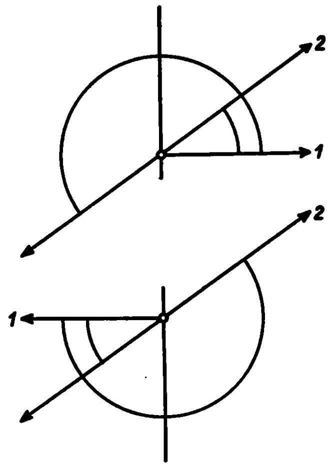

Fig. 6

We are dealing here with an ordered pair of halflines, the first of which lies on x_1-axis and the second on m. Two of the four goniometric angles thus obtained are equal to each other and the other two (also equal to each other) differ from them by π (Fig. 6).

More generally, the angle between an ordered pair of lines l,m in the sense of analytic geometry is the goniometric angle (in the oriented plane) of an ordered pair of halflines, the first of which lies on l and the second on m. This angle is determined mod π. Here the plane is oriented by the choice of coordinate system (the bilateral consisting of the positive x_1-axis and the positive x_2-axis).

In addition to these three concepts for an angle there is still a fourth, commonly used in elementary solid geometry, where it is remarkable that we speak not of the angle between a pair of (unoriented) halflines but of a pair of (unoriented) lines. The lines may be skew, but in order to determine the angle they are translated into the same plane.

Let us set up a table for these four concepts of an angle (Fig. 7).

The reader should not assume from this table that in analytic geometry, for example, it would be impossible to consider any other concept of an angle. On the contrary, we have already seen in §5 that an oriented line can be defined in analytic geometry, so that the goniometric concept is quite possible there. Similarly, in solid geometry we could very well consider the angle between halflines (see 14 below); the table (Fig. 7) merely represents the usual procedure in elementary instruction.

13. The angles of elementary geometry form an ordered set in which

<table>
<tr><td colspan="4">The angle in</td></tr>
<tr><td>Goniometry</td><td>Elementary Geometry</td><td>Analytic Geometry</td><td>Solid Geometry</td></tr>
<tr><td colspan="4">between an</td></tr>
<tr><td>ordered</td><td>unordered</td><td>ordered</td><td>unordered</td></tr>
<tr><td colspan="4">pair of</td></tr>
<tr><td>halflines</td><td>halflines</td><td>lines</td><td>lines</td></tr>
<tr><td colspan="4">in an</td></tr>
<tr><td>oriented</td><td>unoriented</td><td>oriented</td><td>unoriented</td></tr>
<tr><td colspan="4">plane is defined</td></tr>
<tr><td>mod 2π</td><td>between 0° and 180°</td><td>mod π</td><td>between 0° and 90°</td></tr>
</table>

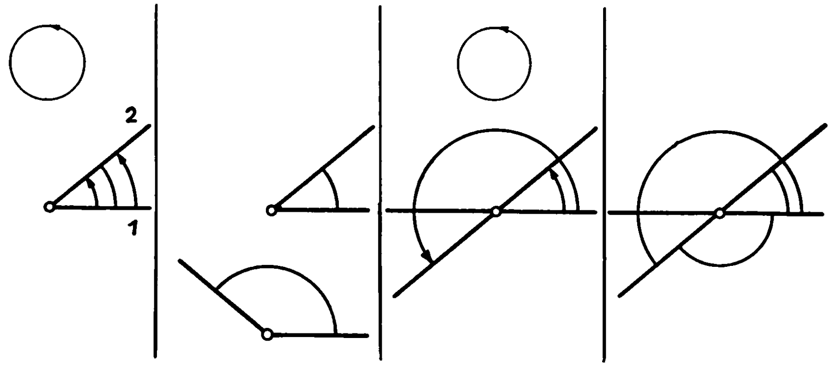

Fig. 7

addition is not yet unrestrictedly possible, and calculation with angles greater than 180° is rather hard to justify. The system of angles mod 360° is a double covering of the system mod 180° and is cyclically ordered in the same way as the lines or halflines in a pencil. The ∞-fold covering produces exactly the kind of angles needed in the statement that the sum of the angles in n-gon is $(n - 2)\pi$.

14. If a space is already oriented, its planes are not necessarily oriented thereby. On the contrary, by a rotation in space it is possible to take an oriented plane into the plane with opposite orientation. Thus we cannot meaningfully define an angle mod 2π for an ordered pair of halflines of a plane in space; the angles α and $2\pi - \alpha$ are necessarily indistinguishable. Similarly, we cannot define the angle mod π between two coplanar lines in space; i.e., we cannot distinguish between an angle and its adjacent angle.

The situation is quite different if we confine our attention to skew lines or halflines l,m in oriented space. A plane ε parallel to l and m can be oriented by postulating that an oriented line n (intersecting l and m and pointing from l to m) crosses ε from left to right (see also §6.) The lines l and m can then be translated into this oriented plane so as to determine the various types of angles; in particular, the "goniometric" angle is the angle by which l must be twisted, in the direction prescribed by the orientation of the space, in order to reach m.

In oriented space the angle φ between the line l and an oriented plane α can be reduced by definition to the angle ψ between l and the normal to α oriented toward the left side of α, and the angle between two oriented planes can be defined analogously.

Spaces of higher dimension give rise to complications. The relative position of two nonparallel planes in four-dimensional space can no longer be described in terms of *one* angle.

15. Up to now we have considered an angle as a magnitude, in agreement with Euclid's first notion of it as the "inclination of two lines." But other procedures are possible. Compare, for example, our treatment of a line segment, not as a length, but as the set of points between its two endpoints, which completely determine the segment. Similar possibilities are available for angles, if we wish to avoid considerations of magnitude altogether.

Thus two intersecting lines l,m determine four angles (*sets* of points) in the plane, all of them logically on a equal footing. But if in an oriented plane we consider l,m as an ordered pair of oriented lines we can assign a *unique* angle to this pair, namely, the set of points to the left of l and (at the same time) to the right of m. All the angles defined in this way are convex, since they are the intersections of two (convex) halfplanes, but it is easy to see how we may introduce nonconvex angles (Fig. 8).

Area and Volume

16. In elementary instruction, areas and volumes are introduced numerically, i.e., as numbers for which certain rules of calculation are pre-

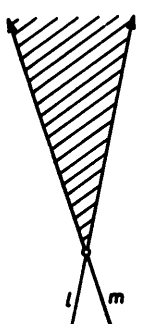

Fig. 8

scribed. But in Euclid, and to some extent in the schools today, the interest lies not in calculating areas and volumes but in comparing them. Among Euclid's axioms (more precisely, his κοιναὶ ἔννοιαι—common notions), in addition to those concerning the general notion of magnitude there is one that reads: "Things that can be superposed on each other are equal."

Thus we consider congruent figures to be equal in area or volume. But we also make use of Euclid's axiom: "If equals are added to equals, the wholes are equal," and are thereby led to Hilbert's concept of *decomposable equality* (Zerlegungsgleichheit): two figures that can be decomposed into pairwise congruent figures are said to be equal, like the rectangle and rhombus in Fig. 9. But this concept is not yet adequate if we wish to prove, for example, that parallelograms with equal bases and altitudes are equal in area. The method is successful for the parallelograms $ABDC$ and $ABD'C'$ in Fig. 10, each of which is the sum of the same trapezoid and of congruent triangles, but it will no longer work for Fig. 11, where we must argue differently: the parallelograms can be obtained by subtraction from the trapezoid $ABD'C$; in the first case by subtraction of the triangle BDD' and in the second of the congruent triangle ACC'. Here we are making use of Euclid's axiom "If equals are subtracted from equals, the remainders are equal," and are thereby led to Hilbert's concept of *supplementwise equality* (Ergänzungsgleichheit): two figures are said to be supplementwise equal if by the adjunction of decomposably equal figures they can be supplemented in such a way as to become decomposably equal. (The case of Fig. 11 could also be dealt with by reducing it to Fig. 10 by means of a step-by-step insertion between the two parallelograms of a sequence of parallelograms each of which is in the same position with respect to the next as the two parallelograms in Fig. 10; but then it would be necessary to make use of the Axiom of Archimedes, without which the concept of supplementwise equality is actually more inclusive than that of decomposable equality.)

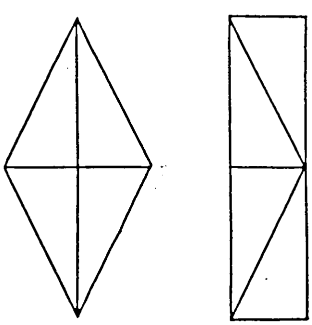

Fig. 9

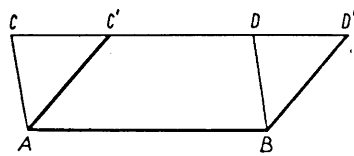

Fig. 10

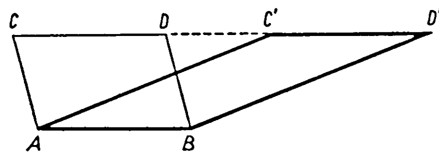

Fig. 11

Since the relation of supplementwise equality has the properties of an equivalence, we can combine supplementwise equal figures into a class, which we shall call the *area* (or in three dimensions, the *volume*) of these figures. For these areas (equivalence classes) we can now define an order relation and an addition, whereupon they will form a system of magnitudes. But this system is not yet altogether satisfactory, since up to now we do not even know whether or not all geometric figures have the same area. In fact, some extremely pathological cases can arise. For example, it is possible to decompose the surface of a sphere into three congruent sets such that two of them, rearranged in a suitable way, again produce the whole surface of the sphere.[5]

But if as "figures" we admit only planar polygons we can show that all such figures are supplementwise equal to rectangles with a fixed altitude and that two such rectangles can be supplementwise equal to each other only if they have the same base. The base of the corresponding rectangle can then be taken as a measure for the area of the figure, whereby we return to the elementary notion of area.

But in space, with its polyhedral surfaces, this method is no longer successful. Two pyramids with bases of equal area and with equal altitudes are no longer necessarily supplementwise equal to each other.[6] In order to establish a theory for the volumes of polyhedra it is customary in the schools to refer to a principle usually named after Cavalieri but already to be found in Democritus and Archimedes; namely, if two three-dimensional figures are such that their intersections with any plane parallel to a given plane are equal in area (supplementwise equal), then the figures themselves are equal in volume (*Cavalieri equal*). This concept, together with the concept of supplementwise equality in space, is sufficient for the theory of volumes of polyhedra in space.[7]

The *Cavalieri principle* can in its turn be based on a passage to the limit, and such limiting processes are necessary if we wish to consider figures bounded by curved lines.

The problem of showing that the areas of circles C, C' are to each other as the squares of their radii r, r' is hardly more difficult than the proof that the areas of rectangles with the same height are to each other as their bases. For example, if we had $C:C' > r^2:r'^2$, there would exist positive integers m,n such that

$$mC > nC', \text{ but } mr^2 \leqq nr'^2 \tag{3}$$

(see 10), and we could find a regular polygon inscribed in C with an area V

[5]F. Hausdorff, Grundzüge der Mengenlehre, 1. Aufl., Berlin-Leipzig 1914, S. 469. J. von Neumann, Fundamenta Math. (1929), 73–116.

[6]M. Dehn, Math. Ann. 55 (1902).

[7]W. Süss, Math. Ann. 82 (1921), 297–305.

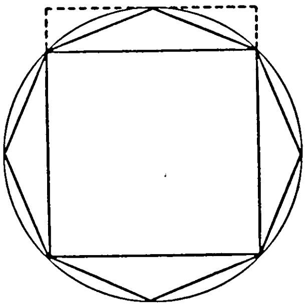

Fig. 12

such that also

(4) $$mV > nC'.$$

Such a polygon can be found in the following way: let V_i be the area of the inscribed regular polygon with 2^i-sides. Then it is easy to see (Fig. 12) that

$$C - V_{i+1} < \tfrac{1}{2}(C - V_i),$$

and thus

$$C - V_p < 2^{-(p-2)}(C - V_2).$$

The axiom of Archimedes now guarantees the existence of a q such that

$$2^{q-2}(mC - nC') \geqq m(C - V_2).$$

For this q,

$$(mC - nC') > m(C - V_q),$$

and thus

$$mV_q > nC',$$

so that (4) is satisfied with $V = V_q$. Then if V' is the area of the regular 2^q-gon inscribed in C', we have

$$mV > nV',$$

and, on the other hand,

$$V : V' = r^2 : r'^2,$$

in contradiction to the second half of (3). The assumption $C:C' > r^2:r'^2$ is thus refuted, and similarly for the opposite inequality.

Here we have made use of continuity only in the sense of the axiom of Archimedes, but if we wish to show that for a given circle, for example, there exists a rectangle, with prescribed altitude, that is equal in area to the circle, we must make use of Dedekind continuity,[8] and in fact in analysis, the area of arbitrary figures is treated systematically from a modern point of view.

17. Up to now we have based the concept of area on congruence ("Things that can be superposed on each other are equal"). But this is not the only possibility, and from the algebraic standpoint it is not even the most convenient. The concept of area is less closely related to congruence than to affine transformations, under which the ratio of areas is invariant.

We first replace the requirement of invariance of area under congruence by *invariance of area under translation,* since translation is itself an affine-invariant concept. The area of a parallelogram $ABCD$ in the plane is then completely determined by the vectors AB and AD; it is a function $f(\mathfrak{a}, \mathfrak{b})$ of the two vectors $\mathfrak{a}, \mathfrak{b}$ forming the sides. We next require, as is natural, that if one side is multiplied by a factor c (multiplication of a vector by a number is also an affine-invariant concept) the area is thereby multiplied by the same number, and then it is desirable to admit negative factors, which lead to the concept of negative areas, removing the difficulty that the side of a parallelogram determines not one vector but rather two (opposite) vectors. But what does it mean intuitively that the parallelograms $ABCD$ and $ABC'D'$ (Fig. 13) have opposite areas? We see that in the plane they determine opposite orientations, so that the area must be a function of an *ordered* pair of vectors and must change sign with interchange of the vectors:

$$f(\mathfrak{a}, \mathfrak{b}) = -f(\mathfrak{b}, \mathfrak{a}), \tag{5}$$

and we must also have

$$f(c\mathfrak{a}, \mathfrak{b}) = f(\mathfrak{a}, c\mathfrak{b}) = cf(\mathfrak{a}, \mathfrak{b}). \tag{6}$$

Furthermore, the parallelograms $ABDC$ and $CDFE$ taken together (Fig. 14) have the same area as $ABFE$; for we may subtract the triangle BDF, move it to the position ACE and then add it again. Now the three parallelograms have one side ($AB = CD$) in common, and the fourth side of $ABFE$ is the

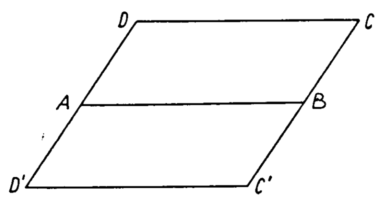

Fig. 13

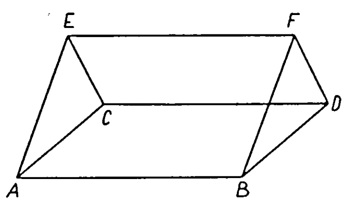

Fig. 14

[8] See also H. Hasse and H. Scholz, Die Grundlagenkrisis der Mathematik. Pan-Bücherei, Gruppe: Philosophie, Nr. 3, 1928.

sum of the corresponding sides of $ABDC$ and $CDFE$; i.e., for the vector AE we have $AE = AC + CE$. So we must also require:

$$f(\mathfrak{a}, \mathfrak{b} + \mathfrak{b}') = f(\mathfrak{a}, \mathfrak{b}) + f(\mathfrak{a}, \mathfrak{b}') \tag{7}$$

and

$$f(\mathfrak{a} + \mathfrak{a}', \mathfrak{b}) = f(\mathfrak{a}, \mathfrak{b}) + f(\mathfrak{a}', \mathfrak{b}). \tag{8}$$

The equations (5) through (8) can be summed up as follows: area, regarded as a function of two vectors, is antisymmetric and linear in each of its arguments.

But this function is not completely determined until we have chosen a unit of area. To do this, we take an arbitrary parallelogram, with the vectors $\mathfrak{e}$ and $\mathfrak{f}$ for sides, and set and

$$f(\mathfrak{e}, \mathfrak{f}) = 1, \tag{9}$$

by definition. The unique existence of such a function f is shown in algebra (IB3, §3.4).

18. The procedure in three-dimensional space (or in n-dimensional) is entirely analogous, the analogue of a parallelogram being a parallelepiped (in n dimensions, a parallelotope). By definition, the volume is an antisymmetric function of three (or n) vectors, linear in each of its arguments, and uniquely determined by means of a standard figure.

Groups

19. How many of the elements of a triangle are necessary to determine it completely? In elementary geometry the answer is three, but in plane analytic geometry it is the six coordinates of the three vertices. How does the contradiction arise?

With the six coordinates of the vertices we determine the triangle not only in shape and size but also in position (with respect to the coordinate system). But in elementary geometry we often regard a triangle as already constructed if, in a class of congruent triangles, we have found one triangle that satisfies the requirements of a given problem. But even in elementary geometry the usage varies. Consider the two theorems:

1. A triangle is completely determined by the lengths of its three sides.
2. Two triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $BC = B'C'$, $CA = C'A'$.

If the word "triangle" is to have precisely the same meaning in these two sentences, then the second sentence, while not in contradiction to the first, is trivial and superfluous; for by the first theorem the two triangles ABC and $A'B'C'$ are already identical with each other.

But in fact the word "triangle" means something quite different in the two theorems. In the first theorem (and more generally in many construction problems) congruent triangles are regarded as not essentially different, or, as it is usually expressed nowadays, the word "triangle" means a *class* of congruent triangles, in the sense of the second theorem.

Construction of classes is a logical process of widespread usefulness. Usually it rests on an equivalence relation, i.e., on a two-place relation $(\cdots \sim \cdots)$ with the properties that $a \sim a$ and that $a \sim c$, $b \sim c$ implies $a \sim b$. In a set in which such a relation has been established among the pairs of elements, we can combine equivalent elements into a "class." Since congruence is an equivalence, congruent figures in the plane can be combined into classes. A congruence class of segments is simply a length, and a congruence class of triangles is a "triangle" in the sense of (most) construction problems. Similarity and equality of area are other concepts often regarded as equivalences.

20. From the logical point of view a geometry is a system of elements (i.e., the elements of a set) and relations among them. The elements may (intuitively) be points, lines, circles, angles, distances, and so forth. The relations may be one-place (X is a point), two-place (X is incident with Y), three-place (Y lies between X and Z), or four-place (the distance from X to Y is the same as from Z to U), and so forth.

To every geometry there belongs a group, its *automorphism group*, i.e., the totality of all mappings of the set of elements onto itself under which all the relations are preserved.

For example, if we regard the plane as a set of points and for every ρ consider the relation "X and Y are at a distance ρ from each other," we obtain the group of *rigid mappings* (direct and opposite isometries), in which two points at a distance ρ are taken into two points at the same distance ρ. But if for our relation we take "X and Y are the same distance apart as Z and U," we obtain the group of *similarities*, namely, the transformations that leave invariant the ratios of distances.

Conversely, a geometric concept of equivalence (see 19) often depends on a *group of transformations* G. Two figures Φ and Φ' are said to be equivalent if there exists a transformation f in G taking one of them into the other; thus

$$\Phi \sim \Phi' \text{ if and only if } f\Phi = \Phi'$$

for a suitably chosen $f \in G$. From the axioms for a group it follows that this relation is actually an equivalence.

21. We have already mentioned the group B of rigid mappings obtained by transforming the plane (or space) as a rigid body, i.e., by requiring that distances remain invariant. If for a triangle ABC (or tetrahedron $ABCD$) we prescribe the position of its (congruent) image $A'B'C'$ ($A'B'C'D'$), the transformation is completely determined, but if only the segment AB (the triangle ABC) has a prescribed image $A'B'$ ($A'B'C'$), then for the image

$C'(D')$ of a point $C(D)$ not on the line AB (the plane ABC) two positions are still possible, one on each side of AB (ABC). However, if we require that in addition to distances the transformation f must leave invariant the orientation of the plane (or space), there is only one possibility for $C'(D')$; by prescribing the image of AB (ABC) we have already completely determined the transformation f. The set of rigid mappings that preserve orientation forms a subgroup B_0 of B. (For the definition of B_0 it is of no importance *how* we orient the plane (or space); all that matters is the fact that we *can* orient it.)

22. "This segment is 3 cm long" and "This parallelogram has an area of 12 cm^2" are statements invariant under the group B. Nevertheless, such statements are hardly regarded as being part of geometry (but rather of geodesy). When they occur in geometry, they are regarded as references to a certain unit of measurement (1 cm, 1 cm^2), which may in fact be chosen in a completely arbitrary way (on a blackboard usually about 10 times as large as in a notebook). Consequently the group B of rigid mappings is much less important than the group A of *similarities,* i.e., transformations that leave invariant the *ratios* of distances (and therefore angles, and ratios of areas). If to a triangle ABC (tetrahedron $ABCD$) we assign its (similar) image $A'B'C'$ ($A'B'C'D'$), it is again true that the corresponding $f \in A$ is completely determined. Here also we can add the requirement of invariance of the orientation of the plane (or space) and thus arrive at a subgroup A_0 of A.

23. By parallel projection we can map a plane onto another plane. A pair of parallel lines is then taken into a pair of parallel lines, and parallel segments are multiplied by the same factor. If by a further sequence of parallel projections we finally bring the images back into the original plane, we obtain a mapping in the plane which preserves parallelism and the ratios of parallel segments. Such a mapping is said to be *affine*. The affine mappings of the plane onto itself form a group F. If we prescribe the image $A'B'C'$ of a triangle ABC, the corresponding affine mapping $f \in F$ is thereby completely determined. For it follows from the invariance of ratios of segments that every point on the lines AB and AC has a predetermined image on $A'B'$ and $A'C'$, while an arbitrary point X can be regarded as the vertex of a parallelogram AB_1XC_1 (B_1 on AB, C_1 on AC) and its image X' as a vertex of the corresponding parallelogram $A'B_1X'C_1$ (Fig. 15).

The group A of similarities is a subgroup of the group F of affine transformations and A is certainly a proper subgroup of F, since under the mappings of A all ratios of segments remain invariant, whereas under F (in general) only the ratios of parallel segments remain invariant. The group F has another subgroup F_0 consisting of those affine transformations that preserve orientation. Ratios of areas of parallelograms remain invariant under all the transformations of F, since their definition depends only on parallelism and on the ratios of intervals, both of which are affine-invariant.

Affine transformations in space are defined in exactly the same way as in

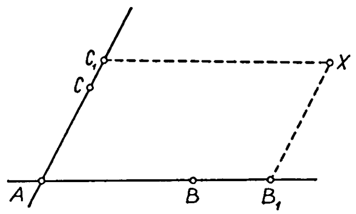

Fig. 15

the plane; a tetrahedron and its image completely determine an affine transformation. Ratios of volumes of parallelepipeds are invariant, but this statement is not necessarily true for areas of parallelograms in nonparallel planes or for nonparallel segments.

24. If a plane is mapped by a central projection (i.e., a perspectivity) onto another plane, straight lines will be mapped into straight lines, as will again be the case if we project back (from another center) onto the original plane. Yet there are difficulties here. Even for one projection there will be points that have no images, and points in the image plane that are not images of any point, i.e., when the projecting ray is parallel to one of the planes. In order to avoid these "exceptions," we supplement the plane by *ideal* points (parallel lines being considered to pass through the same ideal point) and thereby obtain the *projective plane.*

Let us take four points A, B, C, D in the plane, no three of them on the same straight line, draw the six lines joining them in pairs, construct the intersections of these lines, and then proceed by successively joining points and taking the intersections of lines. The configuration thus obtained is called the *Möbius net* (for A, B, C, D). It does not contain every point in the plane; for example, if A, B, C, D have rational coordinates, then only points with rational coordinates can be obtained. But the points of the net come arbitrarily close to every point of the plane.

In a mapping φ of the projective plane that takes lines into lines in such a way that the points A, B, C, D have prescribed images A', B', C', D' (no three of them on the same straight line), every point of the Möbius net for A, B, C, D, will have a unique image in the net for A', B', C', D'. If φ is continuous, the φ-image of every point in the plane is uniquely determined.

A continuous mapping of the projective plane onto itself that takes lines into lines is called a *projectivity.* If we prescribe the images of four points in general position, the corresponding projectivity is completely determined.

In space the situation is precisely analogous; a projectivity is completely determined by prescribing the images of five points in general position.

Adjunction of the ideal elements to the projective plane (space) destroys the order properties of the ordinary plane (space). Order on the projective line is the same as in a pencil of lines; in other words, it is cyclic. A point does not divide the projective line into two parts, a line does not divide the projective plane, and a plane does not divide projective space.

It is possible to construct a model of the projective plane that lies entirely in a bounded part of the Euclidean plane, in the following way. From the center M of a sphere let us project the projective plane onto the sphere, delete the northern hemisphere as being superfluous, identify diametrically opposite points of the equator (since they correspond to the same point of the projective plane) and then project vertically onto the equatorial plane, thereby providing a model of the projective plane in the form of a circular disk with identification of opposite points on its boundary.

In the same way the ball (solid sphere) with identification on its boundary is a model of projective space.

25. From Euclid up to the end of the 19th century, every textbook in geometry began with definitions like "A point is that which has no part," and these definitions were followed by axioms (or postulates) like "A straight line may be drawn from any point to any point" and "If equals are added to equals, the wholes are equal." In general, such so-called definitions played no role in any proof, and many more axioms were used than were actually stated. A reasonably complete system of axioms for Euclidean geometry is to be found for the first time in Pasch (1882),[9] who explicitly requires that "the process of deductive proof must everywhere be independent, not only of the figures, but of the meaning of the geometric concepts. Poincaré (1902), in his description of Hilbert's "Grundlagen der Geometrie," expresses himself more brusquely: we must be able to insert the geometric axioms into a machine, which will then produce the whole of geometry.

The question "What is space?" was much debated in the 19th century and was finally settled by Hilbert. The answer is that this question is of no concern to the geometer. What are points, lines, and circles? The answer is that the meaning of these words is implicitly determined by the axioms in which the words occur. Whether there exists anything in nature that satisfies the axioms and what it looks like are questions for the physicist, not the mathematician. The mathematician requires from his system of axioms only that it shall not produce contradictions; but the physicist requires that it shall have useful applications to the external world.

The axiomatic method goes further; it investigates the mutual relationship of the axioms. For example, can we omit Axiom 7, i.e., can we deduce it from the other axioms? Or is it independent of the others? To prove its independence, we must construct a consistent system in which all the axioms except Axiom 7 are valid and Axiom 7 itself is false. It was in this way that the independence of the parallel axiom was proved by means of non-

[9]Vorlesungen über neuere Geometrie. Leipzig 1882. 2. Aufl. Berlin 1926.

Euclidean geometry, and the method has given rise to a great number of new geometries.

The axiomatic method has proved valuable in many fields; it brings into clear relief exactly what is necessary in the proof of a theorem and often allows us to combine many branches of mathematics into one.

But we must note that the natural approach does not begin with axioms. We discover certain relations in the physical world, or in some already mathematicized system, and register their properties. When we have carried this process sufficiently far, we select as "axioms" some of the properties that appear to be of fundamental importance, and then we operate with this system of axioms in a purely deductive way; that is, we draw conclusions from it. The resulting mathematical system can then be applied, either in the situation from which it was originally derived, or elsewhere. Discrepancies between the consequences of the mathematical theory and the observations of the physicists do not demonstrate any error, either in mathematics or in physics; they only show that the mathematical theory does not fit the observable world.