A Wave Theory of the Electron*†

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Modern physics, in its attempts to solve the quantum theory dilemma, is in turn attacking all of our most cherished conceptions. The wave theory of light, the conservation of energy, our mechanics, our electrodynamics, and even our methods of calculus have been assailed, separately and collectively, as either entirely wrong or at least inapplicable to systems of small dimensions, that is to the electron. The usual quantum mechanics are based, however, on the conception of the electron as a *point*. It is therefore not astonishing that we are sometimes unable to give the exact location of this point.[1] It is, to be sure, explained that we are dealing merely with the geometrical center of a *spherical electron*. But the spherical electron itself is a fiction assumed for lack of a better knowledge. Indeed it would seem *a priori* probable that a better knowledge of the structure of the electron would lead to changes in the mechanics and electrodynamics of its motion which would afford at least a partial solution of the dilemma in which we find ourselves. A first step in this direction has been taken by de Broglie,[2] and that part of the present paper dealing with stationary states is little more than a new presentation of his ideas.

De Broglie has shown that the standard quantum formula

$$W = h\nu \tag{1}$$

is not invariant to changes of coordinates in space-time, but that we do obtain an invariant relation if we also write

$$\mathbf{p} = h\boldsymbol{\mu} \tag{1'}$$

where $|\mu|$ is the wave-number of a wave train of frequency ν, and μ is a vector normal to the wave-surfaces whose magnitude is $|\mu|$. Equations (1)

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[1] Max Born, Problems in Atomic Dynamics, p. 69.

[2] Louis de Broglie, Ann. de Physique, 10, III, p. 22.

and (1') express the proportionality of two tensors, the factor of proportionality being Planck's constant h. If, now, W and $\mathbf{p}$ are the energy and momentum of a spherical electron, ν and μ will be the frequency and wave-number of a wave train whose exact nature is not yet well understood. We shall consider this wave train to be the electron,[3] the laws obeyed by W and $\mathbf{p}$ being consequences of the mode of propagation of this wave train. We shall, of course, have to make some assumption as to this mode of propagation. We might introduce arbitrarily a wave equation as Schrödinger has done[4] but as this step is not necessary for our purpose, we shall merely assume a mode of propagation such as to give the usual equations of mechanics for W and $\mathbf{p}$.

Consider the electron to go from A and B, and imagine all possible wave trains going from A to B. The phase of any wave train at B, at time t, will be

$$\phi = 2\pi \int_A^B (|\mu|\, ds - \nu\, dt). \tag{2}$$

The values of $|\mu|$ are to be determined from the kinetic energy in the same manner as the p (divided by h) would be determined for a point-electron, although the kinetic energy no longer has quite the same significance. The integral is to be taken along the "ray." If these wave trains were in phase at A at time t_0, they will, in general, not be in phase at B at time t. There will, however, be a certain wave train for which the variation of the above integral is zero, and this wave train has the property that wave trains following nearly the same path will be very nearly in phase with it at B. Other wave trains will have their phases at B distributed over wide ranges. These will, as in optics, destroy one another, and need therefore not be considered. Indeed, diffraction effects, in which the latter waves would play a part, cannot occur with the electron-waves as there are no edges to diffract them. Interference effects, however, do occur. We need therefore only consider a certain wave-bundle[5] clustered around that particular wave train which is determined by

$$\delta\phi = 2\pi \int_A^B \delta(|\mu|\, ds - \nu\, dt) = 0. \tag{3}$$

This central wave train follows the path which we would, from mechanics, expect the point-electron to follow, for the above equation, in mechanical quantities, expresses the Principle of Least Action. The waves destroy each other everywhere excepting in the neighborhood of this path.

[3] E. Schrödinger, Phys. Rev., 28, p. 1049.
[4] E. Schrödinger, Ann. der Physik, 79, 6, p. 489.
[5] E. Schrödinger, l. c., p. 500.

Let now ν and μ be the frequency and wave-number of the central wave train. Its phase is given at all points (not necessarily on the ray) by the vector integral

$$\phi = 2\pi \int (\mu \cdot \mathbf{ds} - \nu \, dt) \tag{4}$$

taken along *any* path *within* the region occupied by the wave train. If the component wave trains of the bundle are distributed symmetrically about this central wave train the resultant disturbance can be represented by a sum of the form

$$\begin{aligned} \sum \Big[\frac{\xi}{2} \sin 2\pi \int \Big(\mu \cdot \mathbf{ds} + \Delta_\xi \mu \cdot \mathbf{ds} - \nu \, dt - \frac{\partial \nu}{\partial \mu} |\Delta_\xi \mu| \, dt \Big) \\ + \frac{\xi}{2} \sin 2\pi \int \Big(\mu \cdot \mathbf{ds} - \Delta_\xi \mu \cdot \mathbf{ds} - \nu \, dt + \frac{\partial \nu}{\partial \mu} |\Delta_\xi \mu| \, dt \Big) \Big] \\ = \sum \xi \cos 2\pi \int \Big(\Delta_\xi \mu \cdot \mathbf{ds} - \frac{\partial \nu}{\partial \mu} |\Delta_\xi \mu| \, dt \Big) \sin 2\pi \int (\mu \cdot \mathbf{ds} - \nu \, dt) \end{aligned} \tag{5}$$

where $\Delta_\xi \mu$ is the vector difference between the wave-number vector of the wave train denoted by ξ and that of the central wave train. In this form we see that we can speak without ambiguity of the frequency, wave-number, and phase of the electron, for the wave-bundle is equivalent to a simple wave train whose phase is given by (4) and whose amplitude is

$$\psi = \sum \xi \cos 2\pi \int \Big(\Delta_\xi \mu \cdot \mathbf{ds} - \frac{\partial \nu}{\partial \mu} |\Delta_\xi \mu| \, dt \Big). \tag{6}$$

This represents a "wave group" travelling with the velocity

$$v = \frac{\partial \nu}{\partial \mu} = \frac{\partial W}{\partial p} = \dot{q}. \tag{7}$$

The coefficients ξ are functions of the electric and magnetic fields and of the frequency ν so that the "group" may have different "shapes" at different times. It would, indeed, seem to be considerably elongated in the direction of the motion. It is this "group," essentially, which constitutes the electron, for it moves with the velocity q which is observed experimentally as the "velocity of the electron." The charge, though it is probably spread over the group without a definite boundary, perhaps proportionally to the amplitude ψ, forms an inseparable unit e[6]; and its field obeys Maxwell's equations.

Equations (1) and (1′) can be more explicitly written

$$\begin{aligned} h\nu - eV &= mc^2 \\ |h\mu - e\mathbf{A}| &= c\sqrt{m^2 - m_0^2} \end{aligned} \tag{8}$$

[6] De Broglie, l. c., p. 73.

where V is the electrostatic potential due to *outside* charges, and $\mathbf{A}$ is the magnetic vector potential. In this paper we shall only consider cases in which there is no magnetic field, and may therefore set $\mathbf{A} = 0$. The velocity of the electron is then

$$v = \frac{\partial \nu}{\partial \mu} = \frac{\partial \nu}{\partial m}\frac{\partial m}{\partial \mu} = c\sqrt{1 - \left(\frac{m_0}{m}\right)^2} = \beta c,$$

whence

$$\frac{m_0}{m} = \sqrt{1 - \beta^2}. \tag{9}$$

If m is eliminated between equations (8) we obtain

$$c^2\mu^2 = \left(\nu - \frac{eV}{h}\right)^2 - \nu_0^2, \tag{10}$$

where

$$h\nu_0 = m_0c^2$$

Let us now apply these ideas to motion in a circle, such as in the circular orbits of the hydrogen atom. If we use polar coordinates μ will be expressed in waves per radian. The electron will move with the angular velocity $\partial\nu/\partial\mu$, and, if the orbit is sufficiently large, it will, by classical electrodynamics, radiate light of frequency

$$N = \frac{1}{2\pi}\frac{\partial \nu}{\partial \mu} \tag{11}$$

so that the energy W, and hence the frequency ν, of the electron will slowly decrease. As in classical theory the orbit will be a spiral of small angle. If, however, the orbit is small the "wave electron" may reach all the way around it so that the electron effectively becomes a ring electron and therefore does not radiate. It cannot, however, do this in all small orbits for in some of these it will interfere with and destroy itself. The waves will, on the other hand, add if

$$\int_0 \mu\, d\theta = 2\pi\mu = n = \text{a whole number.} \tag{12}$$

The existence of stationary states thus seems to be explained, and the Bohr quantum condition appears as a kind of resonance condition; and as there is no radiation and therefore no damping the resonance phenomena should be very sharp.

Moreover the normal state of the hydrogen atom has a magnetic moment of one Bohr magneton, as recent experiments[7] seem to require.

[7] T. E. Phipps and John B. Taylor, paper presented at the Chicago meeting of the American Physical Society.

De Broglie has shown[8] how to take account of the motion of the nucleus and this point will not be discussed here.

Multiple periodic motions can be treated similarly. Let us consider them first from the usual point of view, that of the point electron. The orbit does not close but passes, to within any degree of approximation, through every point of a certain limited region of space. If we attach to every point a vector $\mathbf{p}$ equal to the momentum of the point electron as it passes that point we shall have defined a vector field. As $\mathbf{p}$ is multiple-valued it is convenient to consider space as composed of several sheets, as a Riemann surface, on each of which $\mathbf{p}$ is single-valued. This vector field has been studied by Einstein,[9] and he has shown that it is derivable from a potential, the integral J^* of the Hamilton-Jacobi partial differential equation. Consequently the integral $\int \mathbf{p}\cdot\mathbf{dq}$ has the same value when taken around closed curves which can be brought into coincidence by a continuous process. In particular this integral will be zero when taken around a curve which does not enclose an excluded region or a singular point. On the other hand it will, in general, have different values when taken along paths which cannot be brought into coincidence continuously. Einstein uses this property to write the Sommerfeld quantum conditions in an invariant form.

We have seen that the wave electron follows the same path as the point electron above, and that the wave extends a certain distance on both sides of this path. The conditions for noninterference is then that the waves along portions of the orbit which are close to each other be in phase. The space must be considered to consist of several sheets, for the non-interference condition does not apply to portions of the wave train passing the same point but going in different directions. If the non-interference condition is satisfied, and the wave train is long enough, a coherent wave will fill the entire space occupied by the mechanical orbit. The phase of this wave will be given by (4), or for stationary states by

$$\phi = 2\pi\left(\int \mu\cdot\mathbf{ds} - \nu t\right). \tag{4'}$$

For the wave to be coherent the integral, taken between fixed limits, must be independent of the path, at least to within a whole multiple of 2π. In particular for closed paths it must be zero, or a whole multiple of 2π. That is:

$$\int_0 \mu\cdot\mathbf{ds} = 0, n, n', \cdots \text{whole numbers}. \tag{12'}$$

[8] De Broglie, l. c., p. 34.
[9] A. Einstein, Ber. d. Deutschen Phys. Ges. 1917, p. 82.

If μ is replaced by $\mathbf{p}/h$ this becomes the quantum condition as formulated by Einstein.

The simplest example of this type of motion is found in the hydrogen atom. The wave is confined to the neighborhood of a plane, for the same reason that the orbit of the point electron is plane, and we shall assume a solution with cylindrical symmetry. Equation (10) for this case is

$$c^2\mu^2 = \left(\nu + \frac{e^2}{hr}\right)^2 - \nu_0^2 \tag{13}$$

and we shall suppose $\nu < \nu_0$ as the contrary assumption leads to hyperbolic orbits. As the center is a singular point one of the conditions (12′) will be given by an integral taken along a path around the center. We shall choose a circular path of radius r, and let μ_θ be the component of μ along this path. We then have

$$\int_0 \mu_\theta \, d\theta = 2\pi r \mu_\theta = k$$

whence

$$\mu_\theta = \frac{k}{2\pi r}. \tag{14}$$

Substituting in (13) we obtain

$$\mu_r = \pm\sqrt{\mu^2 - \mu_\theta^2} = \pm\sqrt{\frac{\nu^2 - \nu_0^2}{c^2} + \frac{2e^2\nu}{c^2hr} + \left(\frac{e^4}{c^2h^2} - \frac{k^2}{4\pi^2}\right)\frac{1}{r^2}}. \tag{15}$$

Inasmuch as $2\pi e^2/ch = a < 1 \leq k$, the expression under the radical has two positive roots, and μ_r is real when r is comprised between these roots. These roots therefore determine the radii, r_1 and r_2 of two circles which limit the wave. Because of the double sign before the radical the Riemann surface between these radii must be considered double. There is then another type of path along which integral (12′) must be taken: from r_1 to r_2 on one sheet and back to r_1 on the other. Choosing a path along a radius this gives

$$\int \sqrt{\frac{\nu^2 - \nu_0^2}{c^2} + \frac{\alpha\nu}{\pi cr} + \frac{\alpha^2 - k^2}{4\pi^2 r^2}} = n' = n - k.$$

Calculating this integral in the way Sommerfeld[10] has shown we find

$$\frac{\alpha\nu}{\sqrt{\nu_0^2 - \nu^2}} - \sqrt{k^2 - \alpha^2} = n - k. \tag{16}$$

[10] A. Sommerfeld, Atombau und Spektrallinien, Zusatz 6.

This determines the frequency ν in terms of the constants α, ν_0, and the quantum numbers n and k. What is usually spoken of as the energy is

$$h(\nu - \nu_0) = -\frac{\alpha^2\nu^2}{\nu_0 + \nu} \frac{h}{\left(n - \frac{\alpha^2}{2k} - \cdots\right)^2}$$

$$= -\frac{Rch}{n^2}\left(1 + \frac{\alpha^2}{nh} + \cdots\right)\left(1 - \frac{3}{4}\frac{\alpha^2}{n^2} + \cdots\right).$$

This is the usual expression for the energy levels of the elliptic orbits.

Substituting the value of ν from (16) in (15) we can calculate r_1 and r_2. Then μ is fully determined by (14) and (15). It is then possible to calculate the phase

$$\phi = 2\pi\left(\int \mu_r\, dr + \frac{k}{2\pi}\theta - \nu t\right) \tag{17}$$

and the surfaces of equal phase (wave surfaces) are given by

$$\int \mu_r\, dr + \frac{k}{2\pi}\theta = \text{constant}. \tag{18}$$

These are shown in the accompanying figure, the heavy lines representing the wave surfaces on one of the sheets and the light lines those on the other. The wave surfaces have been shown one-half wave-length apart so that their number must be divided by two to obtain the azimuthal quantum

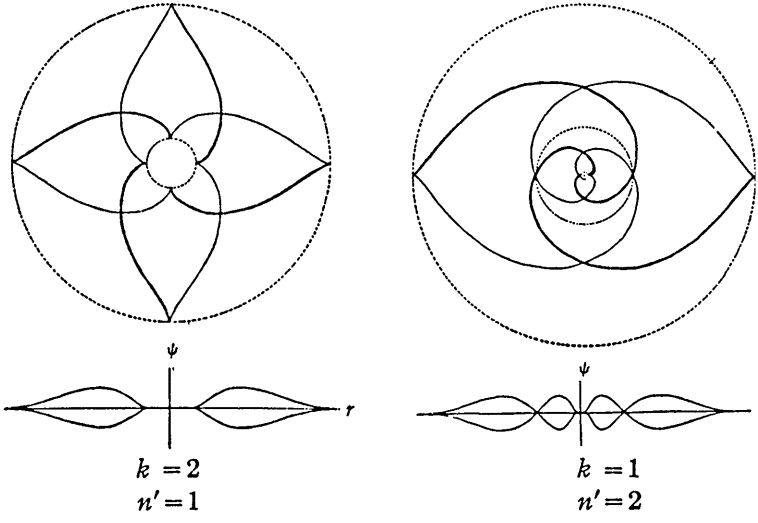

FIG. 1
Waves in hydrogen atom for $n = 3$.

number. They are of course orthogonal to the mechanical orbits corresponding to the same quantum states. These waves on the two sheets do not, however, exist independently but are superposed, so that the resultant disturbance is of the form

$$\sin 2\pi\left(\int \mu_r\, dr + \frac{k}{2\pi}\,\theta - \nu t\right) - \sin 2\pi\left(-\int \mu_r\, dr + \frac{k}{2\pi}\,\theta - \nu t\right)$$
$$= 2 \sin 2\pi \int \mu_r\, dr \cos (k\theta - 2\pi\nu t). \quad (19)$$

Here μ_r represents the positive value of (15). The phase difference between the waves on the two surfaces is indeterminate, for although a change of phase at one boundary will appear in the radial quantum integral its effect will be annulled at the other boundary. We have therefore assumed that particular phase difference which gives zero displacement on the boundaries. Equation (19) shows that the superposition of the waves on the two imaginary surfaces results in a wave with radial wave surfaces, travelling circularly, and having k wave-lengths to the circle.

Although this wave is a travelling wave as regards its circular component, it is a standing wave as regards its radial component, so that the amplitude varies with the radius as $\sin 2\pi \int \mu_r\, dr$. This variation is shown in the lower curves of the figure. The number of circles of maximum amplitude gives directly the radial quantum number n'.

The major difficulty of the quantum theory would seem to be, however, not the explanation of the stationary states, but of the "quantum jumps." The radiation of atoms has frequently been compared to "beats" between vibrations within the atom, and this has seemed to excuse to a certain extent the assumption which we shall make: that during the passage of an electron from one stationary state to another there is a transition stage in which the waves corresponding to the two stationary states are superposed, perhaps with reduced amplitudes. There is, indeed, no reason to suppose that while it is absorbing or emitting radiation the wave electron is as simple as has been pictured above. New alternating electric forces, not derivable from a potential V (*cf.* Equation (8)), have come into play, and they may alter the distribution of the charge in a way which may only be guessed. The final result only is known: that the electron has passed from one stationary state to another. Turning to optics for a hint as to what the process may be, we find something similar occurring in the reflection of light from a mirror. An initial steady state, the incident beam, enters a region of interference in front of the mirror, and issues again as a final steady state, the reflected beam. The energy in the region of interference cannot properly be assigned to either wave train. The electric and magnetic

vectors are here not in phase. Indeed it is quite a distinct state of which we would probably be entirely unaware did we not have an exact theory of light and independent evidence of the existence of the mirror, as only specially designed experiments make it directly apparent. It is nevertheless an essential part of the phenomenon, for it is only during its existence that transfer of momentum to the mirror occurs.

We would regard the "quantum jump" as an entirely similar process during which charge passes gradually and continuously from one steady state of electron wave (corresponding to the incident beam) to another (the reflected beam) through a region of interference. In this region the waves will form "groups" moving with the velocity

$$v = \frac{\Delta\nu}{\Delta\mu} \tag{20}$$

and this is seen to differ from Equation (7) only in that a differential ratio has been replaced by a difference ratio. Indeed the limit of this transition stage when the initial and final states differ but very slightly is identical to our picture of the electron as a bundle of very slightly differing waves. The principle of correspondence is then evident.

These considerations may now be applied to transitions between the stationary states described in the preceding pages, but let us first consider a simpler case, where there is but one electron and the potential V is uniform. The electron waves are plane, and move with a constant velocity, and this state will continue until there is some disturbance, such as the arrival of a light wave. After the light wave has passed, a steady state of the same general character will be resumed. If the initial and final states of the electron are the simple wave trains

$$\sin 2\pi(\mu_1 \cdot \mathbf{s} - \nu_1 t) \qquad \text{and} \qquad \sin 2\pi(\mu_2 \cdot \mathbf{s} - \nu_2 t) \tag{21}$$

the transition stage will be of the form

$$\sin 2\pi\left(\frac{\mu_1 + \mu_2}{2} \cdot \mathbf{s} - \frac{\nu_1 + \nu_2}{2} t\right) \cos 2\pi\left(\frac{\mu_1 - \mu_2}{2} \cdot \mathbf{s} - \frac{\nu_1 - \nu_2}{2} t\right). \tag{22}$$

The first factor represents a wave train intermediate between the initial and final states. The second factor cuts it up into groups by planes moving uniformly in the direction of $\mu_1 - \mu_2$. The surfaces of zero displacement are shown in Fig. 2. As the distribution of amplitude in the electron wave is unknown the boundaries of the region of interference have been left blank.

It is with this system of wave groups that the incident light wave

$$E_1 = E \cos 2\pi(\mathbf{M}_1 \cdot \mathbf{s} - N_1 t) \tag{23}$$

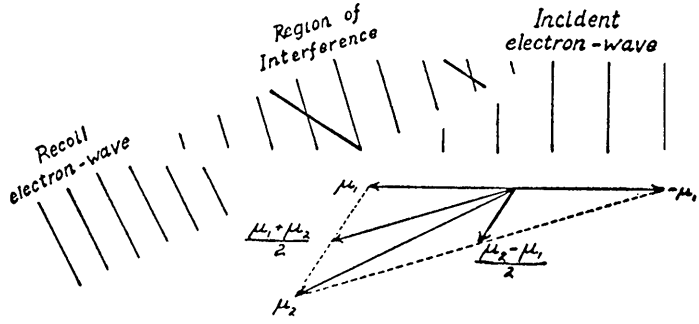

FIG. 2
Electron-waves in Compton Effect

will react. The groups, as they contain charge, will act as reflecting surfaces. Not the groups defined by the sine factor of (22), for these are moving faster than light, but the groups defined by the cosine factor. If the reflected light beam is

$$E_2 = -E \cos 2\pi(\mathbf{M}_2 \cdot \mathbf{s} - N_2 t) \tag{23'}$$

the total field is

$$2E \sin 2\pi\left(\frac{\mathbf{M}_2 + \mathbf{M}_1}{2} \cdot \mathbf{s} - \frac{N_2 + N_1}{2} t\right) \sin 2\pi\left(\frac{\mathbf{M}_2 - \mathbf{M}_1}{2} \cdot \mathbf{s} - \frac{N_2 - N_1}{2} t\right) \tag{24}$$

and this must be zero on the reflecting surfaces. The same conclusion is reached if we consider that the uniform motion of the electron wave groups requires that the field be zero. This condition is satisfied if

$$N_2 - N_1 = \nu_1 - \nu_2$$
$$\mathbf{M}_2 - \mathbf{M}_1 = \mu_1 - \mu_2. \tag{25}$$

These relations are not sufficient to solve the problem, as we have been obliged to assume known the final state of the electron. They are conditioned by the fact that the reflected light must have the velocity c, but this still leaves two quantities undetermined in the final electron state (recoil electron). They are perhaps related to the phases of the incident waves.

If the wave quantities in Equations (25) are replaced by the corresponding mechanical quantities the conservation of energy and momentum relations of the ballistic theory of the Compton effect are obtained, and this without the necessity of assuming the light to be quantized. It has been attempted to explain the Compton effect as a Doppler effect, but this

explanation has been considered untenable because the reflecting center would have to move in quite a different direction from the electron. The two sets of groups furnished by our theory remove the difficulty, for while the electron moves in the direction of $\mu_1 + \mu_2$ with the velocity $(\nu_1 + \nu_2)/|\mu_1 + \mu_2|$, the reflecting surface moves in the direction of $\mu_1 - \mu_2$ with the velocity $(\nu_1 - \nu_2)/|\mu_1 - \mu_2|$. This latter surface can be considered to reflect both the electron and the light waves.[11]

We may now consider jumps between states corresponding to circular orbits, such as have been considered on page 10. Suppose the electron to be in the state designated by n, and suppose that for some reason the state $(n - \Delta n)$ becomes slightly excited. The two waves will interfere, forming Δn groups in the region between their orbits. As the amplitude of the $(n - \Delta n)$ wave is supposed to be at first small, the accumulations of electricity in these Δn groups will be slight, but they will nevertheless radiate according to the classical laws. As the energy is thus decreased, charge must necessarily fall from the higher to the lower level, from the nth to the $(n - \Delta n)$th orbit, the groups becoming at first more marked and later fading away until all the charge has fallen to the $(n - \Delta n)$th orbit. The frequency of rotation of the groups is (*cf.* 11 and 12)

$$\frac{1}{2\pi}\frac{\Delta\nu}{\Delta\mu} = \frac{\Delta\nu}{\Delta n}. \tag{26}$$

This frequency is, by the theorem of the mean, intermediate between the frequencies of rotation of the electron in the two orbits and therefore is the frequency of rotation which the electron would have, were it in some intermediate orbit. This is in accord with the principle of correspondence, but it was never clear why, according to that principle, only the Δnth harmonic should be radiated. Here it is clear. There are Δn groups, and hence the frequency radiated is

$$N = \Delta n \frac{\Delta\nu}{\Delta n} = \Delta\nu = \frac{\Delta W}{h}. \tag{27}$$

This type of jump is barred, however, by the principle of selection unless $\Delta n = \pm 1$. This would appear to be related to the essential indivisibility of the electron, only such transitions being allowed in which one electron wave group is formed.

Transitions between the more general type of stationary states of the hydrogen atoms are to be treated in the same way. Because of the stationary character of the radial component of the wave the azimuthal quantum

[11] Since this article went to press E. Schrödinger, Ann. d. Phys., 82, p. 257, 1927, has published the same result.

number will determine the number of groups, there being Δk groups moving circularly with the linear velocity (*cf.* 14).

$$v = \frac{\Delta\nu}{\Delta\mu_\theta} = 2\pi r \frac{\Delta\nu}{\Delta k}. \tag{28}$$

The frequency of rotation will therefore be $\Delta\nu/\Delta k$. And the frequency radiated will be

$$N = \Delta k \frac{\Delta\nu}{\Delta k} = \Delta\nu = \frac{\Delta W}{h}. \tag{29}$$

Again the correct frequency is obtained, and the indivisibility of the electron points to $\Delta k = \pm 1$.

The following item is the truly classic paper by Professor Allis and Professor Morse which they wrote while working with Sommerfeld in Munich in 1931. Here, in the early days of wave mechanics, they were able to explain the apparently most anomalous behavior which Ramsauer had demonstrated so conclusively in which the collision cross section or alternately the mean free path were, in some instances, sensitive functions of the electron energy. The Morse-Allis theory which calculated the diffraction of bombarding electrons by the external electrons surrounding the nuclei of a large number of atoms was not only the first but for many years the only wave mechanical calculation of a gas-discharge phenomenon.