CHAPTER ONE

The Integral Laws in Free Space

The development of electromagnetism in the nineteenth century went hand in hand with a very significant modification of the point of view from which the pertinent experimental evidence was interpreted and pieced together. The original point of view of "action at a distance," characteristic of Coulomb's law, had led to considering forces of electric and magnetic origin as exerted directly by electric charges on electric charges, and by magnetic poles or current elements on other magnetic poles or other current elements. It was Faraday, in the first half of the nineteenth century, who first conceived of the space surrounding electric charges as filled with "lines of force," indicating everywhere the direction and (through their density) the magnitude of the force that would be acting on a positive unit charge if such a charge were present. Faraday, furthermore, thought of the space—whether empty or occupied by polarizable matter—as an elastic medium under stress, tension being present along the lines of force, and pressure being exerted in all directions normal to them. Mutual forces between charges could then be conceived as being "transmitted" by the medium.

Faraday's line of thought shifted the focus of attention from the properties of geometric configurations of charges and conductors to those of the surrounding medium and of the field of force hypothesized within it. Maxwell, in the second half of the nineteenth century, was much impressed by the importance of this shift of emphasis, and set out to express Faraday's ideas in a precise mathematical form. He stated in the preface to the first edition of his famous *A Treatise on Electricity and Magnetism* [1]: [1]

[1] Numbers set in brackets refer to references at end of chapter.

When I had translated what I considered to be Faraday's ideas into a mathematical form, I found that in general the results of the two methods [that of Faraday and that of action at a distance, which was the most popular among the theoretical physicists and mathematicians of the time] coincided, so that the same phenomena were accounted for, and the same laws of action deduced by both methods, but that Faraday's methods resembled those in which we begin with the whole and arrive at the parts by analysis, while the ordinary mathematical methods were founded on the principle of beginning with the parts and building up the whole by synthesis. I also found that several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of ideas derived from Faraday than in their original form. The whole theory, for instance, of the potential, considered as a quantity which satisfies a certain partial differential equation, belongs essentially to the method which I have called that of Faraday.

Maxwell's interest in the inherent mathematical properties of electric and magnetic fields, as contrasted with those that depend on the geometry and strength of their sources, led him to the formulation of his famous field equations and to the theoretical discovery of electromagnetic waves. Although electromagnetic waves can also be interpreted as the result of "delayed action at a distance," their discovery by Maxwell as a necessary consequence of the properties of electromagnetic fields constitutes the single most striking example of the much greater power of the field point of view.

In deference to the mechanistic attitude of the nineteenth-century physicists, Maxwell kept alive Faraday's conception of free space as an appropriate elastic medium through which electromagnetic actions are transmitted, although his formulation of the field equations did not depend in the least upon it. Only the repeated failure to observe any one of the expected physical consequences of the existence of such a medium, the ether, led modern physicists to disregard such an hypothesis as unwarranted and, furthermore, as totally unnecessary. Yet, because of the similarity between the mathematics of electromagnetism and that of elasticity, the concept of an elastic medium is still useful in providing helpful, suggestive analogies.

The historical approach, beginning with Coulomb's law, is followed in most elementary treatments of electromagnetism because it permits one to develop slowly the abstract concept of field while discussing the experimental laws leading to Maxwell's formulation of the field equations. On the other hand, it seems more appropriate, in a second and more profound study of electromagnetism, to postulate Maxwell's field equations as the laws of electromagnetism, from which the simpler laws of Coulomb, Ampere, Faraday, etc., can be derived as special cases. This approach has the advantage of making a clear-cut separation between the sources of an electromagnetic field, the field itself, and the action exerted by the field on charges, currents, and neutral matter. The important fact, in this regard, is that in any given region of space the same field can be produced by a variety of source distributions outside the region. However, the field within the region must satisfy conditions entirely independent of the sources located outside the region. These conditions, which may be looked upon as physical realizability conditions, are expressed by Maxwell's field equations, so that the solutions of Maxwell's equations represent the physically realizable fields. Thus we see that the field approach permits us to split any design problem into three parts:

1. The determination of the class of fields able to produce the desired type of action on charges, currents, and matter;
2. the selection within such a class of a physically realizable field, i.e., a field that satisfies Maxwell's equations;
3. the determination of primary sources (charges and currents) and of secondary sources (polarizable and magnetizable matter) able to produce the desired field.

This field-synthesis point of view will guide our thinking in most of this volume.

1.1 Review of Basic Postulates and Definitions

Basic postulates and definitions are always a troublesome subject in the exposition of any physical theory. Educationally speaking, we are faced with a vicious circle. On the one hand, the exposition of the theory should be preceded by a thorough discussion of the postulates on which it is based and by precise definitions of the physical quantities involved. On the other hand, both postulates and definitions cannot be properly justified or even stated precisely without exploring their consequences and comparing them with the available experimental evidence; thus it would seem that postulates and definitions should be discussed after the presentation of the theory rather than before it. Furthermore, questions concerning their consistency, necessity, and sufficiency are often very difficult and involve not only the theory that stems from them but also other related physical theories.

Serious difficulties of this type confront us in connection with the field theory of electromagnetism. We are thus forced to compromise and be satisfied with postulates and definitions that are not so clear and precise as we should like them to be, and which appear somewhat arbitrary. Some of the questions that are left open will be answered later on; others are beyond the scope of this text.

The evidence available from a wide variety of experiments on electromagnetic forces is consistent with the following basic postulates:

1. There exist two kinds of electric charges: a positive charge and a negative charge.

2. Electric charge is conserved; in the sense that whenever any positive charge appears, an equal amount of negative charge also appears. Conversely, whenever any positive charge disappears, an equal amount of negative charge also disappears. Thus the algebraic sum of all charges is constant in any isolated system.

3. All charges are integral multiples of the electronic charge, whose magnitude is given by

$$e = 1.60 \times 10^{-19} \text{ coulomb} \tag{1.1}$$

4. An electric charge in motion may be acted on by a force independent of its velocity and also by a force proportional to its velocity and directed at right angles to it. More precisely, the total force $\boldsymbol{F}$ known as the Lorentz force can be expressed in the form

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \mu_0 \boldsymbol{H}) \tag{1.2}$$

where q represents the charge and $\boldsymbol{v}$ its velocity. The vector $\boldsymbol{E}$, the electric-field intensity, and the vector $\boldsymbol{H}$, the magnetic-field intensity, are thereby defined in terms of the force, the charge, and its velocity relative to the observer. The quantity μ_0 is the permeability of vacuum, a constant whose value depends on the system of units.

In the mks rationalized system of units, used throughout this volume, the force is measured in newtons, the velocity in meters per second, and the charge in coulombs. The coulomb is the basic electric unit which, together with the meter, the kilogram, and the second, permits the definition of all other electromagnetic units, as discussed in Appendix 2. Its definition requires, of course, an additional relation independent of Eq. 1.2. The unit of electric-field intensity is specified by Eq. 1.2 in terms of the units of force and charge. In practice, the electric-field intensity is measured in volts per meter, a volt being a joule per coulomb. The unit of $\mu_0 \boldsymbol{H}$ is specified, similarly, in terms of the units of force, velocity, and charge. The dimensions and the value of the permeability of vacuum

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/meter} \tag{1.3}$$

are selected in such a way that the magnetic-field intensity be measured in amperes per meter, that is in coulombs per meter-second, as we shall see in Sec. 1.3.

We shall need in our study the concepts of charge density, current, and current density. The charge density ρ at any point P is defined as the ratio of the charge δq contained in a small region about P to the volume δV of the region, in the limit when the region shrinks to the point P; i.e.,

$$\rho = \lim_{\delta V \to 0} \frac{\delta q}{\delta V} \tag{1.4}$$

Conversely, the charge density is a scalar function of position such that the total charge in any volume V shall be representable as the volume integral

$$q = \int_V \rho \, dv \tag{1.5}$$

Strictly speaking, this definition is inconsistent with postulate 3 above, because the limit of Eq. 1.4 cannot exist if the charge δq must remain an integral multiple of the electronic charge. Conversely, the total charge in a finite region cannot be an integral multiple of the electronic charge for all regions if the charge density is a finite function of position. On the other hand, the quantization of charge implied by postulate 3 is so fine compared with the charge involved in the large-scale phenomena with which we shall be concerned that the inaccuracies resulting from the assumption of a smooth charge distribution with a finite density are completely negligible.

The current I flowing through a surface S is defined as the limit of the ratio of δq, the amount of charge that crosses S in the time δt, to the time interval δt, when δt approaches zero, i.e.,

$$I = \lim_{\delta t \to 0} \frac{\delta q}{\delta t} \tag{1.6}$$

Current is measured in amperes, one ampere being equal to a coulomb per second. The sign of I is arbitrarily defined as positive for a current flowing in the direction of motion of positive charges or opposite to the direction of motion of negative charges.

The current density $\boldsymbol{J}$, a vector, is defined in turn as follows. Let us consider a small element of surface δa, and indicate with $\boldsymbol{n}$ a unit vector normal to it. Clearly, the current δI flowing through δa is a maximum when the direction of $\boldsymbol{n}$ coincides with the direction of motion of the charge; with $\boldsymbol{n}$ so oriented, the magnitude of $\boldsymbol{J}$ is defined as

$$|\boldsymbol{J}| = \lim_{\delta a \to 0} \frac{\delta I}{\delta a} \tag{1.7}$$

The direction of $\boldsymbol{J}$ coincides with the direction of motion of positive charge and is opposite to the direction of motion of negative charge, in agreement with the above convention regarding the sign of I. Thus the current density in an electron beam has a direction opposite to the direction of motion of the electrons. The current density can also be defined in an equivalent manner as a vector function of position such that the current through any surface S shall be representable as the surface integral

$$I = \int_S J_n \, da \tag{1.8}$$

where da is a differential element of surface, and J_n is the component of $\boldsymbol{J}$ normal to the surface.

It is clear that the above definitions of current and current density are just as inconsistent with postulate 3 as the definition of charge density. Again the inconsistency may be disregarded as long as we are dealing with large-scale phenomena.

The definition of charge density and current density, together with the law of conservation of charge (postulate 2), implies that, for any surface S enclosing a volume V,

$$\oint_S J_n \, da = -\frac{d}{dt} \int_V \rho \, dv \tag{1.9}$$

where J_n is the component of $\boldsymbol{J}$ normal to S, and outwardly directed. The left-hand side of this equation represents the current flowing out of the closed surface S, i.e., the net outgoing positive charge per unit time. The right-hand side is the negative time rate of change of the net charge within V. We shall use this equation as a formal statement of the law of conservation of charge.

1.2 Convection and Conduction Currents

The current through a given surface was defined, in the preceding section, as the amount of charge crossing the given surface per unit time, without reference to any other characteristics of the motion of the charge. On the other hand, it is convenient for the purposes of our discussion, to classify currents according to their physical origins in three categories: convection currents, conduction currents, and polarization currents. Convection currents and conduction currents result from the free motion of electric charges; for this reason, they are often referred to as free currents. Polarization currents result from the relative displacement of charged particles in the atomic structure of matter, when such particles remain bound to the atom or molecule to which they belong. The current resulting from the motion of such bound charges is discussed in Sec. 5.2, as part of our study of dielectric polarization. We shall focus our attention here on convection currents and conduction currents.

We regard a current as being of the convection type when it results from the motion of charge whose density and velocity are explicitly stated. Thus, for instance, the current in a vacuum tube is regarded as a convection current because it originates from the motion of a well-identified space charge. The same is true for the current of an electron beam in a cathode-ray tube, and for the current resulting from the motion of a charged conductor. If ρ is the density of the moving charge at a given point in a stationary system of coordinates, and $\boldsymbol{v}$ is its velocity, the corresponding convection-current density is

$$\boldsymbol{J} = \rho \boldsymbol{v} \tag{1.10}$$

Conduction currents result from the drift of free electrons and ions in matter under the influence of an electric field, as, for instance, in metals and electrolytic solutions respectively. The motion of such charged particles is opposed by frictionlike forces within the conducting material that balance the forces exerted by the electric field. In metals and in electrolytic solutions these frictionlike forces are proportional to the velocity of the charged particles over a large range of values, with the result that the latter must be proportional to the electric-field intensity in order for the particles to be in dynamic equilibrium. It follows that the current density, which is proportional to the velocity of the particles, becomes proportional to the electric-field intensity, i.e.,

$$\boldsymbol{J}_c = \sigma \boldsymbol{E} \tag{1.11}$$

where σ is the conductivity of the material at the point at which $\boldsymbol{J}_c$ and $\boldsymbol{E}$ are measured. This equation is readily recognized as expressing Ohm's law in terms of field vectors.

It is important to note that the presence of conduction current does not imply the presence of a net charge density. In a metal, for instance, conduction current results from the drift of free atomic electrons in the presence of stationary atoms, which are positively charged because of the loss of electrons. The net charge density may or may not be equal to zero; in any case, it bears no relation to the current density beyond that required by the law of conservation of charge. Furthermore, the actual density of the moving charge and its velocity are of no interest from a macroscopic point of view; we are only concerned with their product which constitutes the current density. By comparison, in the case of a convection current, both the charge density and its velocity are individually of interest.

1.3 The Field Equations in Free Space

The electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ have been defined in Sec. 1.1 in terms of the Lorentz force exerted on a moving charge. In the first six chapters we shall focus our attention on the properties of these two fields without reference to their original definition, following the field approach to electromagnetism developed by Faraday and Maxwell. We shall return to their significance in terms of electromagnetic forces in Chapter 7 in order to develop the concepts of electromagnetic energy and electromagnetic power from the work done by such forces.

Let us begin by reconsidering the integral form of Maxwell's equations in free space, which culminates most elementary discussions of electromagnetism. For this purpose, let us consider an arbitrary, two-sided, simply connected [1] surface S, bounded by a closed contour C, as illustrated in Fig. 1.1. The direction of the arrow along the contour is related to that of the unit vector $\boldsymbol{n}$, normal to the surface, by the right-handed-screw rule; i.e., if the surface is continuously deformed into a plane, a right-handed screw turning in the direction indicated on the resulting contour should move axially in the direction of the unit vector $\boldsymbol{n}$, normal to the plane. The two fundamental equations of Maxwell can be written in the form:

$$\oint_C E_t\,ds + \frac{d}{dt}\int_S \mu_0 H_n\,da = 0 \tag{1.12}$$

$$\oint_C H_t\,ds - \frac{d}{dt}\int_S \epsilon_0 E_n\,da = \int_S J_n\,da \tag{1.13}$$

[1] A simply connected surface is a surface without holes, i.e., a surface bounded by a contour consisting of a single continuous line. We shall see later on that any surface with holes (multiply connected) can be reduced for our purpose to a simply connected surface by means of appropriate cuts. An example of a one-sided surface is the Moebian strip constructed by joining the two ends of a twisted strip of paper in such a way that one edge of the strip is made to coincide with the other edge.

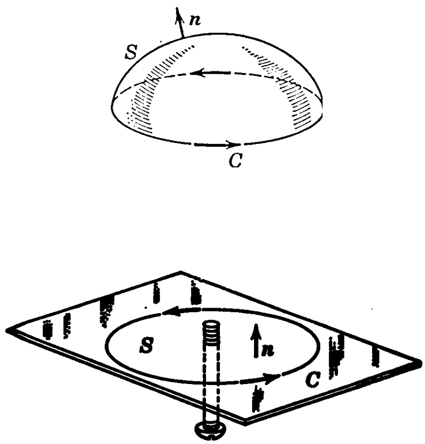

Fig. 1.1. The use of the right-handed-screw rule in determining reference directions on a surface and on the contour bounding it.

where E_t and H_t are the components of E and H tangent to the contour C in the direction indicated by the arrow, and E_n and H_n are the components of the same vectors normal to the surface S and in the direction of the unit vector n; da represents a differential element of the surface S, and ds represents a differential element of the contour C. The value of the constant ϵ_0, the permittivity of vacuum, is obtained from the equation

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.998 \times 10^8 \text{ meters/second} \tag{1.14}$$

where c is the velocity of light in vacuum, as determined by measurements. This equation yields for ϵ_0 the value

$$\epsilon_0 = 8.854 \times 10^{-12} \tag{1.15}$$

Equation 1.14 is a direct consequence of Maxwell's equations, although it cannot be derived at this point. Since it relates the values of μ_0 and ϵ_0 to the velocity of light in vacuum, a measurable physical quantity, only one of these two constants of vacuum can be selected arbitrarily in devising a system of units. As stated in Sec. 1.1, the value of μ_0 is selected, in the mks rationalized system, in such a way that the magnetic-field intensity is measured in amperes per meter, as evidenced by Eq. 1.13; this selection fixes both the value and the dimensions of ϵ_0.

The first equation, Eq. 1.12, expresses Faraday's induction law, namely, that the electromotive force around any closed contour must equal the negative time rate of change of the magnetic flux linking the contour. In fact, the contour integral represents the work that would be done by the electric field in moving a unit positive charge once around the contour, i.e., the electromotive force. The surface integral represents the flux of the magnetic field through the given surface S. Since the contour integral depends only on the field and the contour C, the surface integral must also depend only on the field and on the contour; in particular, the time rate of change of the flux must be the same for all two-sided surfaces bounded by C. We shall see in Sec. 3.1 that this requirement implies that the magnetic flux through any closed surface is always equal to zero, i.e.,

$$\oint_S \mu_0 H_n \, da = 0 \tag{1.16}$$

This equation is sometimes stated as a separate field law known as Gauss' law for the magnetic field. Actually it is a direct consequence of Eq. 1.12.

The second field equation, Eq. 1.13, is a statement of Ampere's circuital law, modified by the addition of the term involving the time rate of change of the flux of $\epsilon_0 E$. The contour integral is the magnetomotive force around the contour; the surface integral of the current density represents, of course, the net current flowing through S. The role played by the second term on the left-hand side, sometimes misleadingly referred to as the "displacement current," becomes evident when we move it to the right-hand side and require that the sum of the two surface integrals be the same for all two-sided surfaces bounded by the same contour, just as we did in connection with Eq. 1.12. This requirement could not be met by the current term alone; for instance, the surfaces S and S' in Fig. 1.2 would yield different current fluxes, since S cuts through a wire leading to a capacitor whereas S' passes between the plates of the capacitor without cutting through any wire.

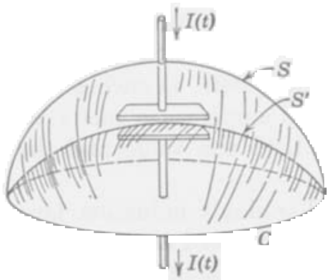

Fig. 1.2. An example of two surfaces bounded by the same contour C through which different amounts of current flow.

Thus the circuital law as stated originally by Ampere cannot be correct for time-varying fields.

We shall see in Sec. 3.1 that the requirement that the sum of the two surface integrals in Eq. 1.13 be the same for all surfaces bounded by the same contour implies that the outward flux of $\epsilon_0 \boldsymbol{E}$ through any closed surface must be equal to the net charge q in the volume V enclosed by the surface: i.e.,

$$\oint_S \epsilon_0 E_n \, da = \int_V \rho \, dv = q \tag{1.17}$$

Again, this equation is sometimes stated as a separate law, known as Gauss' law for the electric field. Actually it is a direct consequence of Eq. 1.13 and of the law of conservation of charge expressed by Eq. 1.9. Historically, however, the discovery of Gauss' law preceded the formulation of the second basic field equation. Maxwell noted that Ampere's circuital law (similar to Eq. 1.13 but without the term involving $\epsilon_0 E_n$) was mathematically inconsistent for time-varying fields because the flux of $\boldsymbol{J}$ could depend on the particular surface S selected, as discussed above. He then showed, on the basis of Gauss' law and the law of conservation of charge, that mathematical consistency could be obtained by adding the term involving $\epsilon_0 E_n$. It is important to note that the addition of this term was the key to the theoretical discovery of electromagnetic waves.

The name "displacement current" originated from Maxwell's argument about an additional current term being required, for mathematical consistency, and from his views about free space being some sort of a material medium. Actually, nothing is displaced in free space, and the new term introduced by Maxwell in Eq. 1.13 should be thought of as being parallel to the corresponding term in Eq. 1.12. Thus, a finite magnetomotive force is associated with a time-varying flux of $\epsilon_0 \boldsymbol{E}$, just as a finite electromotive force is associated with a time-varying flux of $\mu_0 \boldsymbol{H}$.

1.4 Usefulness and Limitations of Integral Laws

It is important to stress that the integral laws discussed in the preceding section must be satisfied for *every closed contour C and every closed surface S*. Clearly, it would be very difficult to ascertain whether any particular pair of fields $\boldsymbol{E}$ and $\boldsymbol{H}$ does or does not satisfy such laws for all possible contours and surfaces, and it would be even more difficult to find directly a pair of fields that would satisfy them for a given distribution of charges and currents. We shall see in Chapter 3 that the problem can be considerably simplified by substituting for the integral laws equivalent differential laws. Yet, there are important special cases in which the integral form of the field laws is not only adequate but also more illuminating. These special cases are characterized by particular geometric symmetries, such as spherical or cylindrical. This point is best explained in terms of specific examples.

Let us consider a time-independent charge q, uniformly distributed within a sphere of radius a centered at the origin. We wish to determine the electric field produced by this charge, both inside and outside the sphere. We note, first of all, that because of the spherical symmetry of the charge distribution, the electric field must have everywhere a direction radial from the origin. This follows from the fact that a radial direction is the only direction that can have a complete spherical symmetry. For the same reason, the intensity of the electric field must be constant over any concentric spherical surface. Then, if we take any such spherical surface as the closed surface S of Eq. 1.17, and indicate with r its radius, this equation becomes

$$4\pi r^2 \epsilon_0 E_n = \begin{cases} q\left(\dfrac{r}{a}\right)^3 & \text{for } r < a \\ q & \text{for } r \geq a \end{cases} \tag{1.18}$$

from which we obtain

$$E_n = \begin{cases} \dfrac{q}{4\pi\epsilon_0 a^3}\, r & \text{for } r < a \\ \dfrac{q}{4\pi\epsilon_0 r^2} & \text{for } r \geq a \end{cases} \tag{1.19}$$

where E_n is the electric-field intensity in the outward direction.

It is important to observe that Eq. 1.17 together with the spherical symmetry requirement was sufficient to determine uniquely the electric-field intensity. This means that Eq. 1.12 must be automatically satisfied, or, in other words, that the constraint imposed on the electric field by this equation is already implied by the spherical-symmetry requirement. It can be shown, in fact, that, in the absence of any time-varying magnetic field, Eq. 1.12 is satisfied by any radial, spherically symmetrical electric field. This property of spherically symmetrical fields is very readily proved with the tools of vector analysis, discussed in the following chapter.

Let us consider next the case of an infinite straight wire of radius a, carrying a steady current I, parallel to the axis of the wire and uniformly distributed through its cross section. We wish to determine the magnetic field produced by this current distribution, both inside and outside the wire. We shall show, first, that the magnetic field must be tangent to any circular cylinder coaxial with the wire. For this purpose, we observe, first of all, that, because of the circular cylindrical symmetry of the current distribution, the magnetic field can depend only on the radial distance r from the axis of the wire, besides on the magnitude and the direction of the current. In particular, if the magnetic field includes a radial component, this component must have the same direction (either toward the axis of the wire or away from it) at all points. Furthermore, if the magnetic field has a component parallel to the axis of the wire, this component must be constant over any straight line parallel to the wire.

Let us consider then Eq. 1.16, using for S the surface of any circular cylinder of finite length, coaxial with the wire. The flux of $\boldsymbol{H}$ entering the cylinder from either end surface must be equal to the flux leaving the cylinder from the opposite end surface, because the component of the magnetic field parallel to the wire must be independent of the position along the wire. Thus this component cannot contribute to the surface integral. On the other hand, any flux through the circular part of the surface can only be caused by a radial component; furthermore, because of the circular symmetry requirement on the radial component, this flux can vanish only if this component is equal to zero. Thus, we can conclude that Eq. 1.16 requires the magnetic field to be tangent to any circular cylinder coaxial with the wire.

Let us consider next Eq. 1.13 and use as contour C a circle concentric with the wire, drawn on a plane normal to it, as illustrated in Fig. 1.3. Assuming that the current flows upward from the paper, Eq. 1.13 requires the current flowing through the circle to be equal to the line integral, in the direction indicated by the arrow, of H_t, the component

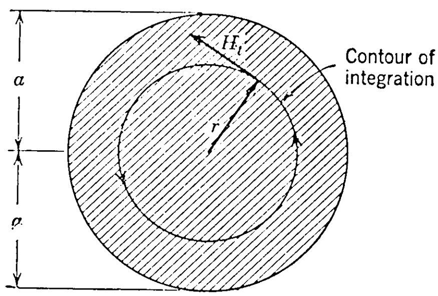

Fig. 1.3. Contour of integration used in connection with Eq. 1.20.

of the magnetic field tangent to the circle. Since H_t must be constant over the circle, because of the circular symmetry of the current distribution, Eq. 1.13 yields for a circle of radius r,

$$2\pi r H_t = \begin{cases} I\left(\dfrac{r}{a}\right)^2 & \text{for } r < a \\ I & \text{for } r \geq a \end{cases} \tag{1.20}$$

from which we obtain

$$H_t = \begin{cases} \dfrac{I}{2\pi a^2} r & \text{for } r < a \\ \dfrac{I}{2\pi r} & \text{for } r \geq a \end{cases} \tag{1.21}$$

It remains to be shown that, if there is any component of the magnetic field parallel to the axis of the wire, the magnitude of such a component must be constant throughout the entire space and independent of the current in the wire; in other words, this component plays the role of an "arbitrary additive constant." For this purpose, let us use for C the rectangular closed path illustrated in Fig. 1.4, drawn on a plane containing the axis of the wire. The two radial sides of the rectangle do not contribute to the line integral in Eq. 1.13 because the magnetic field has no radial component. Furthermore the entire line integral must vanish because no current flows through the

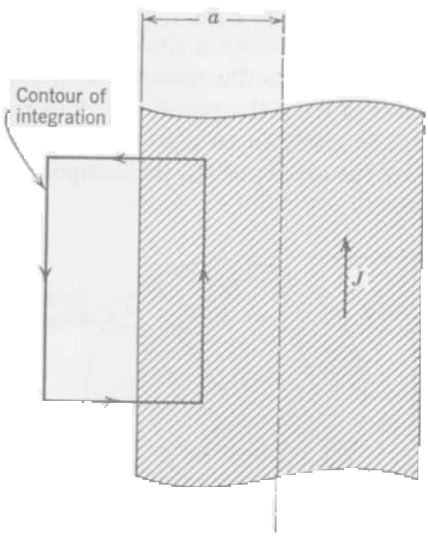

Fig. 1.4. Contour of integration used in determining the magnetic field parallel to the wire.

rectangular closed contour. It follows that the contributions to the line integral of the two remaining sides must be equal in magnitude and opposite in sign. Since this must be true for all similar rectangular paths, the component of the magnetic field parallel to the axis of the wire must be constant through the entire space. This uniform field is independent of the current in the wire and must be thought of as being produced by currents at infinity.

The above two examples illustrate how static fields can be determined from their sources with the help of the integral laws when the sources have special geometric symmetries. The simplicity of the procedure results from the selection of surfaces and contours on which the pertinent field components are known to be constant because of the symmetry of the sources. In the two examples considered above, the fields turn out to depend on a single spacial coordinate; there are cases, however, in which fields that depend on two spacial coordinates can be determined directly from the integral laws by following a similar procedure. In other words, the adequacy of the integral laws for the solution of a particular problem depends on the possibility of finding appropriate contours and surfaces rather than on the dimensionality of the field, although these two characteristics of the problem are related to some extent.

1.5 Matter as a Field Source

Matter is known to consist of positively charged nuclei surrounded by electrons. The negatively charged electrons revolve in orbits around the nuclei, and carry a total charge equal in magnitude and opposite in sign to that of the nuclei. Thus matter, in its normal state, is macroscopically neutral.

The field produced by atomic charges is, clearly, extremely complex. In dealing with large-scale phenomena, however, we can disregard its fine structure, and focus our attention on the smoothed field obtained by averaging the actual field over volumes large compared to atomic dimensions, yet small compared with the dimensions of the system under consideration. We shall use the adjective "macroscopic" in referring to these smoothed fields.

Usually, no macroscopic electric field is produced by neutral matter, mainly because of the mutual cancellation of the fields produced by neighboring atoms, and the averaging effect of thermal agitation. However, when the atomic structure of a material is modified, or the averaging action of thermal agitation is counteracted by an external electric field (or by other external forces), the contributions of the individual atoms may add up to yield a macroscopic field comparable in intensity to the external applied field. Two distinct situations arise: the first one characteristic of conducting materials when electrons or ions are relatively free to move about under the influence of an electric field, as in metals and electrolytic solutions; the second one characteristic of dielectric materials, when positive and negative charges are held together by such strong forces that they cannot be pulled apart completely, but only slightly displaced.

The ordered drift of charges resulting in the first situation constitutes macroscopically a conduction current which is found to depend at each point on the local electric-field intensity and on the local structure of matter, as discussed in Sec. 1.2. Such free charges may accumulate within a conducting body and on its surface, giving rise thereby to a net macroscopic charge distribution. In the second situation, in which charges are held together by strong forces, only small changes of their relative mean positions can result. When a net macroscopic electric field results from such displacements, the material is said to be electrically polarized. We shall see in Chapter 5 that the state of polarization of a material can be taken into account by associating to the material a distribution of electric dipoles whose moment density at each point is a function of the local state of matter. The macroscopic charge and current densities resulting from such dipole distributions and from their time rates of change will then be incorporated in the field equations as polarization components of ρ and $\boldsymbol{J}$.

Electrons are known to possess a magnetic-dipole moment in addition to a negative electric charge. This is evidenced by the magnetic field produced by them as well as by the force and torque exerted on them by an external magnetic field. This magnetic-dipole moment is associated to an angular momentum, or spin, and, therefore, is usually regarded as resulting from the current loop formed by the spinning electric charge. In some substances the spin dipole moments of the various electrons cancel completely within each atom or molecule; in others they do not cancel completely, so that each of their atoms or molecules has a resultant net dipole moment. These atomic or molecular magnetic dipoles are usually randomly oriented, mainly because of thermal agitation, so that they produce no net macroscopic field. However, a partial orientation in a particular direction may occur under the influence of an external magnetic field, or as a result of strong interatomic or intermolecular forces. In such cases the individual contributions of each atom or molecule add up to yield a finite macroscopic field and the material is said to be magnetized.

We shall see in Chapter 5 that the state of magnetization of a material can be taken into account macroscopically by associating with it a distribution of magnetic dipoles, whose moment density at each point is a function of the local state of matter. This representation is entirely analogous to that of electrically polarized materials. However, it is not immediately clear how a distribution of magnetic dipoles should be incorporated in the field equations as a field source. We shall see in Chapter 5 that, if each dipole of the distribution is regarded as a microscopic current loop, the entire dipole distribution is equivalent to a macroscopic current distribution whose density can be treated as a magnetization component of $\boldsymbol{J}$. However, the use of such a current model for magnetized materials makes it impossible to develop a macroscopic theory of electromagnetism that is both self-consistent and in agreement with experimental evidence. We shall see, on the other hand, that a satisfactory model can be obtained by treating magnetic dipoles in a manner entirely analogous to electric dipoles, just as if they consisted of magnetic charges with properties analogous to those of electric charges. This will require introducing in the field laws a magnetic-charge density ρ^* and a magnetic-current density $\boldsymbol{J}^*$, analogous to ρ and $\boldsymbol{J}$; Eqs. 1.12 and 1.16 will then become

$$\oint_C E_t\, ds + \frac{d}{dt}\int_S \mu_0 H_n\, da = -\int_S J_n^*\, da \qquad (1.22)$$

$$\oint_S \mu_0 H_n\, da = \int_V \rho^*\, dv \qquad (1.23)$$

The purpose of the above qualitative remarks about the role of matter as a source of electromagnetic fields is to introduce at this early stage the point of view that will characterize our treatment of macroscopic electromagnetic phenomena, and to justify the fact that, in the first four chapters, we shall confine our attention to free-space fields. It is convenient, for our purposes, to regard the phenomena of electric polarization and of magnetization of matter as consisting of two distinct parts: the action of an electromagnetic field in changing the state of polarization and magnetization of matter, and the action of polarized and magnetized matter in producing an electromagnetic field. The first part involves the functional relations between the state of polarization and the state of magnetization of matter on the one hand, and the electromagnetic field acting on matter on the other hand. These functional relations are often referred to as "constituent relations of matter." The physical origin of these relations and the microscopic characteristics of matter that are responsible for them are outside the scope of our discussion. We shall treat them instead empirically as experimentally determined properties of matter. The second part, namely, the role of polarized and magnetized matter as a source of electromagnetic fields is not only within the scope of our discussion but is a central part of it.

Another way of describing the same point of view is to say that matter behaves like a "controlled source," in the sense that matter is a source of electromagnetic field, but, at the same time, its source strength is a function of the field itself. In other words, the phenomena of polarization and magnetization involve a sort of "feedback control," whose characteristics are assumed to be given, or otherwise experimentally determinable, for each material.

This point of view leads us to consider any macroscopic electromagnetic field in matter as a free-space field in the presence of source distributions, which are either directly specified, or are expressible in terms of the field itself with the help of the constituent relations of the material involved. It follows that all field properties that do not involve the feedback link represented by the constituent relations can be studied without any reference to whether the sources are independent of the field or result from polarization and magnetization of matter. In particular, the field laws for macroscopic fields are the same within matter as outside matter, as long as the source densities which appear in the field laws are understood to include the components contributed by matter. We must keep in mind in this regard that, whereas electric sources can be present in the absence of polarized matter, magnetic sources can arise only from magnetized matter. For this reason, magnetic sources are not usually included in the free-space field equations; we have followed this convention in Sec. 1.3, and we shall continue to follow it in the next three chapters to avoid generating any misunderstanding as to the physical nature of magnetic charges and currents.

In view of the above arguments, we shall focus our attention first on free-space fields, produced by specified source distributions. Metallic conductors, however, will be included from the start in our discussion because they provide a wealth of interesting illustrations, and because of the simplicity of the constituent relation between electric field and conduction current, namely, Ohm's law. Polarizable and magnetizable materials will be taken up in detail in Chapter 5.

1.6 Summary and Conclusions

This chapter has been devoted to a review of the basic postulates of electromagnetism and of the laws governing the behavior of electromagnetic fields in free space. The main purpose of this review was to place in evidence the foundations on which we shall build our more advanced discussion of electromagnetic phenomena, free from the intermediate steps necessary in a first presentation to develop gradually the abstract concept of electromagnetic field. These foundations are

1. The law of conservation of charge.
2. The expression for the Lorentz force on a moving charge in terms of which the electric field and the magnetic field are defined.
3. The field laws in integral form that relate the electric field and magnetic field to each other and to the charge and current distributions.

These fundamental laws form a self-consistent set of relations in terms of which observable mutual forces between charges, whether stationary or in motion, can be described and predicted. The description and prediction of macroscopic forces between material bodies will require the additional postulation in Chapter 5 of macroscopic models for polarized and magnetized matter.

An important characteristic of the law of conservation of charge and of the field laws as expressed in Secs. 1.1 and 1.3 is that they relate contour integrals, surface integrals, and volume integrals of electromagnetic quantities. They are not equations of the type in which physical quantities are related to space or time derivatives of other physical quantities at the same point in space. On the other hand, such integral relations must be valid for all closed contour and associated surfaces, and for all closed surfaces and associated volumes. This arbitrary nature of the contours and surfaces suggests, as it is actually the case, that there should exist equivalent point relations between the same electromagnetic quantities and their time and space derivatives. For instance, the elementary derivation of plane waves found in many texts [2, Sec. 15.1] provides a good illustration of how the field equations in integral form yield point relations between the time derivative of one field vector and the space derivatives of the other.

Differential point relations describe the variations of vector fields from point to point rather than their properties over extended regions of space. As a result, they are much more convenient than integral relations, both conceptually and mathematically. The next chapter is devoted to the development of the mathematical tools necessary to express the laws of electromagnetism in the appropriate differential form.

1.7 Selected References

The following selected references should be helpful in reviewing the elementary aspects of electromagnetism, in acquiring a better historical perspective of the development of electromagnetism, and in developing a clearer understanding of the basic postulates and definitions discussed in this chapter, and a better appreciation of the problems involved in their choice.

1. J. C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed., reprinted by Dover Publications, New York, 1954. The preface to the first edition, dated February 1, 1873, gives the reader a good appreciation of Maxwell's point of view and of his fundamental contribution to the theory of electromagnetism.
2. N. H. Frank, *Introduction to Electricity and Optics*, 2nd ed., McGraw-Hill, New York, 1950. This is the physics textbook most appropriate in content and point of view for reviewing the elementary aspects of electromagnetism with which the reader is expected to be familiar.
3. A. Sommerfeld, *Electrodynamics*, Academic Press, New York, 1952. Section 1 of Part I is an historical review including some interesting biographical notes on the great men of electromagnetism. This review is particularly illuminating because Sommerfeld lived through the period in which electromagnetic theory became of age. Sections 2, 7, and 8 present a careful discussion of units and dimensions.
4. J. C. Slater and N. H. Frank, *Electromagnetism*, McGraw-Hill, New York, 1947. The introduction to Chapter 1 provides a good discussion of the development of electromagnetism.

PROBLEMS

Problem 1.1. An electron moves with a velocity v along the z-axis of a Cartesian coordinate system. A uniform magnetic field of magnitude H is applied in the positive x-direction. What electric field is required to force the electron to follow a straight path along the z-axis?

Problem 1.2. An electron (charge $e = 1.6 \times 10^{-19}$ coulomb, mass $m = 9.1 \times 10^{-31}$ kg) moves in a uniform magnetic field $H = 10^6$ amp/m in a plane at right angles to the direction of $\mathbf{H}$. Show that the electron moves in a circular path of radius r, and find r for an electron velocity of $v = 10^4$ m/sec.

Problem 1.3. In a cathode-ray oscilloscope, the electrons emitted from a heated filament are accelerated through a potential difference of 1000 v. The electrons then pass between two parallel deflecting plates, 2×2 cm, spaced 0.5 cm apart.

The electrons finally strike a fluorescent screen 30 cm from the rear edge of the deflecting plates.

What is the deflection of the spot on the fluorescent screen when a potential difference of 10 v is applied to the deflecting plates? Neglect the fringing field near the edges of the deflecting plates.

Problem 1.4. A cloud of charged particles is distributed in a hollow spherical cavity carved out of a perfectly conducting material. The charge density is

$$\rho(r) = \rho_0\left(1 - \frac{r^2}{R^2}\right)\frac{\text{coulombs}}{\text{meter}^3}$$

where ρ_0 is a constant, R is the radius of the cavity, and r is the distance from the center of the cavity. Find the electric-field intensity at every point within the cavity. What is the surface-charge density on the surface of the cavity?

Problem 1.5. A spherical drop of fluid carries a charge of q coulombs. Assume that the charge is uniformly distributed throughout the volume.

(a) Calculate the electric field and the potential both inside and outside the sphere.

(b) Two identical drops as above coalesce to form a single spherical drop. What is the potential at the surface of the new drop?

Problem 1.6. Given a very large plane sheet of charge (not a conductor) with uniform surface-charge density σ, find the *difference* of the electric-field vectors on either side of the sheet, far from the edges of the sheet.

Problem 1.7. The static electric field between two infinite parallel conducting plates held at a potential difference V_0 is perpendicular to the plates, and uniform.

(a) Show that the field satisfies Eq. 1.12 for all rectangular paths normal and parallel to the plates.

(b) Show that the field satisfies Eq. 1.17 for all parallelepipeds with faces normal and parallel to the plates. Show that the same equation is satisfied for any spherical surface.

(c) Find the surface-charge density on the plates by applying Eq. 1.17 to an appropriate surface.

Problem 1.8. (a) Find the electrostatic field produced by a point charge q.

(b) Show that Eq. 1.17 of the text is satisfied for a closed surface whose sides are formed by a circular cone with the apex at the point charge, and whose two endfaces are two spherical caps of radii R_1 and R_2 respectively ($R_2 > R_1$).

(c) Show that the field satisfies Eq. 1.12 for any planar contour consisting of two arcs of circles centered at the charge and two segments of straight lines passing by the charge.

Problem 1.9. Two infinite coaxial metallic cylinders are uniformly charged with a density λ_i per unit length on the inner cylinder (outer radius r_0), and a charge λ_0 per unit length on the outer cylinder (inner radius R_i and outer radius R_0). Determine the electric field between the cylinders, and in the outside space, and show that it satisfies both Eqs. 1.12 and 1.17.

Problem 1.10. A direct current is uniformly distributed over the cross section of a straight, infinitely long, circular cylindrical copper conductor of radius r_0. An equal amount of current flows in the opposite direction through a coaxial conductor of inner radius R_i and outer radius R_0, and it is uniformly distributed over its cross section. Find the magnetic field both inside and outside the conductors, and show that it satisfies Eqs. 1.13 and 1.16.

Problem 1.11. The Supreme Council of Lower Slabovia has decreed that in honor of its famous scientist Popin, a new unit of flux be introduced, the popin (abbreviation "pop," dimensional symbol P).

$$1 \text{ pop} = 10 \text{ webers}$$

The coulomb has been abolished. Derive a table of dimensions as used by the Slabovian scientists. Use only the four fundamental dimensions, meter, second, kilogram, and popin. Find the explicit values and dimensions of the electric permittivity ϵ_0 and the permeability μ_0, as used by the Slabovians.