Chapter 1

Introduction

The complexity of Boolean functions has been studied for almost 40 years. The field has developed into a theory in this, perhaps short, period mainly because of the success in defining both a set of complexity measures (those for circuit complexity and for Turing machine complexity) and a surprising hierarchy of very robust complexity classes. Moreover, characteristics of most of the defined classes have been understood by showing complete problems for them. Relations between some of the complexity classes have been discovered, and new models have been developed. The main frustration of the theory has been, however, the inability of showing a separation of any two classes (excluding those obtained by diagonalization methods*). To state it simply, the main problem remains unsolved: Though it is known that most functions are complex [Sh49], we do not have an example of a simple function (say in NP) that requires super-linear circuit size, or super-logarithmic circuit depth.

The reason for our inability to obtain non-trivial lower bounds is, perhaps, that although the circuit model is elegantly simple, our understanding of the way it computes is, at most, vague. There seems to be a need to develop more intuitive ways of looking at computation. A new approach may give some clues as to where to look for the heart of complexity and, at the same time, shed some light on how to prove lower bounds.

In this thesis we would like to propose a new approach to circuit depth: The *Communication Complexity approach* [†]. The approach is based on an equivalence between the circuit depth of a given function, and the communication complexity of a related problem. The bottom-line of the new approach is that it looks at

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*Diagonalization methods are not strong enough to separate such classes as P and NP (see [BGS75]).

[†]Yannakakis independently discovered this equivalence which is implicit in [KPPY].

a computation device as a *separating* device; that is, a device that separates the words of a language from the non-words. The characterization of circuit depth in terms of communication complexity is reminiscent of, but somehow more explicit and intuitive than, the well-known relationship between circuits and alternating machines [Ru80]. Among other things, the new approach allows us to view computation in a *top-down* fashion. Also, the approach makes explicit the idea that *flow of information* is a crucial term for understanding computation.

We will demonstrate that the communication complexity approach is very intuitive, and that it captures, in a strong way, the essence of circuit depth. We will do so by:

- Giving new, simpler proofs to old results which become clearer in this new setting.

- Proving a super-logarithmic monotone depth lower bound for the function *st*-connectivity.

In 1985, work of Andreev [An85] and Razborov [Ra85a], later improved by Alon and Boppana [AB], lead to exponential monotone size lower bounds for such functions in NP as $CLIQUE$. These results separate the monotone analogues of P and NP. Though these results can be used to obtain exponential (in $\log n$) monotone depth lower bounds as well, the depth lower bound is always logarithmic in the size bound. That is, the techniques apply to size rather than to depth. Our contribution is to present monotone depth lower bounds which are super-logarithmic in the size of the best circuit for the function considered. In this way, our results complement those by Andreev and Razborov. We present a tight $\Theta(\log^2 n)$ depth bound for *st*-connectivity [‡], a function which has $O(n^3 \log n)$ size monotone circuits.

[‡] We present an improved and simplified version of an early result giving a $\Omega(\log^2 n / \log \log n)$ bound. This was possible after J. Hastad and, independently, R. Boppana formulated and proved lemma 5.1.1.

As a consequence, we get both a super-polynomial ($n^{\Omega(\log n)}$) size lower bound for monotone formulas computing st-connectivity, and a separation of the monotone analogues of NC^1 and AC^1.

This thesis is organized as follows:

In § 2, we give an overview of the relevant definitions and previous work of both circuit complexity and communication complexity. In this chapter we treat these fields as two unrelated ones. We present a slightly different treatment to communication complexity from that in the literature. The main difference is that we consider mainly search problems, as opposed to decision problems.

In § 3, we develop our main thesis by defining and proving the equivalence between circuit depth (or formula size) and a related search problem in communication complexity. In this chapter we also vary the search problem in order to capture the essence of monotone circuit depth. We finish the chapter by giving some general consequences of the new approach.

In § 4, we demonstrate that the communication complexity approach is very intuitive by *i)* Presenting new, more intuitive proofs for some old results; and *ii)* Defining some new concepts which come about naturally in the communication approach. In § 4.1, we present new proofs of some results concerning slice functions, and the relation between monotone and non-monotone computation. In § 4.2, we show that the new approach may help us, not only to understand better some known upper bounds, but also to improve upon the known ones. In this section we present a couple of such examples. In § 4.3, we introduce the concept of a universal relation (closely related to that of a universal circuit). We give both deterministic as well as randomized protocols for these universal relations. We also show that, while the universal relation has efficient randomized protocols, its monotone version does not. Finally, in § 4.4, we present a new proof of a depth analogue of a theorem of Khrapchenko. We believe that this example best exemplifies the power of the

new setting.

In § 5, we demonstrate the usefulness of the new approach by presenting two monotone depth lower bounds. In § 5.1 we present the depth lower bound for *st*-connectivity. This is our main technical contribution. We would like to emphasize that most of the ideas behind the proof, and even the flow of the argument, were suggested by the new approach. In § 5.2, we present a recent result of Razborov [Ra88] which uses communication complexity to give a monotone lower bound for *MINIMUM COVER.*

In our last chapter, § 6, we comment upon some points regarding the approach in general, and our proofs in particular. We also propose some open problems which, we feel, will lead the way towards proving a general depth lower bound.

Preliminary results from this work have been published in [KW88]. The material contained in § 5.2 did not appear in the thesis of the author but is included in order to make this work more complete.