

Introduction

The principal example of a linear noisy network, and the one of greatest practical importance in electrical engineering, is a linear noisy amplifier. The noise performance of such amplifiers involves many questions of interest. One very significant question is the extent to which the amplifier influences signal-to-noise ratio over a narrow band (essentially at one frequency) in the system of which it is a part. We shall address ourselves exclusively to this feature, without intending to suggest that other features of the much larger noise-and-information problem are less important. The term “spot-noise performance” or merely “noise performance” will be used to refer to the effect of the amplifier upon the single-frequency signal-to-noise ratio. It is essential to emphasize right at the beginning the very restricted meaning these terms will have in our discussions.

We undertook the study reported here in the hope of formulating a rational approach to the characterization of amplifier spot-noise performance, and to its optimization by external circuit operations upon the terminals. Fortunately, a characterization has resulted which is based on a single hypothesis about the essential function of an amplifier and which turns out to avoid pitfalls previously associated with the effect of feedback upon noise performance. In developing the aforementioned noise characterization of amplifiers and in pursuing the relevant optimization problem, we encountered a number of illuminating features relating power and noise in linear multi-terminal-pair networks. Indeed, it eventually became clear that the major issues could be presented most simply by postponing until last the questions we had originally asked first. The result is a work of broader scope than was originally envisaged,

and one for which the title "Circuit Theory of Linear Noisy Networks" seems appropriate.

Since the introduction by Friis¹ and Fränz² of the concept of spot-noise figure F for the description of amplifier noise performance, this figure has played an essential role in communication practice. The noise figure is, however, merely a man-made definition, rather than a quantity deduced from clearly defined postulates or laws of nature. The possible consequences of this fact were never questioned deeply, although it has always been known that the (spot-) noise figure F does not constitute a single absolute measure of amplifier noise performance.

In particular, the noise figure is a function of the impedance of the source connected to the amplifier input. Thus in giving an adequate conventional description of amplifier noise performance, the source impedance, as well as the noise figure, must be specified.

Usually, when regarded as a function of source impedance alone, the noise figure has a minimum value for some particular choice of this impedance. If with this source impedance the gain of a given amplifier *remains sufficiently high*, its noise figure will prescribe the noise figure of any amplifier cascade in which it is used as the first stage. In this way, it is possible to build an amplifier cascade with any desired high gain, and with a noise figure set by the minimum (with respect to source impedance) of the noise figure of the original amplifier.

If a cascade is to be composed of several individual amplifiers, each of which alone has a "high enough" gain when driven from the source impedance that yields its minimum noise figure, the previous argument shows that the amplifier with the lowest minimum noise figure should be used as the first stage. Any other choice would result in a higher over-all noise figure for the cascade.

The foregoing discussion seems to suggest that the minimum value (with respect to source impedance) of the noise figure of an amplifier may be used as an absolute measure of its noise performance and as a basis for comparison with other amplifiers. The validity of the argument, however, is based upon the two previously mentioned restrictions:

1. Each stage has "*high enough*" gain when driven from the "optimum" source that yields the minimum noise figure.
2. Only the *source impedance* of each stage is varied in controlling the noise performance.

The inadequacy of this viewpoint becomes clear when stage variables other than source impedance and stage interconnections other than the

¹ H. T. Friis, "Noise Figure of Radio Receivers," *Proc. I.R.E.*, **32**, 419 (1944).

² K. Fränz, "Messung der Empfängerempfindlichkeit bei kurzen elektrischen Wellen," *Z. Elektr. Elektroak.*, **59**, 105 (1942).

simple cascade become important in amplifier applications. The question of the quality of noise performance then becomes much more complicated. For example, when degenerative feedback is applied to an amplifier, its noise figure can be reduced to as close to unity as desired (for example, bypassing the entire amplifier with short circuits yields unit noise figure). But its gain is also reduced in the process. Indeed, if identical stages with the feedback are cascaded to recover the original single-stage gain before feedback, the resulting noise figure of the cascade cannot be less than that of the original amplifier.³ Moreover, with degenerative feedback the gain may easily be so greatly reduced that, as a first stage in a cascade, this amplifier alone no longer determines the over-all noise figure of the cascade. The minimum-noise-figure criterion considered above as a measure of amplifier noise performance breaks down. It appears that an absolute measure of amplifier noise performance must include, in addition to the specification of noise figure and source impedance, at least the specification of the gain.

The foregoing reasoning led us to the investigation presented in this study. Taking our clues from the results previously found by Haus and Robinson⁴ for microwave amplifiers, and the method of active-network description presented by Mason,⁵ we searched for a measure of amplifier noise performance that would not only include the gain explicitly, as discussed earlier, but could also be minimized by external circuitry in a nontrivial way. Moreover, we believed that the minimum thus obtained should be a quantity characteristic of the amplifier itself. It should, for example, be invariant under lossless feedback, a type of feedback that does not appear to change the essential "noisy" character of the amplifier because it certainly adds no noise and can always be removed again by a realizable inverse lossless operation.

The precise form of a suitable noise-performance criterion has actually been known for many years, although its deeper significance somehow escaped attention. Indeed, the most glaring example of the correct criterion arises from the familiar problem of cascading two (or more) low-gain amplifiers having different noise figures F_1 and F_2 and different available gains $G_1(>1)$ and $G_2(>1)$.

The question is: If the available gain and noise figure of each amplifier do not change when the order of cascading is reversed, which cascade order leads to the best noise performance for the pair? Usually, "best noise performance" has been taken to mean "lowest noise figure" for the

³ A. van der Ziel, *Noise*, Prentice-Hall, New York (1954).

⁴ H. A. Haus and F. N. H. Robinson, "The Minimum Noise Figure of Microwave Beam Amplifiers," *Proc. I.R.E.*, 43, 981 (1955).

⁵ S. J. Mason, "Power Gain in Feedback Amplifiers," *Trans. IRE, Professional Group on Circuit Theory, CT-1*, No. 2, 20 (1954).

pair, though in view of the answer obtained on that basis, the criterion should have been viewed with a little suspicion. Thus if F_{12} and F_{21} are the respective noise figures of the cascade when amplifier No. 1 and amplifier No. 2 are placed first, we have

$$F_{12} = F_1 + \frac{F_2 - 1}{G_1} \quad (1.1a)$$

$$F_{21} = F_2 + \frac{F_1 - 1}{G_2} \quad (1.1b)$$

The condition that F_{12} be less than F_{21} is

$$F_1 - F_2 < \frac{F_1 - 1}{G_2} - \frac{F_2 - 1}{G_1}$$

or

$$(F_1 - 1) - (F_2 - 1) < \frac{F_1 - 1}{G_2} - \frac{F_2 - 1}{G_1}$$

or

$$\frac{F_1 - 1}{1 - \frac{1}{G_1}} < \frac{F_2 - 1}{1 - \frac{1}{G_2}} \quad (1.2)$$

That is, amplifier No. 1 should come first if Eq. 1.2 is satisfied.

Equation 1.2 implies that in a cascaded system of amplifiers, where the earliest stages are obviously the most critical in regard to noise performance, the "best" amplifier is the one having the lowest value not of F but of the quantity

$$M = \frac{F - 1}{1 - \frac{1}{G}} \quad (1.3)$$

It is with M that we shall be most concerned, and we shall call it the Noise Measure of an amplifier.

In terms of M , and the fact that the available gain of a cascaded pair of amplifiers is $G = G_1 G_2$, Eq. 1.1a becomes

$$M_{12} = M_1 + \frac{\Delta M}{G_1} \left(\frac{1 - \frac{1}{G_2}}{1 - \frac{1}{G}} \right) = M_1 + \Delta M \left(\frac{G_2 - 1}{G - 1} \right) \quad (1.4)$$

where $\Delta M \equiv M_2 - M_1$ is the difference between the noise measures of the second and first amplifiers of the cascade. Equation 1.4 shows that as long as G_1 and G_2 are greater than 1 the noise measure of a cascade of two amplifiers lies between the noise measures of its component amplifiers. In the particular case when the noise measures of the two amplifiers are equal, the resulting noise measure of the cascade is that of either amplifier, even if the available gains of the individual amplifiers are different.

Furthermore, since the available gain $G = G_1 G_2$ is supposed to remain the same for either order of cascading, the result (Eq. 1.2) and the definition (Eq. 1.3) show that the lowest noise measure for a cascaded pair of amplifiers results from placing at the input the amplifier with the lowest individual noise measure.

Compared with the noise figure alone, which always deteriorates in a cascade (Eqs. 1.1) and which does not suffice to determine which amplifier should come first, the noise measure alone is evidently a more satisfactory and self-consistent single criterion of amplifier noise performance. Moreover, since noise measure and noise figure become essentially the same for amplifiers with sufficiently high gain, the final performance evaluation of a practical multistage amplifier always rests numerically (if not in principle) upon the familiar noise-figure criterion.

From such reasoning, we evolved a criterion for amplifier noise performance. The criterion is based on the plausible premise that, basically, amplifiers are supposed to provide "gain building blocks" without adding excessively to system noise. In its final stage of evolution, the criterion can be described as follows.

Suppose that n different types of amplifiers are compared. An unlimited number of amplifiers of each type is assumed to be available. A general *lossless* (possibly nonreciprocal) interconnection of an arbitrary number of amplifiers of each type is then visualized, with terminals so arranged that in each case an over-all *two-terminal-pair network* is achieved. For each amplifier type, both the lossless interconnecting network and the number of amplifiers are varied in all possible ways to produce two conditions simultaneously:

1. A very high available gain (approaching infinity) for the over-all two-terminal-pair system when driven from a source having a positive real internal impedance.
2. An absolute minimum noise figure F_{\min} for the resulting high-gain system.

The value of $(F_{\min} - 1)$ for the resulting high-gain two-terminal-pair network is taken specifically as the "measure of quality" of the amplifier type in each case. The "best" amplifier type will be the one yielding the smallest value of $(F_{\min} - 1)$ at very high gain.

The proof of this criterion will be developed through the concept of noise measure. Inasmuch as the general criterion involves (at least) arbitrary lossless interconnections of amplifiers, including feedback, input mismatch, and so forth, a rather general approach to the noise measure is required. In particular, we must show that the noise measure has a real significance of its own which is quite different from and much deeper than the one suggested by its appearance in Eq. 1.2. There it appears only as an algebraic combination of noise figure and available gain that happens to be convenient for describing amplifier cascades. Here the properties of M with regard to lossless transformations are becoming involved.

Consideration of these properties brings us into the entire general subject of external network transformations of noisy linear networks. Among these, lossless transformations form a group in the mathematical sense. The quantities invariant under the group transformations must have a physical significance. Investigation of these invariants forms a substantial part of the present study. To be sure, for the special case of a two-terminal-pair amplifier, the optimum noise performance, through its related noise measure, turns out to be one of the invariants; but several other interpretations of the invariants prove equally interesting, and the development of the entire subject is simplified by presenting them first.

The simplest formulation and interpretation of the invariants of a linear noisy network result from its impedance representation. The following chapter is therefore devoted to a discussion of network transformations, or "imbeddings," in terms of the impedance-matrix representation. The concept of exchangeable power as an extension of available power is then introduced.

In Chapter 3, the n invariants of a linear noisy n -terminal-pair network are found as extrema of its exchangeable power, with respect to variations of a lossless n -to-one-terminal-pair network transformation. It is found that an n -terminal-pair network possesses not more than these n invariants with respect to lossless n -to- n -terminal-pair transformations. These n invariants are then exhibited in a particularly appealing way in the canonical form of the network, achievable by lossless transformations and characterized by exactly n parameters. This form is introduced in Chapter 4.

Through Chapter 4, the invariants are interpreted only in terms of the extrema of the exchangeable power. New interpretations are considered next. They are best introduced by using other than the impedance-matrix description. Accordingly, in Chapter 5, general matrix representations are studied, where it is pointed out that usually a different

matrix description leads to a different interpretation of the invariants.

In the case of an active two-terminal-pair network, a particularly important interpretation of the invariants is brought out by the general-circuit-parameter-matrix description. This interpretation relates directly to the optimum "noise measure" of the network used as an amplifier and, therefore, to the minimum noise figure of the amplifier at arbitrarily high gain. Chapter 6 is devoted to this noise-measure concept and to the range of values that the noise measure may assume for a two-terminal-pair amplifier subjected to arbitrary passive network transformations. In particular, the minimum value of the noise measure of the amplifier is found to be directly proportional to one of the two invariants of the amplifier.

A study is made of those arbitrary passive interconnections of two-terminal-pair amplifiers which result in an over-all two-terminal-pair amplifier. The conclusion is that the noise measure of the composite amplifier cannot be smaller than the optimum noise measure of the best component amplifier, namely, the amplifier with the smallest optimum noise measure.

The general theorems having established the *existence* of an optimum value of the noise measure of amplifiers, it remains in Chapter 7 to discuss in detail the network realization of this optimum for two-terminal-pair amplifiers. Some practical ways of achieving it are presented. Among these, the realization of optimum noise performance for a maser may be of greatest current interest.

With proof of the existence and realizability of a lower limit on the noise measure, and therefore of the noise figure at high gain, the major objective of the present work is accomplished. It is demonstrated that the quality with regard to noise performance of a two-terminal-pair amplifier can be specified in terms of a single number that includes the gain and that applies adequately to low-gain amplifiers.

We have previously published various separate discussions of some of these topics in different contexts.⁶⁻¹⁰ Each of these discussions has suffered from unnecessary complications because space limitations forced

⁶ H. A. Haus and R. B. Adler, "Invariants of Linear Networks," *1956 IRE Convention Record*, Part 2, 53 (1956).

⁷ H. A. Haus and R. B. Adler, "Limitations des performances de bruit des amplificateurs linéaires," *L'Onde Electrique*, **38**, 380 (1958).

⁸ H. A. Haus and R. B. Adler, "Optimum Noise Performance of Linear Amplifiers," *Proc. I.R.E.*, **46**, 1517 (1958).

⁹ R. B. Adler and H. A. Haus, "Network Realization of Optimum Amplifier Noise Performance," *IRE Trans. on Circuit Theory*, **CT-5**, No. 3, 156 (1958).

¹⁰ H. A. Haus and R. B. Adler, "Canonical Form of Linear Noisy Networks," *IRE Trans. on Circuit Theory*, **CT-5**, No. 3, 161 (1958).

them to be divorced from each other. It seemed, therefore, desirable to present the entire picture at greater leisure, particularly because the mathematical and logical complexity of the whole subject is thereby actually reduced.