1 Introduction

1.1 Orientation

A brief outline by chapters of the content of this monograph seems appropriate in order to focus attention on the subject matter, the original results, and the framework of the presentation. The first chapter is introductory; it deals mainly with mathematical preliminaries of a general nature.

The second chapter is devoted to the study of operators and specifically discusses questions related to the invertibility of nonlinear operators. This study is made in an algebraic framework, and special emphasis is placed on the properties of causal (nonanticipatory) operators. Causal operators are indeed of particular interest to engineers and physicists. The concept of causality is roughly equivalent to that of a "dynamical system" and is a basic restriction of physical realizable systems. In the algebraic framework employed in this monograph, causal operators are considered as a subalgebra in the algebra of (in general, nonlinear) operators. Another heavily emphasized and exploited concept is that of extended spaces. These consist of functions which are well-behaved on bounded intervals, but which do not satisfy any regularity conditions at infinity. Extended spaces have not been used extensively in analysis; however, they are the natural setting for the study of causal operators, and they form a very elegant conceptual framework for the study of dynamical systems described, for instance, by an ordinary differential equation with specified initial conditions or by Volterra integral equations.

The analysis in the third chapter is devoted to the derivation of some specific positive operators, which yield the inequalities leading to some general frequency-power formulas and stability conditions.

The basic concepts related to feedback systems are introduced in the fourth chapter. Only the analysis problem is considered, and the main questions investigated are well-posedness, stability, and continuity. This theory is developed in the framework of input-output descriptions of systems, and thus—following the modern trend of mathematical system theory—departs somewhat from the classical methods, which consider undriven systems with initial disturbances.

The fifth chapter discusses the Nyquist criterion and the circle criterion. These yield graphical conditions for stability and instability of linear (possibly time-varying) systems in terms of frequency-response data.

The sixth chapter is devoted to the study of some more complex stability criteria, which apply to systems with a linear time-invariant system in the forward loop and a periodically time-varying gain or a monotone memoryless nonlinearity in the feedback loop.

The final chapter discusses linearization techniques and shows that properly defined linearizations can indeed be successfully used for the analysis of the continuity of feedback systems. This linearization, however, is of a dynamical type and leads to time-varying systems even when the original system is time invariant. The final chapter also contains a simple and rather general class of counterexamples to Aizerman's conjecture.

1.2 Mathematical Preliminaries

This section introduces some notation and definitions which will be freely used throughout this monograph. More details may be found, for instance, in Refs. 1, 2.

A *set* (or *space*) is a collection of objects with a common property. The set, S, of objects with property P is denoted by $S \triangleq \{x \mid x$ has property $P\}$. A *subset* S_1 of a set S, denoted $S_1 \subset S$, is defined as $S_1 = \{x \mid x \in S$ and x has property $P_1\}$ and is sometimes denoted by $S_1 = \{x \in S \mid x$ has property $P_1\}$. The sets R, R^+, I, and I^+ denote respectively the real numbers, the nonnegative real numbers, the integers, and the nonnegative integers.

The *union* of the sets S_1 and S_2, denoted by $S_1 \cup S_2$, is defined as $S_1 \cup S_2 \triangleq \{x \mid x \in S_1 \text{ or } x \in S_2\}$. The *intersection* of the sets S_1 and S_2, denoted by $S_1 \cap S_2$, is defined as $S_1 \cap S_2 \triangleq \{x \mid x \in S_1 \text{ and } x \in S_2\}$. The *Cartesian product* of two sets, denoted by $S_1 \times S_2$, is defined as $S_1 \times S_2 \triangleq \{(x_1,x_2) \mid x_1 \in S_1, x_2 \in S_2\}$.

A *map* F (or *operator* or *function*), from a set S_1 into a set S_2 is a law which associates with every element $x \in S_1$ an element $Fx \in S_2$. S_1 is called the *domain* of the operator. If $S_1' \subset S_1''$ ($S_1'' \subset S_1'$) and if F' and F'' are maps from S_1' into S_2 and S_1'' into S_2 such that $F'x = F''x$ for all $x \in S_1'$ ($x \in S_1''$), then F' is called the *restriction* (an *extension*) of F'' to S_1' (S_1''). A *sequence* is a map from I (I^+) to a set S and will be denoted by $\{x_n\}$, $n \in I$ ($n \in I^+$).

A *metric space* is a set X and a map, d, from $X \times X$ into R^+ such that for all $x, y, z \in X$, the following relations hold: $d(x,y) = d(y,x) \geqslant 0$; $d(x,y) + d(y,z) \geqslant d(x,z)$ (the *triangle inequality*); and $d(x,y) = 0$ if and only if $x = y$.

A sequence $\{x_n\}$, $n \in I^+$, of elements of a metric space X is said to *converge* to a point $x \in X$ if $\lim_{n\to\infty} d(x_n,x) = 0$. This *limit point* x is denoted by $\lim_{n\to\infty} x_n$.

A subset, S, of a metric space, X, is said to be *open* if for any $x \in X$ there exists an $\epsilon > 0$ such that the set $N_\epsilon(x) \triangleq \{y \in X \mid d(x,y) < \epsilon\}$ is a subset of S. A subset, S, of a metric space, X, is said to be *closed* if any converging sequence $\{x_n\}$, $n \in I^+$, of points in S converges to a point in S. A sequence $\{x_n\}$, $n \in I^+$, of elements of a metric space S is said to be a *Cauchy* (or *fundamental*) *sequence* if given any $\epsilon > 0$ there exists an integer N such that $d(x_n,x_m) < \epsilon$ for all $n, m \geq N$. A metric space is said to be *complete* if every Cauchy sequence converges. Completeness is one of the most important properties of metric spaces. A subset of a metric space is said to be *compact* if every bounded sequence has a convergent subsequence.

A *vector space* (sometimes called a *linear space* or a *linear vector space*) is a set V and two maps, one called *addition*, denoted by $+$, from $V \times V$ into V, and the other called *multiplication* from the Cartesian product of the *field* of *scalars* K (which will throughout be taken to be the real or complex number system) and V into V such that for all $x, y, z \in V$ and $\alpha, \beta \in K$:

1. $(x + y) + z = x + (y + z)$;
2. there exists a *zero element*, denoted by 0, with $x + 0 = 0 + x = x$;
3. there exists a *negative element*, denoted by $-x$, with $x + (-x) = 0$ ($y + (-x)$ will be denoted by $y - x$);

4. $x + y = y + z$;
5. $(\alpha + \beta)x = \alpha x + \beta x$;
6. $\alpha(x + y) = \alpha x + \alpha y$;
7. $(\alpha\beta)x = \alpha(\beta x) \triangleq \alpha\beta x$;
8. $1 \cdot x = x$.

A vector space is called a *real* or *complex* vector space according to whether the field K is the real or complex number system. R^n denotes the real vector space formed by the n-tuples of real numbers with addition and multiplication defined in the obvious way. A vector space V is called a *normed vector space* if a map (the *norm*), denoted by $\| \ \|$, from V into R^+ is defined on it, such that:

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha| \, \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$ (the *triangle inequality*).

The norm induces a *natural metric* $d(x,y) \triangleq \|x - y\|$, and all statements (e.g. concerning convergence) always refer to this metric, unless otherwise mentioned. Sometimes the norm is subscripted for emphasis, as $\| \ \|_V$, but the subscript will be deleted whenever there is no danger of confusion.

A *Banach space* is a complete normed vector space. This completeness is, of course, to be understood in the topology induced by the natural metric. A very useful class of Banach spaces are the so-called *L_p-spaces.* These consist of Banach space B-valued functions defined on a measurable set $S \subset R$, for which the pth power of the norm is integrable[1] in the case $1 \leq p < \infty$, with the norm defined by

$$\|x\|_{L_p^B(S)} \triangleq \left(\int_S \|x(t)\|_B^p \, dt \right)^{1/p}$$

The space $L_\infty^B(S)$ is defined as the collection of all measurable B-valued functions defined on a measurable set $S \subset R$ which are essentially bounded (i.e., there exists a real number $M < \infty$ such that $\|x(t)\|_B \leq M$ for almost all $t \in S$) with

$$\|x\|_{L_p^B(S)} \triangleq \{\inf M \mid \|x(t)\|_B \leqslant M \text{ almost everywhere on } S\}.$$

[1] The integration and measurability considerations refer to Lebesgue measure and integration when the Banach space B is finite-dimensional. Otherwise, these notions are to be interpreted in the sense of Bochner (see, e.g., Ref. 1, p. 78).

The sequence spaces l_p^B are defined in an analogous way, with the integral replaced by a summation. When B is taken to be the real or complex numbers and S is the interval $(-\infty, +\infty)$ for L_p-spaces or $S = I$ for l_p-spaces, then $L_p^B(S)$ and $l_p^B(S)$ will be denoted by L_p and l_p respectively, when no confusion can occur. The L_p-spaces are very often used in analysis. The triangle inequality for L_p-spaces is known as *Minkowski's inequality*. Another useful inequality for L_p-spaces is *Hölder's inequality*, which states that for $f \in L_p(S)$ and $g \in L_q(S)$ with $1/p + 1/q = 1$ and $1 \leqslant p, q \leqslant \infty$, $fg \in L_1(S)$, and

$$\|fg\|_{L_1(S)} \leqslant \|f\|_{L_p(S)} \|g\|_{L_q(S)}.$$

An *inner product space* is a linear vector space, V, with a map, denoted by $\langle\ \rangle$ and called the *inner product*, from $V \times V$ into the scalars K such that for all $x, y, z \in V$ and scalars α, β, the following relations hold:

1. $\langle x,y\rangle = \overline{\langle y,x\rangle}$ (the overbar denotes complex conjugation);
2. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$; and
3. $\langle x,x\rangle \geqslant 0$ and $\langle x,x\rangle = 0$ if and only if $x = 0$.

The inner product induces a *natural norm* $\|x\| = \sqrt{\langle x,x\rangle}$ and all statements (e.g., concerning convergence) always refer to the metric induced by this norm, unless otherwise mentioned. Sometimes the inner product is denoted by $\langle,\rangle_V$, but the subscript will be deleted whenever there is no danger of confusion.

The *Cauchy-Schwartz inequality* states that $|\langle x,y\rangle| \leqslant \|x\|\ \|y\|$.

A *Hilbert space* is a complete inner product space. This completeness is, of course, to be understood in the topology induced by the natural norm. The standard example of a Hilbert space is R^n with $\langle x,y\rangle_{R^n} = \sum_{i=1}^n x_i y_i$ where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. A very useful class of Hilbert spaces are the L_2-*spaces*. These consist of Hilbert space H-valued functions defined on a measurable set $S \subset R$, for which the square of the norm is integrable, and with

$$\langle x,y\rangle_{L_2^H(S)} \triangleq \int_S \langle x(t),y(t)\rangle_H\, dt.$$

Similarly, $l_2^H(S)$ with $S \subset I$ is a Hilbert space, with

$$\langle x,y\rangle_{l_2^H(S)} \triangleq \sum_{n\in S} \langle x_n,y_n\rangle_H.$$

1.3 Transform Theory

Definitions: If $x \in L_1$, then the function X defined by

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

is called the *Fourier transform* of x. Clearly $X \in L_\infty$ and $\|X\|_{L_\infty} \leqslant \|x\|_{L_1}$; if $x(t)$ is real, then $X(j\omega) = \bar{X}(-j\omega)$. Since this transform need not belong to L_1, it is in general impossible to define the inverse Fourier transform. However, if X itself turns out to belong to L_1 then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

(As always, this equality is to be taken in the L_1 sense, that is, except on a set of zero Lesbesgue measure.) Thus the need of a slightly more general transform in which the inverse transform can always be defined is apparent. This is done by the limit-in-the-mean transform. It is well known that if $x,\ y \in L_2 \cap L_1$ then $X,\ Y \in L_2$ and $\langle x,y\rangle = \langle X, Y\rangle/2\pi$ (*Parseval's equality*). Let $x \in L_2$. Since $L_1 \cap L_2$ is dense in L_2, i.e., since any L_2-function can be approximated arbitrarily closely (in the L_2 sense) by a function in $L_1 \cap L_2$, there exists a sequence of functions $\{x_n\}$ in $L_2 \cap L_1$ which is a Cauchy sequence and which converges to x (in the L_2 sense). Let X_n be the Fourier transform of x_n. It follows from the Parseval relation that $\|x_n - x_m\| = (2\pi)^{-1/2}\|X_n - X_m\|$ and that $X_n \in L_2$. Thus since L_2 is complete, these transforms, X_n, converge to an element X of L_2. This element X is called the *limit-in-the-mean* transform of x. It follows that the limit-in-the-mean-transform maps L_2 into itself and that $\langle x,y\rangle = \langle X, Y\rangle/2\pi$ for all $x, y \in L_2$ and their limit-in-the-mean transforms $X,\ Y$.

One way of defining a limit-in-the-mean transform is by

$$X(j\omega) = \lim_{T\to\infty}\int_{-T}^{T} x(t)e^{-j\omega t}\,dt$$

where the limit is to be taken in the L_2 sense. (It is easily verified that this induces a particular choice for the Cauchy sequence $\{x_n\}$.) The notation that will be used for the limit-in-the-mean transform is

$$X(j\omega) = \text{l.i.m.}\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt.$$

With this definition of transforms, the inversion is always possible, and the *inverse transform formula* states that

$$x(t) = \text{l.i.m.}\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

Let $x \in L_2(0,T)$, $T > 0$. Then the sequence $X = \{x_k\}$, $k \in I$, given by

$$x_k = \frac{1}{T}\int_0^T x(t)e^{-jk2\pi t/T}\,dt,$$

is well defined since $L_2(0,T) \subset L_1(0,T)$, and is called the *Fourier series* of $x(t)$. Clearly $X \in l_\infty$ and $x_k = \bar{x}_{-k}$ whenever $x(t)$ is real. The *Parseval relation* states that if x_1, $x_2 \in L_2(0,T)$ and if X_1, X_2 are their Fourier series, then X_1, $X_2 \in l_2$ and $\langle x_1,x_2\rangle_{L_2(0,T)} = 2\pi\langle X_1,X_2\rangle_{l_2}$.

In trying to obtain the *inverse Fourier series* formula, the same difficulties are encountered as with the inverse Fourier transform, and the same type of solution is presented. This leads to

$$x(t) = \text{l.i.m.}\sum_{k=-\infty}^{+\infty} x_k e^{jk2\pi t/T}.$$

One way of expressing this l.i.m. summation is by

$$x(t) = \lim_{N\to\infty}\sum_{k=-N}^{N} x_k e^{jk2\pi t/T},$$

where the limit is to be taken in the $L_2(0,T)$ sense.

If $x \in l_1$, then the function X defined by

$$X(z) = \sum_{k=-\infty}^{+\infty} x_k z^{-k}$$

exists for all $|z| = 1$ and is called the *z-transform* of x. In trying to extend this notion to sequences in l_2 the same difficulties and the same solution as in the previous cases present themselves. This leads to the *limit-in-the mean z-transform*

$$X(z) = \text{l.i.m.}\sum_{k=-\infty}^{+\infty} x_k z^{-k}$$

and the *inverse z-transform*

$$x_k = \frac{1}{2\pi}\oint_{|z|=1} X(z)z^{-1}\,dz$$

where the integral is interpreted in the usual manner since

$$L_2(|z| = 1) \subset L_1(|z| = 1).$$

A continuous function, x, from R into K is said to be *almost periodic* if for every $\epsilon > 0$ there exists a real number l such that every interval of the real line of length l contains at least one number τ such that

$$|x(t + \tau) - x(t)| \leqslant \epsilon \text{ for all } t.$$

Some properties of almost periodic functions are:

1. Every almost periodic function is bounded and uniformly continuous.
2. Continuous periodic functions are almost periodic.
3. The sums, products, and limits of uniformly convergent almost periodic functions are almost periodic.
4. The limit, as $T \to \infty$, of the mean value
$$\frac{1}{2T} \int_{-T}^{T} x(t + \tau)\, dt$$
exists, is independent of τ for all almost periodic functions x, and converges uniformly in τ.
5. If x_1 and x_2 are almost periodic functions then so is
$$x_1 * x_2 \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t - \tau) x_2(\tau)\, d\tau.$$
Moreover, $x_1 * x_2 = x_2 * x_1$ and $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$ for all almost periodic functions x_1, x_2, x_3.
6. The function
$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) e^{-j\omega t}\, dt$$
vanishes for all but a countable number of values of ω.
7. The space of almost periodic functions forms an inner product space with
$$\langle x_1, x_2 \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t) \bar{x}_2(t)\, dt$$
for x_1, x_2 almost periodic functions. (This inner product space is, however, not complete and not separable.) Let x be an almost periodic function and let $\{\omega_k\}$ be the set of values for which the limit in (6) does not vanish and let x_k be the value of that limit for $\omega = \omega_k$. The sequence $\{x_k\}$ is called the *generalized Fourier series* of $x(t)$. If

$x(t)$ is real, then ω belongs to the set $\{\omega_k\}$ if and only if $-\omega$ does and the values x_k associated with ω and $-\omega$ are complex conjugates. The *inverse Fourier series* is defined as

$$x(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} x_k e^{j\omega_k t}$$

This limit, which exists, is to be taken in the metric induced by the inner product on the space of almost periodic functions.[2]

[2] For more details on transform theory, see Refs. 3, 4.

References

1. Hille, E., and Phillips, R. S., *Functional Analysis and Semi-Groups* (second edition), American Mathematical Society, Providence, 1957.
2. Dunford, N., and Schwartz, J. T., *Linear Operators, Parts I and II*, Interscience, New York, 1958.
3. Titchmarsh, E. C., *Introduction to the Theory of Fourier Integrals*, Oxford University Press, Oxford, 1937.
4. Paley, R., and Wiener, N., *Fourier Transforms in the Complex Domain*, American Mathematical Society, Providence, 1934.