Introduction

This book is intended to provide the reader with a rigorous introduction to the theory and application of estimation and association techniques. Skills taught in this book will prepare the student for solving practical problems in this technical area.

Estimation and association involves the extraction of information from noisy measurements. Example applications include signal processing, tracking, navigation, and so on [1, 2, 3]. The extraction of parameter values from signals in order to estimate such attributes as time-of-arrival and sensor pointing angle is called parameter estimation. A sensor signal may have come from a moving object. Determining the kinematics of a moving object is called state estimation. Associating measurements with state estimates in a multiple object environment is a joint estimation and association problem that is known as tracking [2–5]. Example applications include sensor surveillance systems for air traffic control, guiding space vehicles toward a planet, extracting information regarding a moving object with multiple-degree-of-freedom motions, and so on.

The authors of this book, together with their colleagues, have been applying the theory and techniques of estimation and association to real-world problems for the past 40 years. They have taught classes to Lincoln Laboratory staff members who are involved in applying these skills as well as solving problems of their own. The content of this book represents their collective experience in applying estimation and association techniques. The technical level of this book is equivalent to a first year graduate course in a control or system engineering curriculum. The students are required to be familiar with the state-variable representation of systems, and basic probability theory including random variables and stochastic processes. This book can also be used for self-study by practitioners in the area of state estimation and association.

Theory and techniques developed in this book are for discrete time systems. Although all physical systems are continuous in time, the measurements are taken in discrete time and the computational system that exploits the measurements operates in discrete time. Furthermore, unique discrete time equivalence to continuous time systems can be easily derived and implemented. The use of discrete time models enables us to solve the problem without resorting to more abstract mathematics such as measure theory and Ito calculus [6]. Homework problems are included at the end of each chapter. The purpose is two-fold: (a) to develop students confidence in their derivation skills so that they are able to apply them to new problems, and (b) to build computer models so that they will have a useful set of tools for problem solving.

The theory and application of estimation has been a rich field of research for decades. The landmark papers by Kalman [4], and Kalman and Bucy [5] gave the optimal solution for state estimation of linear systems having Gaussian system and measurement noise processes. The Kalman filter (KF) algorithm using state space modeling makes it suitable for implementation with digital computers. Kalman's paper also laid the foundation for the concept of observability of a linear system, and its relationship with the Fisher information matrix and the Cramer–Rao bound (CRB) [1] for all unbiased state estimators. For this reason, it has gained enormous interest from practicing engineers. However, most of the real-world application problems are nonlinear. After Kalman's publication, considerable effort was devoted to finding the optimal filter for nonlinear systems (the counterpart of the KF for linear systems) [6]. All these studies came to the same conclusion: the solution of the optimal filter requires an infinite dimensional representation that cannot be practically constructed. Consequently, follow-on efforts focused on searching for suboptimal but practical solutions.

The approach used in this book has two features: (a) it formulates the estimation problem as an optimization problem using measurement data and a priori knowledge of the system, and (b) it develops CRB solutions for each estimation problem addressed. The first feature stresses that the solution to the estimation problem provides a best fit to the measurement data, the system model, and the a priori knowledge. It will be shown that solution algorithms for most of the estimation problems can be obtained this way. The CRB has been well known in signal processing for estimating parameters embedded in the signal [1]. It has been applied to a wide range of state estimation problems at Lincoln Laboratory [7]. In keeping with the second feature, the CRB models for parameter and state estimation are derived for the examples considered or are included as part of the homework problem assignments.

In many engineering applications, noisy measurements are obtained on some unknown variables. Variables of interest can collectively be represented as a vector. Measurements can be arranged as a measurement vector or a set of measurement vectors. In the case where the vector of interest is constant or random, it is referred to as a parameter vector. In the case where the vector of interest is time-varying and follows a set of differential equations for a continuous-time system, or difference equations for a discrete-time system, it is termed a state vector. A parameter vector is a special case of the state vector. The concept of a state vector is identical to the state vector used in the state space representation of control systems [8]. A state vector can be deterministic or random, depending on whether the system is deterministic or driven by a random process.

The estimation problem is to find a solution to the unknown vector using measurements and knowledge about the vector of interest. The measurements used in an estimator are assumed to have come from a single object or dynamic system. This assumption may not be true when multiple objects are closely spaced in sensor measurements. The problem of state association is to determine whether a measurement or a set of measurements comes from the same object.

This book has 10 chapters. Chapters 1 to 6 focus on solving the problem of estimation with a single sensor observing a single object. Chapter 7 expands consideration from a single sensor observer to multiple sensors. Chapters 8 through 10 address the problem of association by expanding the problem to multiple objects and multiple sensors. Concluding remarks and three appendices are offered at the end. They are introduced individually below.

Chapter 1: Parameter Estimation

In this text, a parameter vector can be a constant vector or a random vector with known distribution, but is never a random process. The foundation of estimation can be understood most easily by solving the problem of parameter estimation. The estimate of an unknown vector is obtained by selecting the vector that optimizes a performance criterion or a cost function given the noisy measurements. Six performance criteria are introduced in this chapter, namely, least squares, weighted least squares, maximum likelihood, maximum a posteriori probability, conditional mean, and linear least squares expressed as functions of the measurements [1, 4]. Explicit estimator solutions for linear measurements with Gaussian measurement noise are developed and the equivalence of all six estimators is discussed. It is shown that the a posteriori density function of the parameter vector conditioned on measurements contains all the information for estimating this parameter vector, regardless of whether the measurement relationship is linear or nonlinear, and the conditional mean is the minimum norm solution in the parameter space. For the linear measurement relationship, the closed form solution can be found. For nonlinear measurements, a numerical solution to the weighted least squares estimator is derived. The Cramer–Rao bounds for all cases are derived. The relationship between weighted least squares estimator, minimum variance estimator, and the conditional mean estimator is shown in the appendix.

Chapter 2: State Estimation for Linear Systems

A state vector is the solution of a first order vector differential equation for a continuous system, or difference equation for a discrete system [8]. When the initial condition is a random variable and/or when the system is driven by a random system noise process, the state vector represents a random process. For linear systems with Gaussian system and measurement noise, the a posteriori density of the state conditioned on measurements remains Gaussian, and the state estimate can therefore be completely characterized by the conditional mean and covariance. This result is known as the Kalman filter [4]. The techniques used in Chapter 1 to derive the parameter estimator are extended in this chapter to derive the KF solution for linear systems. These include the conditional mean, weighted least squares, and Bayesian recursive evolution of the a posteriori density function. The concept of smoothing is introduced, and the chapter ends with derivations for the CRB for all cases of interest.

Chapter 3: State Estimation for Nonlinear Systems

Many physical systems and measurement devices are nonlinear. As mentioned before, the conditional mean is the minimum norm estimate, and the a posteriori density function of a state conditioned on measurements contains all the information necessary for estimation. For linear systems with Gaussian noise, the a posteriori density remains Gaussian. This property is, however, no longer true for nonlinear systems even when the input and measurement noise processes are Gaussian. The recursive Bayesian relationship governing the time evolution of the a posteriori density for arbitrary nonlinear systems was published within a few years of Kalman filter [9, 10], but its exact solution for estimation remains open. For this reason, only approximated solutions for the nonlinear estimation problems have found applications. The approximated solutions include the use of the first order Taylor series expansion (the extended Kalman filter) and the addition of the second order term in the Taylor series expansion (the second order filter) [11]. Both filters are aimed at providing approximated conditional means and covariance solutions for the state estimator. Additional nonlinear estimation techniques presented in this chapter include the extension of the numerical solution for estimating a random parameter vector with nonlinear measurements of Chapter 1 to the problems of Chapter 3 to yield a single-stage iterative solution for computing state updates in extended Kalman filter (EKF). A special case for nonlinear estimation occurs when the system is deterministic. An iterative weighted least squares estimator for a deterministic nonlinear system using all the measurements is derived [12]. The numerical examples in [12] shows that the covariance of this algorithm achieves the CRB.

The EKF has gained considerable attention from practitioners due to its simplicity and direct relationship with KF: the KF algorithm becomes an EKF algorithm when the system and measurement matrices of a linear system are replaced by Jacobian matrices of the nonlinear system. The EKF does not solve all nonlinear estimation problems, nor does it provide the best answer even when it does work. Its similarity in functional form with the KF makes much of the KF analysis extendable to the EKF. For this reason, linear systems are used for discussion throughout this book. Exceptions will be noted. To conclude this chapter the CRB for nonlinear state estimation is developed.

Chapter 4: Practical Considerations in Kalman Filter Design

The previous three chapters provide the basic tools for estimation: problem definition, solution derivation, and solution algorithms. In this chapter, practical issues in filter design are discussed. Filter construction is based upon a mathematical representation of the physical process of interest. For most engineering problems, mathematical models do not exactly represent the actual physical process resulting in a less than optimal filter performance. Model differences can occur in the system equation, measurement equation, system input, measurement noise, and so on [13]. This chapter starts with a discussion about tools for estimator performance monitoring that includes the CRB, the measure of statistical behavior of the filter residual process, and the measure of filter consistency in terms of actual and computed covariance. The rest of the chapter provides detailed discussions for addressing the system model mismatch problem, measurement error uncertainty, and systems with uncertain inputs. For each subject, filter compensation methods are suggested. The issue of systems with uncertain inputs is related to tracking objects having unknown or unexpected maneuvers. A maximum likelihood estimator with its associated generalized likelihood ratio (GLR) detection algorithm is developed for systems with sudden input changes [14]. A discussion on the advantages and limitations of this approach is provided. An extension to this approach is considered by assuming that the state may be generated by one of several models (e.g., maneuvering and nonmaneuvering), and that leads into the next chapter, multiple model estimation algorithms.

Chapter 5: Multiple Model Estimation Algorithms

When the underlying true system could be one of several different models, the estimation solution is a bank of KFs with each matched to a specific model within the set of models.[1] The probability (termed the hypothesis probability) that a given filter represents the true system can be computed for each filter using the filter residual. The filter with the highest hypothesis probability is deemed to represent the truth (e.g., a target is maneuvering or not maneuvering). It can be shown that the conditional mean estimate is the weighted sum of the output of all estimators (for both state and covariance) with the hypothesis probabilities as weighting factors. This solution is optimal for linear systems when the truth stays the same as one of the models used in the filter bank [15], referred to as the constant model case. In practical problems, the true system may be changing among models. For example, a target may switch back and forth between maneuvering and nonmaneuvering at multiple instances of time. This is referred to as a switching model case. The solution to the switching model case is unbounded, that is, the true system could switch to a different model in multiple instances of time making the number of possibilities grow exponentially [16]. Approximate solutions are derived for the case when the model switching history has finite memory, for example, for a Markov process. Solutions for this problem include generalized pseudo-Bayesian (GBP) algorithm and interacting multiple model (IMM) algorithm [16, 17]. The IMM is an approximation of GBP but is simpler in implementation. In several applications, the trade-off of these two algorithms is in favor of IMM because the performance gain of GBP over IMM is small. The derivations of all these cases are included in this chapter along with a set of numerical examples.

Chapter 6: Sampling Techniques for State Estimation

In Chapter 3, several solutions to the nonlinear estimation problem were presented in the form of an approximated conditional mean and covariance computation. As stated before, the a posteriori density function of state conditioned on measurements contains all the information for estimation. In this chapter, several numerical methods for

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1 The same concept as the bank of Doppler filters for radar signal processing.

computing the a posteriori density function are presented [18–20]. Two deterministic sampling techniques are introduced. The grid-based point-mass filter was introduced in 1969 [18], which is intuitive but computationally expensive. The second method used the unscented transformation for Gaussian noise known as the unscented Kalman filter (UKF) [19]. UKF is computationally more efficient and has gained considerable popularity since the late 1990s. Random sampling techniques by means of Monte Carlo sampling collectively known as particle filter methods [20] are introduced in this chapter. The goal is to find a numerical approximation to the a posteriori density function and hence, the conditional mean can be computed with a Monte Carlo integral. The concept of sequential Monte Carlo (SMC) sampling is based on point mass (or particle) approximation of the probability density functions involved in the estimation problem. Although the concept of this technique was first developed in the 1950s, it gained popularity in 1990s due to the availability of high-speed computers that makes its realization more feasible. Still, due to the large computational requirement, the SMC has not been used in conventional tracking/filtering problems, but is used for problems with smaller dimension and for nonconventional problems such as when the motion of the object of interest is less analytical (e.g., tracking hand motion, tracking an object moving in a maze, etc.) or the measurement equation is nonlinear and nonanalytical (e.g., hard limiting, hysteresis, etc.). Several Monte Carlo sampling techniques can be applied to these problems, such as rejection sampling, importance sampling, and Markov chain Monte Carlo (MCMC) method, among others. In this chapter, discussions are focused on techniques involving importance sampling. SMC is a current area of research and some approaches are mentioned in this chapter [21, 22].

Chapter 7: State Estimation with Multiple Sensor Systems

Problems addressed in Chapters 1 through 6 consider a single sensor observing a single object. Expansion to multiple sensors and then to multiple objects are the focus of the remainder of this book. Chapter 7 introduces the problem of estimation using multiple sensors. There are many advantages to using multiple sensors. For example, sensors distributed over a wide geographical area can provide a diversity of viewing geometry in which differences in look angle result in improved estimation accuracy. One example of a multisensor system is the air traffic control system that employs a number of radars of integrated together with a communication system. Data in a multisensor system must be fused in order to achieve the potential benefits. An example of multiple sensor architecture is to send all sensor measurements to a centralized processor for processing [23]. This is called measurement fusion. The state estimate obtained this way is considered to be global because it contains all available data on the object. Another possible architecture is to have each individual sensor process its own data to obtain its local estimate of object state. It is called a local estimate because it only uses the sensor data available locally. All local estimates are then sent to a centralized processor to obtain a joint estimate. This is termed state fusion [24]. When the transmission of local estimates does not have to be done frequently, state fusion architecture could result in a smaller communication requirement. All these algorithms are presented and discussed in this chapter. Proof that the joint estimate is less accurate than the global estimate is presented in the appendix.

Chapter 8: Estimation and Association with Uncertain Measurement Origin

Fundamental theories and algorithms for the problem of estimation were developed in Chapters 1 through 7. In all cases, an estimator was used to process a set of measurement vectors with the assumption that all measurements came from the same object. In the case where multiple individual objects are closely spaced, the assignment of measurements to a state may become ambiguous. Multiple approaches to this problem have been discussed in the literature, ranging from using a single scan to multiple scans of data, treating each track individually or jointly, making decision for pairing measurements with tracks as an assignment problem or combining all measurements probabilistically for track update, and so on. One approach to solving this problem is to exhaustively enumerate all possible solutions (including accounting for the possibility of missed and false detections) at each measurement time. This approach is known as the multiple hypothesis tracker (MHT), and is the subject of Chapter 9. The focus of Chapter 8 is to present a menu of approaches other than MHT. Mathematical representation in assigning measurements to tracks over multiple scans is developed. A practical solution starting with multiframe track initiation followed by track continuation is presented [25, 26]. An algorithm for solving the assignment problem for a single frame decision (referred to as immediate resolution) is given, and a solution for the multiple frame decision (referred to as delayed resolution) is described. Comparison of results in using single and multiple frame decisions are illustrated in a numerical example. The appendix gives a track initiation algorithm using equations in dish radar coordinates with the state vector expressed in radar centered Cartesian coordinates.

Chapter 9: Multiple Hypothesis Tracking Algorithm

In this chapter, the MHT for a multiple target tracking problem is described [27–29]. In the MHT formulation, when assigning a new measurement with tracks, the following possibilities are always considered: (a) the new measurement is the continuation of an existing track, (b) the measurement is the start of a new track, and (c) the measurement is a false alarm. Give these considerations the case that a track continues without being updated with a measurement is always one of the tracks. A track is a time sequence of measurements obtained over multiple scans. A measurement from a given scan is allowed to be used by multiple tracks. This is because a measurement may be considered as the continuation of an existing track as well as the start of a new track at the same time since the new measurement could originate from multiple objects and be unresolved due to the limited resolution of the measurement sensor. A hypothesis in MHT consists of a set of tracks that use a measurement only for one track within that hypothesis. The number of tracks can grow combinatorially in MHT. Pruning is necessary in MHT in order to limit the growth of tracks and hypotheses. A method for scoring tracks and hypotheses is developed for pruning purposes, and a numerical example for track and hypothesis scoring is discussed.

Chapter 10: Multiple Sensor Correlation and Fusion with Biased Measurements

Two fusion architectures for multiple sensor systems were introduced in Chapter 7, namely, measurement fusion and state fusion. The ability to realize benefits of a multiple sensor system is dependent on (a) the capability to handle track ambiguities, and (b) the capability to handle sensor biases. The purpose of this chapter is to present approaches to correlation and estimation for multiple sensors with biased measurements. The first half of this chapter is focused on illustrating approaches for bias estimation by means of state augmentation in the measurement fusion architecture. The results are illustrated using a space object track example [30]. In this example, the association problem is assumed solved. The approach to jointly solving the state to state correlation problem and the bias estimation problem in the state fusion architecture is the subject of the second half of the chapter [31, 32]. It is first formulated as a joint mathematical optimization problem followed by several suggested solution algorithms.

Concluding Remarks

Throughout discussions in this book, it is made evident that there are areas of unsolved problems in estimation and association. Some of these are discussed in this final chapter.

Three Appendices: Matrix Inversion Lemma (MIL), a List of Notation and Variables, and Terminologies Used in Tracking

Appendix A: MIL provides a well-known identity for matrix inversion of a specific form. It is used repeatedly in estimator derivations. It is included in this appendix with derivation for easy reference.

Appendix B: Throughout the book there are scalars, vectors, matrices, probability density functions, conditional probability density functions, statistical expectation in terms of means and covariances, hypotheses, hypothesis probabilities, indices for multiple sensors and targets, and so on. A list of symbols and notation is provided as a quick reference for readers.

Appendix C: Terminology often used in the tracking community is defined. It is included in this appendix for the purpose of cross-referencing.

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