Chapter 1
The Basic Intertemporal Model
Answer Key to Exercises[1]

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1. Time inconsistency: the role of discounting

This exercise takes as given the time separability of preferences and illustrates the role of hyperbolic discounting in generating time inconsistency. To this end, the exercise first asks you to verify that time-separable preferences with exponential discounting (i.e., the preferences used in the text) are time consistent. It then asks you to work out an example in which the introduction of hyperbolic discounting renders preferences time-inconsistent.

(a) *Exponential consumer*

Suppose that the preferences of the "exponential" consumer are given by:

$$U_0(c_0, c_1, ...) = \sum_{t=0}^{\infty} \delta^t \log(c_t), \tag{80}$$

where c_t is consumption in period t, $\delta \in (0, 1)$, and the subscript 0 on the lifetime utility function indicates that the consumption path is being evaluated as of time $t = 0$.

The flow constraint for period t is given by

$$b_{t+1} = (1 + r)b_t + y - c_t,$$

where y is the constant endowment, r is the exogenously-given world real interest rate, and b_t are net foreign assets held between period t and period $t + 1$. Iterating forward the flow constraint and imposing the condition that

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$$\lim_{t\to\infty} \frac{b_{t+1}}{(1+r)^t} = 0$$

yields the following intertemporal constraint:

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t = (1+r)\left(b_0 + \frac{y}{r}\right). \tag{81}$$

In this context:

i. Define the discount factor at time t (a measure of the degree of the consumer's impatience) as the marginal rate of substitution between consumption at two consecutive dates for a constant consumption path, $\bar{c}$:

$$\text{Discount factor}_t \equiv MRS^t_{t,t+1} = \left.\frac{\partial U(c_0, c_1, ...)/\partial c_{t+1}}{\partial U(c_0, c_1, ...)/\partial c_t}\right|_{c_0=c_1=...=\bar{c}}.$$

Suppose that the consumer is standing at time t. Compute the discount factor for consumption between $t+1$ and $t+2$ (i.e., compute $MRS^t_{t+1,t+2}$). Next suppose that the consumer is standing at $t+1$. Recompute the discount factor for consumption between $t+1$ and $t+2$ (i.e., compute $MRS^{t+1}_{t+1,t+2}$). Verify $MRS^t_{t+1,t+2} = MRS^{t+1}_{t+1,t+2}$.

ii. Denote by c^0_t, $t = 0, 1, ...$ the optimal consumption path chosen at $t = 0$. Compute reduced-forms for c^0_t, $t = 0, 1, ...$

iii. Denote by c^1_t, $t = 1, 2, ...$ the optimal consumption path chosen at $t = 1$. Verify that the optimal consumption path chosen at $t = 1$ is time consistent (i.e., coincides with the path chosen at $t = 0$). (Hint: Compute reduced-forms for c^1_t, $t = 1, 2, ...$ and check that $c^0_1 = c^1_1$, $c^0_2 = c^1_2$, and so forth).

(b) *Hyperbolic consumer*

Suppose now that preferences are of the "quasi-hyperbolic" type (see, for example, Backus, Routledge, and Zin (2004)):

$$U_0(c_0, c_1, ...) = \log(c_0) + \rho \sum_{t=1}^{\infty} \delta^t \log(c_t), \tag{82}$$

where $\rho \in (0, 1)$.

In this context:

i. Suppose that the consumer is standing at time t. Compute the discount factor for consumption between $t+1$ and $t+2$ (i.e., compute $MRS^t_{t+1,t+2}$). Next suppose that the consumer is standing at $t+1$. Recompute the discount factor for consumption between $t+1$ and $t+2$ (i.e., compute $MRS^{t+1}_{t+1,t+2}$). Verify that

$MRS^t_{t+1,t+2} > MRS^{t+1}_{t+1,t+2}$. Notice that this implies that the consumer is more patient about consuming between tomorrow and the day after tomorrow from the standpoint of today than from the standpoint of tomorrow. In this sense, the consumer is more impatient in the "short-run" than in the "long-run", which is the defining characteristic of hyperbolic discounting.

ii. Compute reduced forms for c^0_t, $t = 0, 1, ...$

iii. Compute reduced forms for c^1_t, $t = 1, 2, ...$ Assume, for simplicity, that $b_0 = 0$. Show that the optimal consumption path chosen at $t = 1$ is time inconsistent (i.e., does not coincide with the path chosen at $t = 0$). (Hint: Compute reduced-forms for c^j_1, $j = 0, 1, ...$ and verify that $c^1_1 > c^0_1$.) Explain intuitively the source of the time inconsistency.

Answer

(a) *Exponential consumer*

i.

$$\begin{aligned} MRS^t_{t+1,t+2} &= \frac{\partial U_t/\partial c_{t+2}}{\partial U_t/\partial c_{t+1}} = \delta, \\ MRS^{t+1}_{t+1,t+2} &= \frac{\partial U_{t+1}/\partial c_{t+2}}{\partial U_{t+1}/\partial c_{t+1}} = \delta. \end{aligned}$$

ii. Preferences as of time 0 are given by

$$U_0 = \ln c_0 + \delta \ln c_1 + \delta^2 \ln c_2 + \ ...$$

The intertemporal constraint is given by

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + ... = (1+r)(b_0 + \frac{y}{r}). \tag{118}$$

The first-order conditions are therefore given by:

$$\frac{\delta^t}{c_t} = \frac{\lambda}{(1+r)^t}, \qquad t = 0, 1, ...$$

If $\delta(1+r) = 1$, first-order conditions boil down to

$$c_t = \frac{1}{\lambda},$$

which implies that consumption as of time 0 (denoted by c^0_t) is constant over time. Hence, from the intertemporal constraint (118), it follows that

$$c_t^0\left[1+\frac{1}{1+r}+\frac{1}{(1+r)^2}+...\right]=(1+r)(b_0+\frac{y}{r}).$$

Then

$$c_t^0 = rb_0 + y, \qquad t=0,1,... \tag{119}$$

iii. Preferences as of time 1 are given by

$$U_1 = \ln c_1 + \delta \ln c_2 + \delta^2 \ln c_3 + ..$$

The intertemporal constraint is given by

$$c_1+\frac{c_2}{1+r}+\frac{c_3}{(1+r)^2}+...=(1+r)\left(b_1+\frac{y}{r}\right).$$

Proceeding as above, we conclude that

$$c_t^1 = rb_1 + y, \qquad t=1,2,... \tag{120}$$

To verify the optimum consumption plan chosen at time $t=1$ is time consistent, we need to show that

$$c_t^0 = c_t^1 \text{ for } t=1,2,...$$

Comparing (119) with (120), we can see that time consistency would hold if

$$b_1 = b_0.$$

And, indeed, we can show that this is the case. Using the economy's flow constraint and (119), it follows that:

$$\begin{aligned} b_1 &= (1+r)b_0 + y - c_0, \\ &= (1+r)b_0 + y - r\left(b_0+\frac{y}{r}\right), \\ &= b_0. \end{aligned}$$

We have thus shown that the consumption choices of our exponential consumer are time consistent.

(b) *Hyperbolic consumer*

i.

$$\begin{aligned} MRS_{t+1,t+2}^{t} &= \frac{\partial U_t/\partial c_{t+2}}{\partial U_t/\partial c_{t+1}} = \delta, \\ MRS_{t+1,t+2}^{t+1} &= \frac{\partial U_{t+1}/\partial c_{t+2}}{\partial U_{t+1}/\partial c_{t+1}} = \rho\delta. \end{aligned}$$

ii. Preferences as of time 0 are given by

$$U_0 = \ln c_0 + \rho\delta \ln c_1 + \rho\delta^2 \ln c_2 + ...$$

The intertemporal constraint is given by

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + ... = (1+r)(b_0 + \frac{y}{r}).$$

First-order conditions are therefore given by

$$\begin{aligned} \frac{\rho\delta^t}{c_t} &= \frac{\lambda}{(1+r)^t}, \qquad t = 1, 2, ... \\ \frac{1}{c_0} &= \lambda. \end{aligned}$$

If $\delta(1+r) = 1$, the first-order conditions reduce to:

$$\begin{aligned} \frac{\rho}{c_t} &= \lambda, \qquad t = 1, 2, ... \\ \frac{1}{c_0} &= \lambda. \end{aligned}$$

It follows that

$$c_t = \rho c_0 \text{ for } t = 1, 2, ... \tag{121}$$

Using the intertemporal constraint,

$$c_0 + \frac{\rho c_0}{1+r}\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + ...\right) = (1+r)\left(b_0 + \frac{y}{r}\right).$$

Hence,

$$c_0(1 + \frac{\rho}{r}) = (1+r)(b_0 + \frac{y}{r}). \tag{122}$$

From (121) and (122), it follows that

$$\begin{aligned} c_0^0 &= \frac{1+r}{r+\rho}(rb_0 + y), & \qquad (123) \\ c_t^0 &= \frac{\rho(1+r)}{r+\rho}(rb_0 + y), \qquad t = 1, 2, ... & \qquad (124) \end{aligned}$$

Notice that $c_0^0 > c_t^0$ $(t = 1, 2, ...)$, because the consumer discounts future consumption at a rate that is higher than the real interest rate In other words, the discount factor, $\rho\delta$, is smaller than the interest rate factor, $1/(1+r)$. Put differentely, the consumer is more impatient in the short-run than in the long-run. (See Figure 1, Panel A.)

iii. Preferences as of time 1 are given by

$$U_1 = \ln c_1 + \rho\delta \ln c_2 + \rho\delta^2 \ln c_3 + ...$$

The constraint is given by

$$c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + ... = (1+r)(b_1 + \frac{y}{r}).$$

Proceeding as in the previous point, we find that

$$c_1^1 = \frac{1+r}{r+\rho}(rb_1 + y), \tag{125}$$

$$c_t^1 = \frac{\rho(1+r)}{r+\rho}(rb_1 + y), \qquad t = 2, 3, ... \tag{126}$$

To show that the choice of c_1 is time inconsistent, it is enough to show that

$$c_1^1 > c_1^0. \tag{127}$$

Intuitively, we expect this to be the case because the consumer discounts more heavily the immediate future than the distant future. Hence, from the point of view of $t = 1$, consumption at time 1 is more valuable than it was from the point of view of $t = 0$. Formally, this can be seen from the fact that c_1 receives a higher weight in U_1 than in U_0.

We now need to compute b_1. To this end, we make use of the economy's flow constraint and (123):

$$\begin{aligned} b_1 &= (1+r)b_0 + y - c_0^0, \\ &= y - c_0^0, \\ &= y - \frac{(1+r)}{r+\rho}y, \\ &= b_0 - \frac{1-\rho}{r+\rho}(rb_0 + y). \end{aligned}$$

(As expected, the economy runs a current accoun deficit in period 0.) Substituting this expression for b_1 into (125) and (126), we get

$$c_1^1 = \rho\left(\frac{1+r}{r+\rho}\right)^2 (rb_0 + y), \tag{128}$$

$$c_t^1 = \rho^2\left(\frac{1+r}{r+\rho}\right)^2 (rb_0 + y) \tag{129}$$

Given (131) and (128), showing $c_1^1 > c_1^0$ implies showing that

$$\underbrace{\rho\left(\frac{1+r}{r+\rho}\right)^2(rb_0+y)}_{c_1^1} > \underbrace{\frac{\rho(1+r)}{r+\rho}(rb_0+y)}_{c_1^0},$$

which reduces to:

$$\rho < 1,$$

which holds since, by assumption, $\rho \in (0,1)$. Hence, our intuition was correct and under these hyperbolic preferences, the choice of c_1 is time inconsistent as it depends on when it is made (at $t = 0$ or $t = 1$).
Finally – and to gain further insight – notice that, intuitively, we expect $c_t^1 < c_t^0$ for $t \geq 2$ The reason is that the consumer run a current account deficit in $t = 0$ (and hence has fewer resources) and has chosen a higher level of c_1. Hence, there are less resources to spend on future consumption.

$$\underbrace{\rho^2\left(\frac{1+r}{r+\rho}\right)^2(rb_0+y)}_{c_1^1} < \underbrace{\frac{\rho(1+r)}{r+\rho}(rb_0+y)}_{c_1^0},$$

which again reduces to

$$\rho < 1.$$

Figure 1, Panel A illustrates the time inconsistency generated by hyperbolic preferences by showing how the path of consumption chosen at $t = 0$ differs from the one that would be chosen at $t = 1$.

2. Time inconsistency: the role of non time-separability

This exercise – which complements the previous one – takes as given the presence of exponential discounting and illustrates the role of non time-separability in generating time inconsistency.[1]

(a) *Past consumption affects today's utility*
Let preferences be given by:

$$U_0(c_0, c_1, ...) = \log(c_0) + \sum_{t=1}^{\infty} \delta^t[\log(c_t) + \alpha \log(c_{t-1})].$$

[1] See Calvo (1996) for a detailed discussion of the role of non-separability in generating time-inconsistency.

The sign of α defines the type of preferences. Naturally, if $\alpha = 0$, these are the standard time-separable preferences. If $\alpha < 0$, there is habit persistence (or habit formation) in the sense that higher consumption in period $t-1$ decreases utility in t (capturing the idea that consumers get used or "addicted" to that level of consumption). If $\alpha > 0$, there is durability of consumption goods in the sense that higher consumption at $t-1$ increases utility at t.[2] The intertemporal constraint remains given by (115).

In this context:

i. Compute reduced forms for c_t^0, $t = 0, 1, ...$

ii. Compute reduced forms for c_t^1, $t = 0, 1, ...$Verify that the optimal consumption path chosen at $t = 1$ is time consistent.

(b) *Future consumption yields utility*

Let preferences be given by:

$$U_0(c_0, c_1, ...) = \sum_{t=0}^{\infty} \delta^t [\log(c_t) + \log(c_{t+1})]$$

.In this case the consumer derives utility not only from today's consumption but also from next period's consumption.

i. Compute reduced forms for c_t^0, $t = 0, 1, ...$

ii. Compute reduced forms for c_t^1, $t = 0, 1, ...$ Assume, for simplicity, that $b_0 = 0$. Show that the optimal consumption path chosen at $t = 1$ is time inconsistent. Explain intuitively the source of time inconsistency.

Answer

(a) *Past consumption affects today's marginal utility*

i. Preferences as of time 0 are given by

$$U_0 = \log c_0 + \delta(\log c_1 + \alpha \log c_0) + \delta^2(\log c_2 + \alpha \log c_1) + ...$$

The constraint is

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + ... = (1+r)(b_0 + \frac{y}{r}). \tag{130}$$

[2] Habit persistence has been used extensively in the finance literature as a possible explanation for the equity-premium puzzle (see Constantinides, 1990). In the area of development macroeconomics, habit persistence has been used to explain some of the stylized facts associated with exhange rate-based stabilization (see Uribe (2002)). We will examine these issues in detail in Chapter @.

First-order conditions are given by:

$$\frac{\delta^t}{c_t}(1+\alpha\delta) = \frac{\lambda}{(1+r)^t}.$$

If $\delta(1+r) = 1$, first-order conditions boil down to:

$$c_t = \frac{(1+\alpha\delta)}{\lambda}$$

which implies that consumption as of time 0 (denoted by c_t^0) is constant over time. Hence, from the intertemporal constraint (130), it follows that

$$c_t^0(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + ...) = (1+r)(b_0 + \frac{y}{r})$$

Hence,

$$c_t^0 = rb_0 + y, \qquad t = 0, 1, ... \tag{131}$$

ii. Preferences as of time 1 are given by

$$U_1 = \ln c_1 + \delta(\ln c_2 + \alpha \ln c_1) + \delta^2(\ln c_3 + \alpha \ln c_2) + ..$$

The intertemporal constraint is given by

$$c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + ... = (1+r)(b_1 + \frac{y}{r}).$$

Proceeding as before, we find that

$$c_t^1 = rb_1 + y. \tag{132}$$

Comparing (131) and (132), we see that the choice of c_1 will be time-consistent if

$$b_1 = b_0.$$

And using the flow constraint and the choice of c_0 given by (131), we can check that this indeed the case:

$$\begin{aligned} b_1 &= (1+r)b_0 + y - c_0 \\ &= (1+r)b_0 + y - r(b_0 + \frac{y}{r}) \\ &= b_0 \end{aligned}$$

(b) *Future consumption yields utility*

i. Preferences are given by

$$U_0 = \ln c_0 + \ln c_1 + \delta(\ln c_1 + \ln c_2) + \delta^2(\ln c_2 + \ln c_3) + ...$$

The intertemporal constraint is given by

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + ... = (1+r)(b_0 + \frac{y}{r}) \tag{133}$$

The first-order conditions are therefore given by

$$\begin{aligned} \frac{1}{c_0} &= \lambda \\ \frac{\delta^{t-1}}{c_t}(1+\delta) &= \frac{\lambda}{(1+r)^t}, \qquad t = 1, 2, .. \end{aligned}$$

If $\delta(1+r) = 1$, the first-order condition reduce to:

$$\begin{aligned} \frac{1}{c_0} &= \lambda, \\ \frac{(1+r)(1+\delta)}{c_t} &= \lambda \ , \qquad t = 1, 2, .. \end{aligned}$$

It follows that

$$c_t = (1+r)(1+\delta)c_0, \qquad t = 1, 2, .. \tag{134}$$

Using this last equation and the intertemporal constraint (133), we obtain

$$c_0 \left[1 + \frac{(1+r)(1+\delta)}{r}\right] = (1+r)\left(b_0 + \frac{y}{r}\right).$$

Using this last equation and (134)

$$c_0^0 = \frac{1+r}{r + (1+r)(1+\delta)} (rb_0 + y) \, , \tag{135}$$

$$c_t^0 = \frac{(1+r)^2(1+\delta)}{r + (1+r)(1+\delta)} (rb_0 + y) \, , \qquad t = 1, 2, .. \tag{136}$$

Using the fact that $\delta(1+r) = 1$, these two expression can be simplify to:

$$c_0^0 = \frac{rb_0 + y}{2}, \tag{137}$$

$$c_t^0 = \left(1 + \frac{r}{2}\right)(rb_0 + y) \, , \qquad t = 1, 2, .. \tag{138}$$

As expected $c_0^0 < c_t^0$ (see Figure 1, Panel B) because c_0 only yields utility today, as opposed to all the other consumptions which yield utility in the period in which they occur but also in the period before.

ii. Preferences are given by

$$U_1 = \ln c_1 + \ln c_2 + \delta(\ln c_2 + \ln c_3) + \delta^2(\ln c_3 + \ln c_4) + ...$$

The intertemporal constraint is given by

$$c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + ... = (1+r)\left(b_1 + \frac{y}{r}\right).$$

Proceeding as in the previous point:

$$c_1^1 = \frac{rb_1 + y}{2}, \tag{139}$$

$$c_t^1 = \left(1 + \frac{r}{2}\right)(rb_1 + y), \qquad t = 1, 2, .. \tag{140}$$

Since c_1 has a higher "weight" in U_0 than in U_1, we expect that

$$c_1^0 > c_1^1.$$

Intuitively, tomorrow's consumption is more valuable from today's standpoint than from tomorrow's standpoint because the anticipation of tomorrow's consumption also gives utility today. Before showing this, we first need to compute b_1. To this end, we make use of the economy's flow constraint and (135) to obtain:

$$\begin{aligned} b_1 &= (1+r)b_0 + y - c_0^0 \\ b_1 &= b_0 + \frac{1}{2}(rb_0 + y). \end{aligned}$$

As expected, the economy runs a current account surplus in period 0.
We now replace this solution for b_1 into the expression for c_1^1 and c_t^0 given in (139) and (140) to obtain:

$$c_1^1 = \left(1 + \frac{r}{2}\right)\frac{rb_0 + y}{2} \tag{141}$$

$$c_t^1 = \frac{(2+r)^2}{4}(rb_0 + y) \tag{142}$$

Recall that we want to show that $c_1^0 > c_1^1$. Given (**??**) and (141), this implies showing that

$$\underbrace{\left(1 + \frac{r}{2}\right)(rb_0 + y)}_{c_1^0} > \underbrace{\left(1 + \frac{r}{2}\right)\frac{rb_0 + y}{2}}_{c_1^1},$$

which reduces to

$$1 > \frac{1}{2},$$

and hence holds. Hence, the choice of c_1 is time inconsistent. When the consumer reoptimizes in $t = 1$, he/she will choose a lower level of c_1 because it has become less valuable.
Finally, notice that we also expect that $c_t^0 < c_t^1$, $t \geq 2$, since the consumer has run a current account surplus in period 0 (and hence has more resources than in $t = 0$) and has chosen a lower level of c_1. To check this, we show that

$$\underbrace{\left(1 + \frac{r}{2}\right)(rb_0 + y)}_{c_t^0} < \underbrace{\frac{(2+r)^2}{4}(rb_0 + y)}_{c_t^1},$$

which reduces to

$$0 < \frac{r}{2}$$

and hence holds. Figure 1, Panel B illustrates the time inconsistency generated by preferences in which future consumption yields utility by showing how the path of consumption chosen at $t = 0$ differs from the one that would be chosen at $t = 1$.

2. Roots of the system

Consider the infinite horizon model of Section 2.2 with logarithmic preferences. In this context:

(a) Show that the dynamic system associated with the model has roots $r - \beta$ and r. (If $r = \beta$, then the roots are zero and r.)

(b) Consider a discrete time version of the model. Show that for the $(1 + r)\beta = 1$ case, the roots are 1 and $1 + r$.

Answer

(a) With logarithmic preferences, we can reduce the model to the following linear system of differential equations (assume constant endowment):

$$\dot{b}_t = rb_t + y - c_t, \tag{143}$$
$$\dot{c}_t = (r - \beta)c_t. \tag{144}$$

Expressed in matrix notation:

$$\begin{bmatrix} \dot{b}_t \\ \dot{c}_t \end{bmatrix} = \begin{bmatrix} r & -1 \\ 0 & r - \beta \end{bmatrix} \begin{bmatrix} b_t \\ c_t \end{bmatrix}.$$

Denote by λ the roots of the system. To obtain the roots of the system, we substract λ from the diagonal elements of the matrix associated with the dynamic system and set its determinant to zero:

$$\begin{vmatrix} r-\lambda & -1 \\ 0 & r-\beta-\lambda \end{vmatrix} = 0.$$

The characteristic equation is then given by

$$\lambda^2 - (2r-\beta)\lambda + r(r-\beta) = 0.$$

The roots are therefore given by

$$\lambda = \frac{2r-\beta \pm \sqrt{(2r-\beta)^2 - 4r(r-\beta)}}{2},$$

which can be reduced to

$$\lambda = \frac{2r-\beta \pm \beta}{2},$$

or

$$\lambda = \begin{cases} r \\ r-\beta \end{cases}$$

The two roots are therefore r and $r-\beta$. If $r < \beta$, then we have one positive and one negative (i.e., stable) root.

If $r = \beta$, then we have a zero root and a positive root. Choosing the solution $c_t = rb_0 + y$ for all $t \geq 0$ (which is equivalent to imposing the transversality condition) implies setting to zero the constant corresponding to the positive (i.e., unstable) root and using only the zero root. To see this clearly, notice that $r = \beta$ implies that $\dot{c}_t = 0$. Then, substituting $c_t = rb_0 + y$ into (144) and evaluating the expression at $t = 0$ implies that $\dot{b}_0 = 0$, which implies that $\dot{b}_t = 0$ for all $t > 0$.

(b) In discrete time with $\beta(1+r) = 1$, the linear system of difference equations reduces to

$$\begin{aligned} b_{t+1} &= (1+r)b_t + y - c_t, \\ c_{t+1} &= \beta(1+r)c_t. \end{aligned} \tag{145}$$

It can be easily check that the same logic discussed above holds. In particular, if $\beta(1+r) = 1$, then the roots of this system are 1 and $1+r$. Choosing the solution $c_t = rb_0 + y$ corresponds to setting to zero the constant corresponding to the $1+r$ root and using only the unit root. If $\beta(1+r) = 1$, then $c_{t+1} = c_t$ for all $t = 0, 1, ...$ Then, substituting $c_t = rb_0 + y$ into (145) and evaluating the expression for $t = 0$ yields $b_1 = b_0$, which implies that $b_{t+1} = b_0$ for all $t \geq 0$.

3. Consumption tilting

This exercise analyzes consumption tilting (i.e., optimal consumption plans when β is not necessarily equal to r) in both the finite and infinite horizon settings.

Let the instantaneous utility function be given by:

$$u(c_t) = \frac{c_t^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}, \tag{146}$$

where $\sigma > 0$ denotes the intertemporal rate of substitution in consumption.

(a) *Finite horizon*

Consider the finite horizon problem analyzed in the text, in which the consumer maximizes (1) (with the instantaneous utility function given by (146) subject to (18). In this context:

i. Derive the first-order conditions for the consumer's problem and show how the rate of growth of consumption depends on the relation between β and r. Explain the intuition behind the results.

ii. Derive a closed-form solution for c_t.

(b) *Infinite horizon (based on Calvo (1996))*

In an infinite horizon setting, the existence of a well-defined optimal consumption path when r is different from β cannot be taken for granted. (By well-defined optimal consumption path we mean a consumption path whose present discounted value is finite.)

Consider the basic infinite horizon model described in the text, in which the consumer maximizes (13) (with the instantaneous utility function given by ((146))) subject to (18). For simplicity, let the endowment stream be constant over time and equal to y and $b_0 = 0$. In this context:

i. Derive a condition involving r, β, and σ that guarantees the existence of a well-defined optimal consumption path. In particular, show that $\sigma \leq 1$ is a *sufficient* condition for existence. [Hint: Solve for the optimal consumption path, write the intertemporal budget constraint in terms of c_0, and establish the condition for the integral to converge.]

ii. To illustrate the fact that $\sigma \leq 1$ is not a *necessary* condition to guarantee the existence of a well-defined optimal consumption path, consider the case in which $\sigma = 1.5$. What is the condition involving r and β for which existence is guaranteed?

iii. Restrict your attention to cases in which a well-defined optimal consumption path exists. Show that:

A. If $r = \beta$, $c_t = y$ for all $t \geq 0$.
B. If $r > \beta$, $c_0 < y$ and consumption increases over time.
C. If $r < \beta$, $c_0 > y$ and consumption falls over time.

iv. Check that the same condition that you derived in (i) above guarantees that the utility funcional (13), with $u(c)$ given by (117), converges.

Answer

(a) *Finite horizon*

i. The first-order condition is given by

$$c_t^{-\frac{1}{\sigma}} e^{-\beta t} = \lambda e^{-rt}.$$

Taking logarithms, we obtain

$$-\frac{1}{\sigma} \ln c_t - \beta t = \ln \lambda - rt.$$

Totally differentiating,

$$-\frac{1}{\sigma} \frac{dc_t}{c_t} - \beta dt = -rdt.$$

Rearranging terms,

$$\frac{\dot{c}_t}{c_t} = (r - \beta)\sigma. \tag{147}$$

We see that consumption will grow over time when the real interest rate is larger than the discount rate. In this case, the consumer is discounting the future at a lower rate than the real interest rate that he/she would obtain by foregoing consumption today. Hence, consumption today will be low relative to tomorrow's (i.e., consumption is increasing over time). The opposite is true when the real interest rate is lower than the discount rate.

ii. Notice that equation (147) is a first-order linear differential equation in c_t. The general solution to this differential equation is given by

$$c_t = c_0 e^{(r-\beta)\sigma t}, \tag{148}$$

where c_0 is the level of consumption at $t = 0$ yet to be determined.
To determine c_0, substitute the expression for c_t given by (148) into the intertemporal constraint (10) to obtain

$$c_0 \int_0^T e^{[(r-\beta)\sigma - r]t} dt = b_0 + \int_0^T y_t e^{-rt} dt.$$

Integrating the left-hand side, we get

$$c_0 = \frac{r-(r-\beta)\sigma}{1-e^{[(r-\beta)\sigma-r]T}}\left(b_0 + \int_0^T y_t e^{-rt} dt\right). \tag{149}$$

Substituting (149) into (148), we obtain a reduced form for c_t:

$$c_t = e^{(r-\beta)\sigma t}\frac{r-(r-\beta)\sigma}{1-e^{[(r-\beta)\sigma-r]T}}\left(b_0 + \int_0^T y_t e^{-rt} dt\right)$$

Note that, as a particular case, if $r=\beta$ then

$$c_t = \frac{r}{1-e^{-rT}}\left(b_0 + \int_0^T y_t e^{-rt} dt\right).$$

In other words, if $r=\beta$, c_t is constant over time and equal to permanent income (i.e., the constant level of consumption that exhausts the present discounted value of resources).

(b) *Infinite horizon*

i. Substituting the expression for c_t given by (148) into the intertemporal budget constraint for the infinite horizon case given by (18) yields

$$c_0 \int_0^\infty e^{[(r-\beta)\sigma-r]t} dt = b_0 + \int_0^\infty y_t e^{-rt} dt. \tag{150}$$

The integral on the left-hand side will converge if and only if

$$\begin{aligned} (r-\beta)\sigma - r &< 0, \\ -\beta &< r\frac{(1-\sigma)}{\sigma}. \end{aligned} \tag{151}$$

For $r>0$, a sufficient condition for this to hold is that $\sigma \leq 1$, which is borne out by the empirical evidence (see the discussion in Chapter 3).

ii. Suppose that $\sigma = 1.5$. Then condition (151) holds whenever

$$\beta > \frac{r}{3}.$$

iii. If (151) holds, we can solve (150) to obtain (assuming that $y_t = y$):

$$\begin{aligned} c_0 &= [r-(r-\beta)\sigma]\,\frac{y}{r} \\ c_t &= c_0 e^{(r-\beta)\sigma t} \end{aligned}$$

So if $r>\beta$, $c_0 < y$ and consumption increases over time. If $r<\beta$, then $c_0 > y$ and consumption falls over time.

iv. We want to check that the condition $(r-\beta)\sigma - r < 0$ is a necessary and sufficient condition for

$$\int_0^\infty \left(\frac{c_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \right) e^{-\beta t} dt \tag{152}$$

to converge.
Substituting expression (148) into (152), we obtain

$$\int_0^\infty \left(\frac{\left[c_0 e^{(r-\beta)\sigma t}\right]^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \right) e^{-\beta t} dt.$$

Rearranging this expression, we obtain

$$\frac{c_0^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \int_0^\infty e^{\left[(r-\beta)\sigma\left(1-\frac{1}{\sigma}\right)-\beta\right]t} dt - \frac{1}{1-\frac{1}{\sigma}} \int_0^\infty e^{-\beta t} dt. \tag{153}$$

The second integral always converges. But the first integral will converge if and only if

$$(r-\beta)\sigma\left(1-\frac{1}{\sigma}\right) - \beta < 0,$$

which reduces to

$$(r-\beta)\sigma - r < 0, \tag{154}$$

which is what we wanted to show.
Notice some important particular cases. First, if $r = \beta$, condition (154) always hold since it reduces to $-r < 0$. Intuitively, since the path of consumption is bounded, $u(c)$ is also bounded and therefore the utility functional always converges regardless of the value of σ. Second, as before, $\sigma \leq 1$ is a sufficient condition for the utility funtional to converge.
To gain further insights, solve for (153) to obtain

$$\int_0^\infty \left(\frac{c_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \right) e^{-\beta t} dt = -\frac{c_0^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \left[\frac{1}{\left[(r-\beta)(\sigma-1)-\beta\right]} \right] - \frac{1}{1-\frac{1}{\sigma}} \frac{1}{\beta},$$

where

$$c_0 = \left[r - (r-\beta)\sigma\right] \left(b_0 + \int_0^\infty y_t e^{-rt} dt \right). \tag{155}$$

If $r = \beta$, then

$$\int_0^\infty \left(\frac{c_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \right) e^{-\beta t} dt = \frac{1}{\beta} \frac{c_0^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \tag{156}$$

where

$$c_0 = r\left(b_0 + \int_0^\infty y_t e^{-rt} dt\right). \tag{157}$$

Finally, if $r = \beta$ and $u(c) = \log(c)$, it can be checked that

$$\int_0^\infty \log(c_t) e^{-\beta t} dt = \frac{\log(c_0)}{\beta}.$$

We can obtain this result by taking the limit of both the LHS and RHS of (156) as $\sigma \to 1$. Alternative, we could re-do the whole calculation starting with $\log(c_t)$ in equation (152).

4. Fluctuating real interest rate

Consider the infinite horizon model analyzed in Subsection 2.2 but suppose that the world real interest rate fluctuates over time. In particular – and to fix ideas – assume that the time path of the real interest rate is given by

$$r_t = \begin{cases} r^H & \text{for} \quad 0 \leqslant t \leqslant T, \\ r^L & \text{for} \quad t > T, \end{cases} \tag{158}$$

where $r^H > \beta$ and $r^L < \beta$. Assume logarithmic preferences.

In this context, derive a reduced-form solution for the path of consumption.

Answer

In light of the time-varying real interest rate, the household's lifetime constraint is given by

$$b_0 + \int_0^\infty y_t e^{-\int_0^t r_s ds} dt = \int_0^\infty c_t e^{-\int_0^t r_s ds} dt. \tag{159}$$

Notice that if $r_s = r$, then this lifetime constraint reduces to the standard lifetime constraint. The Lagrangean takes the form:

$$\mathcal{L} = \int_0^\infty \log(c_t) e^{-\beta t} dt + \lambda\left(b_0 + \int_0^\infty y_t e^{-\int_0^t r_s ds} dt - \int_0^\infty c_t e^{-\int_0^t r_s ds} dt\right).$$

The first-order condition for consumption is given by

$$\frac{e^{-\beta t}}{c_t} = \lambda e^{-\int_0^t r_s ds}.$$

Rewrite this first-order condition as

$$\frac{1}{c_t} = \lambda e^{-\int_0^t (r_s - \beta) ds}. \tag{160}$$

Taking logarithms on both sides:

$$\log(c_t) = \log(\lambda) - -\int_0^t (r_s - \beta)ds.$$

Differentiating with respect to time (and applying Leibnitz rule), we obtain the Euler equation:

$$\frac{\dot{c}_t}{c_t} = r_t - \beta.$$

Taking into account the time path of the real interest rate given by (158), we can rewrite the Euler equation as

$$\dot{c}_t = \begin{cases} c_t \left(r^H - \beta\right) & 0 \leq t \leq T, \\ c_t \left(r^L - \beta\right) & t > T. \end{cases} \tag{161}$$

Since $r^H > \beta$ and $r^L < \beta$, consumption will be increasing between 0 and T and decreasing afterwards. Another important piece of information is that the path of consumption will be continuous at time T, as can be inferred from first-order condition (160). We therefore know the time profile of consumption. To derive the specific level of consumption, we need to resort to the lifetime constraint.

First, let us solve the two differential equations given by 161 to obtain:

$$c_t = \begin{cases} c_0 e^{(r^H - \beta)t} & 0 \leq t \leq T, \\ c_T e^{(r^L - \beta)(t-T)} & t > T. \end{cases} \tag{162}$$

where c_0 and c_T denote initial levels of consumption to be determined. Since c_t is continous at time T, it follow from the first equation above that

$$c_T = c_0 e^{(r^H - \beta)T}. \tag{163}$$

Using (163), we can rewrite (162) as

$$c_t = \begin{cases} c_0 e^{(r^H - \beta)t} & 0 \leq t \leq T, \\ c_0 e^{(r^H - \beta)T} e^{(r^L - \beta)(t-T)} & t > T. \end{cases} \tag{164}$$

All that remains is to determine c_0. To this effect, substitute (164) into the lifetime constraint (159)

$$\int_0^T c_0 e^{(r^H - \beta)t} e^{-r^H t} dt + \int_T^\infty c_0 e^{(r^H - \beta)T} e^{(r^L - \beta)(t-T)} e^{-r^L t} dt = W$$

where

$$W \equiv b_0 + \int_0^T y_t e^{-r^H t} dt + \int_T^\infty y_t e^{-r^L t} dt$$

is the household's wealth. After some algebra, we can show that

$$c_0 = \frac{\beta W}{(1 - e^{-\beta T}) + e^{(r^H - r^L)T} e^{-\beta T}}. \tag{165}$$

Equations (164) and (165) fully characterize the path of consumption. As a particular case, notice that if $r^H = r^L = r$, then $c_0 = \beta W$, which is the solution obtained in (155).

5. Adding labor supply to the basic model

This exercise adds labor supply to the basic infinite horizon model of Section 2.2. Production is thus endogenous. Specifically, consider the economy of Section 2.2 with the following modifications (same notation is used).

Households. Let preferences be given by

$$\int_0^{\infty} \log[c_t - \phi(\ell_t^s)^{\upsilon}] e^{-\beta t} dt,$$

where ℓ^s is labor supply and $\phi(> 0)$ and $\upsilon(> 1)$ are parameters.[3] The household's flow constraint is given by

$$\dot{b}_t = r b_t + w_t \ell_t^s - c_t + \Omega_t,$$

where w is the real wage and Ω_t are the profits from firms (i.e., households own the firms). The corresponding intertemporal constraint is given by

$$b_0 + \int_0^{\infty} (w_t \ell_t^s + \Omega_t)\, e^{-rt} dt = \int_0^{\infty} c_t e^{-rt} dt.$$

Firms Firms face a static problem. Production is given by

$$y_t = \Psi_t \left(\ell_t^d\right)^{\alpha}, \qquad \alpha < 1, \tag{166}$$

where ℓ^d is labor demand and Ψ is a productivity parameter (which may vary over time).

In the context of this model:

(a) Solve the household's maximization problem. Using the first-order conditions, derive a labor supply function (i.e., express ℓ_t^s as a function of w_t). Explain the intuition behind your derivations.

(b) Solve the firms' maximization problem. Explain the intuition behind the results.

(c) After imposing labor market equilibrium (i.e., $\ell_t^s = \ell_t^d$), solve for a perfect foresight path along which Ψ_t is at some high value between time 0 and time T and low afterwards. (For expositional clarity, make sure that you plot the paths of all endogenous variables against time, including the trade balance and current account.) Explain the intuition behind all of your results.

[3] These are the so-called GHH preferences, after the paper by Greenwood, Hercowitz, and Huffman (1998).

(d) What key difference do you notice in the behavior of consumption relative to the model of Section 2.2? Show that if the labor supply elasticity is small (as is the case in practice; see, for instance, Pencavel (1986)) then, from a quantitative point, the behavior of consumption in response to a fluctuating path of Ψ_t will not differ significantly from the behavior of consumption in response to a fluctuating endowment path in the model of Section 2.2.[4]

Answer

(a) The first-order conditions are given by:

$$\begin{aligned} \frac{1}{c - \phi(\ell^s)^{\upsilon}} &= \lambda, \\ \frac{\phi\nu(\ell^s)^{\nu-1}}{c - \phi(\ell^s)^{\upsilon}} &= \lambda w_t. \end{aligned} \tag{167}$$

Substituting the first condition into the second, we obtain:

$$\phi\nu(\ell^s)^{\nu-1} = w_t. \tag{168}$$

Solving for ℓ_t^s, we obtain

$$\ell_t^s = \left(\frac{w_t}{\phi\nu}\right)^{\theta}, \tag{169}$$

where $\theta \equiv 1/(\nu - 1) > 0$ is the labor supply elasticity. Two observations are worth making. First, notice that while condition (167) implies that the marginal utility of consumption will remain constant along a perfect foresight path (as in the model of Section 2.2), consumption itself does not necessarily remain constant along a perfect foresight path. Clearly, if labor supply fluctuates along a perfect foresight path, so will consumption. Second, condition (169) may be viewed as a labor supply function, which says that the higher the real wage, the larger will be the supply of labor. Notice also that there is no wealth effect on labor supply (as indicated by the fact that ℓ_t^s does not depend on λ).

(b) Firms' profits are given by:

$$\Omega_t = y_t - w_t\ell_t^d,$$

[4] In practice, however, total labor hours do fluctuate considerably over the business cycle due to entry and exit from the labor force (the extensive margin), as opposed to changes in hours worked by existing agents (the intensive margin). To generate the observed comovement between total labor hours and the cycle at an aggregate level, one needs to incorporate this "extensive margin" into the model (see King and Rebelo (1999) for a detailed discussion.)

which, using equation (166) can be rewritten as:

$$\Omega_t = \Psi_t \left(\ell^d\right)^{\alpha} - w_t \ell_t^d.$$

The first-order condition is given by:

$$\alpha\Psi \left(\ell^d\right)^{\alpha-1} = w_t. \tag{170}$$

Solving for ℓ^d, we obtain labor demand:

$$\ell^d = \left(\frac{\alpha\Psi_t}{w_t}\right)^{\frac{1}{1-\alpha}}.$$

As expected, labor demand is a decreasing function of the real wage and an increasing function of the productivity parameter, Ψ_t.

(c) Impose labor market equilibrium, $\ell_t^s = \ell_t^d = \ell_t$, and solve for the equilibrium real wage:

$$w_t = \left[(\alpha\Psi)^{\frac{1}{1-\alpha}} \left(\phi\nu\right)^{\theta}\right]^{\frac{1}{\theta+\frac{1}{1-\alpha}}}. \tag{171}$$

Equilibrium labor can then be read from the labor supply, given by (169).

Let us now characterize a PFEP along which Ψ is high from time 0 to time T and low afterwards (Figure 2, Panel A). From equation (171), it follows that the real wage is high before time T and low afterwards (Figure 2, Panel B). From (169), we can derive the path of labor (Figure 2, Panel C). Given the path of labor, the path of consumption follows from first-order condition (167).

What happens to the trade balance? By definition, the trade balance is given by:

$$TB_t = \Psi \left(\ell\right)^{\alpha} - c_t. \tag{172}$$

Since both labor (and hence output) and consumption are higher between 0 and T, it is not obvious how the trade balance compares before and after T. To show that a higher Ψ leads to a larger trade balance, consider a small (i.e., infinitesimal) change in Ψ and differentiate equation (172) to obtain:

$$\frac{dTB_t}{d\Psi} = (\ell)^{\alpha} + \Psi\alpha\left(\ell\right)^{\alpha-1}\frac{d\ell}{d\Psi} - \frac{dc_t}{d\Psi}.$$

But from first-order condition (167), it follows that along a PFEP, $dc_t/d\Psi = \phi\upsilon(\ell)^{\upsilon-1}d\ell/d\Psi$. Hence, after rearranging, we obtain:

$$\frac{dTB_t}{d\Psi} = (\ell)^{\alpha} + \frac{d\ell}{d\Psi}\left[\underbrace{\Psi\alpha\left(\ell\right)^{\alpha-1}}_{=w} - \phi\upsilon(\ell)^{\upsilon-1}\right]$$

Since, as indicated, $\Psi\alpha\left(\ell\right)^{\alpha-1}=w$ (by the firm's first-order condition 170), the term is square brackets is zero by household's labor supply condition (168). Hence:

$$\frac{dTB_t}{d\Psi}=\left(\ell\right)^{\alpha}.$$

This last equation says that, in good times, the trade balance will be higher than in bad times. Assuming that initial net foreign assets are zero, then the trade balance will be positive between 0 and T and negative afterwards (Figure 2, Panel E). Intuitively, by envelope considerations, the indirect effect on output (i.e., via an increase in labor) of a productivity increase is exactly offset by the associated increase in consumption. Only the direct effect of Ψ on output remains.

In sum, periods of high productivity (i.e., good times) will correspond to high labor, high consumption, and a positive trade balance.

(d) The difference lies in the fact that, in the model of Section 2.2, households fully smooth consumption regardless of the path of output. In this case, consumption fluctuates together with output. Periods of high output will coincide with periods of high consumption because labor affects the marginal productivity of consumption.

To formally show that a small labor supply elasticity will lead to a relatively flat path of consumption, use equation (169) to obtain:

$$\hat{\ell}_t=\theta\hat{w}_t,$$

where a hat over a variable denotes proportional change.

From (171), it follows that

$$\hat{w}_t=\frac{1}{1+(1-\alpha)\theta}\hat{\Psi}.$$

Intuitively, the lower is the labor supply elasticity, the larger the increase in the real wage in response to a shift in labor demand.

Using (171), taking into account the last equation, yields:

$$\hat{\ell}_t=\frac{\theta}{1+(1-\alpha)\theta}\hat{\Psi}.$$

This equation says that, for a given change in Ψ, the smaller is the labor supply elasticity, θ, the smaller will the be the change in labor. At the limit (i.e., for θ tending to zero), there would be no change in labor and hence in consumption. In that case, quantitatively speaking, the model would behave very much like the endowment model of Section 2.2.

6. Decentralized economy

This exercise asks you to check that the centralized production economy analyzed in the text can be decentralized. Suppose that there are two agents in the economy: consumers and firms. Consumers own the capital stock and own the firms. There is a market for physical capital in which consumers rent the capital stock to firms at a rate r^k. Firms produce the good using the capital stock and give back profits to consumers. Preferences and technology are the same as in the text.

(a) Write down the consumer's flow constraint and then derive the consumer's intertemporal constraint. Derive the consumer's first-order conditions.

(b) Write down the firm's flow constraint and derive the first-order condition.

(c) Show that the optimality conditions characterizing consumption and the capital stock are the same as in the centralized economy.

(d) Derive aggregate constraints (both the flow constraint, or current account, and the intertemporal constraint) and show that they correspond exactly to those for the centralized economy (equations (41) and (42)).

Answer

(a) In a decentralized version of the production economy, there would be two agents: consumers and firms.

Consumers own the capital stock and rent it to firms at a rental rate r^k. In other words, there is a market where the capital stock is rented. Consumers own the firms and get profits back. As before, b_0 and k_0 are given.

$$b_{t+1} + k_{t+1} = (1+r)b_t + (1+r_t^k)k_t + \Omega_t - c_t$$

Iterating

$$\begin{aligned} b_1 + k_1 &= (1+r)b_0 + (1+r_0^k)k_0 + \Omega_0 - c_0 && (173) \\ b_2 + k_2 &= (1+r)b_1 + (1+r_1^k)k_1 + \Omega_1 - c_1 && (174) \end{aligned}$$

Solving for b_1 from last equation:

$$b_1 = \frac{b_2 + k_2}{1+r} - \frac{1+r_1^k}{1+r}k_1 - \frac{\Omega_1}{1+r} + \frac{c_1}{1+r}$$

Substituting into (173):

$$\underbrace{\frac{b_2+k_2}{1+r}-\frac{1+r_1^k}{1+r}k_1-\frac{\Omega_1}{1+r}+\frac{c_1}{1+r}}_{b_1}+k_1=(1+r)b_0+(1+r_0^k)k_0+\Omega_0-c_0.$$

Rearranging terms, we get:

$$\frac{b_2}{1+r}=(1+r)b_0+k_0+\Omega_0+\frac{\Omega_1}{1+r}+r_0^k k_0+\frac{r_1^k k_1}{1+r}-\frac{rk_1}{1+r}-\frac{k_2}{1+r}-c_0-\frac{c_1}{1+r}$$

To make it comparable to the flow constraint, add and substract k_1 to the RHS and rewrite it as

$$\frac{b_2}{1+r}=(1+r)b_0+\Omega_0+\frac{\Omega_1}{1+r}+r_0^k k_0+\frac{r_1^k k_1}{1+r}-c_0-\frac{c_1}{1+r}-(k_1-k_0)-\frac{(k_2-k_1)}{1+r}$$

Continuing to iterate and imposing the terminal condition

$$\lim_{t\to\infty}\frac{b_{t+1}}{(1+r)^t}=0,$$

we obtain:

$$(1+r)b_0+\sum_{t=0}^{\infty}\frac{\Omega_t}{(1+r)^t}+\sum_{t=0}^{\infty}\frac{r_t^k k_t}{(1+r)^t}-\sum_{t=0}^{\infty}\frac{(k_{t+1}-k_t)}{(1+r)^t}=\sum_{t=0}^{\infty}\frac{c_t}{(1+r)^t}. \tag{175}$$

Maximization:

$$\underset{\{c_t,k_{t+1}\}}{\text{Max}}\, L=\sum_{0}^{\infty}\beta^t u(c_t)+$$
$$\lambda\left[(1+r)b_0+\sum_{t=0}^{\infty}\frac{\Omega_t}{(1+r)^t}+\sum_{t=0}^{\infty}\frac{r_t^k k_t}{(1+r)^t}-\sum_{t=0}^{\infty}\frac{(k_{t+1}-k_t)}{(1+r)^t}-\sum_{t=0}^{\infty}\frac{c_t}{(1+r)^t}\right]$$

The first-order conditions are given by (assuming, as usual, $\beta=r$):

$$\begin{aligned} u'(c_t) &= \lambda, \quad (176)\\ -\frac{1}{(1+r)^t}+\frac{r_{t+1}^k}{(1+r)^{t+1}}+\frac{1}{(1+r)^{t+1}} &= 0. \end{aligned}$$

The last condition implies that the rental rate of capital is constant over time and equal to world real interest rate:

$$r_{t+1}^k=r. \tag{177}$$

In equilibrium, the rental rate will be equal to the world real interest rate. Intuitively, since bonds and capital are perfect substitutes in the consumer's portfolio, they must bear the same rate of return.

(b) Firms will solve the following problem:

$$\begin{aligned}
\underset{k_t}{\text{Max}}\ \Omega_t &= y_t - r_t^k k_t \\
\underset{k_t}{\text{Max}}\ \Omega_t &= A_t f(k_t) - r_t^k k_t
\end{aligned}$$

The first-order condition is given by

$$A_t f'(k_t) = r_t^k. \tag{178}$$

(c) Substituting (177) into (178), we obtain

$$A_t f'(k_t) = r \tag{179}$$

As expected, optimality conditions (176) and (179) are the same as those that we obtained in the text for the centralized case (equations (45) and (46)).

(d) Combining the consumer's flow constraint with the firm's:

$$\begin{aligned}
b_{t+1} + k_{t+1} &= (1+r)b_t + (1+r_t^k)k_t + \Omega_t - c_t \\
b_{t+1} + k_{t+1} &= (1+r)b_t + (1+r_t^k)k_t + \underbrace{A_t f(k_t) - r_t^k k_t}_{\Omega_t} - c_t \\
b_{t+1} + k_{t+1} &= (1+r)b_t + k_t + A_t f(k_t) - c_t \\
b_{t+1} &= (1+r)b_t + A_t f(k_t) - (k_{t+1} - k_t) - c_t
\end{aligned}$$

Since $y_t = A_t f(k_t)$, this is the same flow constraint that we had for the centralized case (equation (41) in the text).

Substituting $\Omega_t = A_t f(k_t) - r_t^k k_t$, into the consumer's intertemporal – given by (175) – we get:

$$\begin{aligned}
(1+r)b_0 \sum_{t=0}^{\infty} \frac{\Omega_t}{(1+r)^t} + \sum_{t=0}^{\infty} \frac{r_t^k k_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{(k_{t+1} - k_t)}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \\
(1+r)b_0 + \sum_{t=0}^{\infty} \frac{A_t f(k_t)}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{(k_{t+1} - k_t)}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \\
(1+r)b_0 + \sum_{t=0}^{\infty} \frac{A_t f(k_t)}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{c_t + (k_{t+1} - k_t)}{(1+r)^t}.
\end{aligned}$$

which, since $y_t = A_t f(k_t)$, is the same as before (equation (42)).

7. Small changes in productivity

This exercise asks you to revisit some of the experiments performed in the text for the model with investment for the case in which the change in productivity is small (i.e., the change is given by dA). This exercise will shed light on the reaction of saving to changes in productivity.

Consider the model with investment described in Section (5). In this context:

(a) Analyze the effects of a (small) unanticipated and permanent increase in A. [In other words, assume that A changes by dA.] In particular, show that there will be no change in saving. Explain the intuition behind the results.

(b) Analyze the effects of a (small) unanticipated and temporary increase in A that lasts for $T(\geq 2)$ periods. Derive a reduced form for the change in the current account in period 0 and show how it depends on T. Explain the intuition behind the results.

Answer

(a) The economy is initially in the stationary perfect foresight equilibrium characterized in Subsection 5.3 with the capital stock given by $\bar{k}$. All changes below will be relative to this initial steady state. Clearly, k_0 is given and does not change. The change in the capital stock from period 1 onwards follows from differentiating first-order condition (55):

$$\frac{dk_0}{dA} = 0, \tag{180}$$

$$\frac{dk_t}{dA} = -\frac{f'(\bar{k})}{\bar{A}f''(\bar{k})} > 0, \qquad t = 1, 2, ... \tag{181}$$

Hence, investment in period 0 will be positive (recall that, by definition, $I_0 \equiv k_1 - k_0$) and will go back to zero in period 1.

What will be the path of output? Output rises in period 0, further increases in period 1, and is flat thereafter. In period 0, the capital stock has not changed yet but A is higher. In period 1, both the capital stock and A are higher relative to their pre-shock levels. Formally, differentiate (37) and use (180) and (181) to obtain:

$$\frac{dy_0}{dA} = f(\bar{k}), \tag{182}$$

$$\frac{dy_t}{dA} = f(\bar{k}) + r\frac{dk_1}{dA} \qquad t = 1, 2, ... \tag{183}$$

By (46), consumption will be constant along the new perfect foresight equilibrium but at a higher level. To compute the change in consumption, differentiate (42) to obtain:

$$\frac{dc}{dA} = \frac{r}{1+r}\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \frac{dy_t}{dA} - \frac{dk_1}{dA}\right]. \tag{184}$$

Using (182) and (183), this expression simplifies to:

$$\frac{dc}{dA} = f(\bar{k}). \tag{185}$$

Intuitively, by envelope considerations, at an optimum the output increase that results from the rise in the capital stock is exactly offset by the cost of increasing the capital stock. Hence, net output increases only by the direct effect of the increase in A.

How does the trade balance respond in period 0? Differentiate (50) and use (182) and (185) to obtain:

$$\begin{aligned} \frac{dTB_0}{dA} &= \frac{dy_0}{dA} - \frac{dc_0}{dA} - \frac{dk_1}{dA} \\ &= -\frac{dk_1}{dA} < 0. \end{aligned}$$

The trade balance worsens in period 0 due to the rise in investment. In subsequent periods, the trade balance will need to improve relative to its pre-shock level to finance period 0 investment. Formally, differentiate (50) and use (183) and (184) to obtain:

$$\frac{dTB_t}{dA} = \frac{dy_t}{dA} - \frac{dc}{dA} = r\frac{dk}{dA} > 0, \qquad t = 1, 2, ...$$

How does saving in period 0 respond to the increase in A? Using equations (53), (182), and (184), it follows that:

$$\frac{dS_0}{dA} = \frac{dy_0}{dA} - \frac{dc_0}{dA} = 0.$$

Saving in period 0 does not change. Hence, since investment in period 0 is positive, there will be a current account deficit:

$$\frac{dCA_0}{dA} = \frac{dS_0}{dA} - \frac{dI_0}{dA} = -\frac{dk}{dA} < 0.$$

Since the economy is stationary again from period 1 onwards, the current account should be zero. To check this, let us first compute saving in period 1. Since the economy ran a deficit in period 1, it will have to service this increase in debt from period 1 onwards. Hence, using (183) and (185):

$$\frac{dS_1}{dA} = -r\frac{dk}{dA} + \frac{dy_1}{dA} - \frac{dc_1}{dA} = 0. \tag{186}$$

Given that investment is zero from period 1 onwards, the current account is zero in period 1. The increase in the stock of debt thus remains unchanged. It follows that saving in period 2 is therefore also zero and, hence, so is the current account. This is, of course, true for any subsequent period.

Finally, notice that all the expressions derived in this exercise could have been obtained directly from the results that we obtained in the text for a discrete change in A by proceeding as follows. Note that, in response to small change in A, the change in k will also be small. Taking a first-order approximation for the change in k, we get

$$f(k_t) = f(\bar{k}) + f'(\bar{k})dk_t, \qquad t = 1, 2, ..., \tag{187}$$

where $dk_t \equiv k_t - \bar{k}$. We will show that, for a small change in A, the net present value of investment (captured by the term in square brackets on the RHS of equation (62) is zero. In other words, we want to show that for a small change in k,

$$\frac{\bar{A}[f(\bar{k}^H) - f(k_0)]}{r} = \bar{k}^H - k_0$$

Using the Taylor approximation for $t = 1$ and noting that $\bar{k}^H = k_1$, $k_0 = \bar{k}$, and $dk_1 \equiv \bar{k}^H - k_0$, this expression reduces to

$$\frac{\bar{A}f'(\bar{k})}{r} = 1,$$

which, of course, holds for the initial equilibrium.[5] Since the net present value of investment is zero, we can rewrite (62) and (63) as (recall that $k_0 = \bar{k}$)

$$\begin{aligned} \bar{c} &= rb_0 + \bar{A}^H f(\bar{k}) \qquad (188) \\ S_0 &= 0 \qquad (189) \end{aligned}$$

These equations tell us that, after a small change in A and thus in k, consumption and saving would be given by (188) and (189). It follows that there is no change in saving and that the (small) change in consumption is given by $f(\bar{k})(\bar{A}^H - \bar{A})$ which is equal to $f(\bar{k})dA$, as indicated by expression (185).

We thus conclude that, in the presence of investment, *a permanent (but small) increase in productivity leads to no change in saving and to an increase in investment.*

(b) Let us begin by deriving the path of the capital stock. From condition (55), it follows that

$$\begin{aligned} &\tfrac{dk_0}{dA} = 0 \\ &\tfrac{dk_t}{dA} = -\tfrac{f'(\bar{k})}{\bar{A}f''(\bar{k})} > 0, \quad t = 1, ..T-1 \\ &\tfrac{dk_t}{dA} = 0, \qquad\qquad\qquad t = T, ... \end{aligned} \tag{190}$$

[5] Notice that it would be incorrect to infer that because the net present value of investment is zero in equilibrium, household will choose not to invest. Clearly, households have an incentive to invest as differentiation of the first-order condition for investment shows. Think also about the analogy with the standard consumer's problem when faced with a change in the price of a good. By envelope considerations, the first-order welfare effect of changing quantities consumed is zero; all the welfare gain comes from the direct effect of a lower price.

The capital stock in period 0 is given and does not change. From period 1 and until period $T-1$, the capital stock is higher. In period T, the capital stock returns to its pre-shock level. As in the case analyzed in the text, the rise in the capital stock in period 1 does not depend on the duration of the shock, T. Hence, it follows that investment is positive in period 0 and negative in period $T-1$.

Let us now derive the path of output. From (37) and (190), it follows that

$$\begin{aligned}
&dy_0 = f(\bar{k})dA > 0, \\
&dy_t = \left[f(\bar{k}) - \tfrac{r}{Af''(\bar{k})}\right] dA > 0, \quad t = 1, 2, .T-1 \\
&dy_t = 0, \quad t = T, ...
\end{aligned} \tag{191}$$

Output thus rises in period 0, increases further in period $t = 1$, and then remains at that higher level up to, and including, period $T-1$. In period T, output returns to its pre-shock level.

To compute the change in consumption, it will be useful to proceed in steps and first compute the changes in the present discounted value of output and net output. Using (191), the present discounted value of output is given by:

$$\begin{aligned}
dPDV(y) &= \sum_{t=0}^{T-1}\left(\frac{1}{1+r}\right)^t f(\bar{k})dA + \frac{1}{1+r}\sum_{t=0}^{T-2}\left(\frac{1}{1+r}\right)^t Af'(\bar{k})dk_t \\
&= \frac{1+r}{r}f(\bar{k})dA\left[1-\left(\frac{1}{1+r}\right)^T\right] + Adk_t\left[1-\left(\frac{1}{1+r}\right)^{T-1}\right]
\end{aligned} \tag{192}$$

To compute the present discounted value of net output (i.e., output net of investment), we must subtract from the present discounted value of output, given by (192) the investment that takes place at time 0 and the disinvestment that takes place at $T-1$:

$$dPDV(\text{net output}) = dPDV(y) - dI_0 - \left(\frac{1}{1+r}\right)^{T-1} dI_{T-1}. \tag{193}$$

Since, by definition, $I_0 = k_1 - k_0$ and $I_{T-1} = k_T - k_{T-1}$ and k_0 and k_T do not change relative to their pre-shock values, then $dI_0 = dk_1$ and $dI_{T-1} = -dk_{T-1}$. Hence, using (190), we can rewrite (193) as

$$\begin{aligned}
dPDV(\text{net output}) &= dPDV(y) - dk_1 + \left(\frac{1}{1+r}\right)^{T-1} dk_T \\
&= \frac{1+r}{r}f(\bar{k})\left[1-\left(\frac{1}{1+r}\right)^T\right] dA > 0.
\end{aligned} \tag{194}$$

Two observations are worth making. First, notice how, due to envelope considerations, the changes in output due to changes in the capital stock cancel each other out with the investment terms. As a result, the change in net output is just the present discounted value of the direct effect of a higher A on output for T periods. Second, the larger is T, the larger is the rise in the present discounted value of net output.

We are now ready to compute the rise in consumption. Once again, consumption will be constant along the new perfect foresight equilibrium path. The change in this constant level of consumption follows from differentiating (42) and using (194):

$$\frac{d\bar{c}}{dA} = \left[1 - \left(\frac{1}{1+r}\right)^T\right] f(\bar{k}) > 0. \tag{195}$$

The larger is T, the larger is the increase in consumption because the bigger is the increase in the present discounted value of net output. For any finite value of T, the rise in consumption in period 0 will be smaller than that in output.[6] Consumption remains below until the shock is reverted. Intuitively – and as we have learned in the first part of this chapter – consumers smooth out the effects of the positive shock over time by consuming less than output during the good times and more than output once the good times are over.

To find out the path of the trade balance, differentiate (50) and use (191) and (195) to obtain:

$$\frac{dTB_0}{dA} = \underbrace{f(\bar{k})\left(\frac{1}{1+r}\right)^T}_{\substack{\text{saving} \\ \text{effect}}} - \underbrace{\frac{dI_0}{dA}}_{\substack{\text{investment} \\ \text{effect}}}. \tag{196}$$

The RHS of equation (196) captures the two key effects that come into play. The saving effect – which is positive and thus tends to improve the trade balance – captures the households' desire to save in order to smooth consumption over time. The larger is T, the smaller will be this effect because the more permanent is the shock. The investment effect – which is also positive and thus tends to worsen the trade balance – captures the increase in investment in response to the increase in productivity. Depending on the relative strength of these two effects, the trade balance could either improve – as in the standard case without investment – or worsen.

From period 1 up to, and including, period $T-2$, the trade balance

[6] For $T \to \infty$ (i.e., when the rise in productivity is permanent), consumption in period 0 would rise by the same amount as output.

will improve relative to its pre-shock level.

$$\begin{aligned}\frac{dTB_t}{dA} &= \frac{dy_t}{dA} - \frac{dc_t}{dA}, \qquad t = 1, 2, .., T-2 \\ &= f(\bar{k})\left(\frac{1}{1+r}\right)^T + r\frac{dk}{dA} > 0.\end{aligned}$$

In period $t = T-1$, the trade balance improves further because there is a fall in investment in anticipation of the fall in A in period T. Formally,

$$\begin{aligned}\frac{dTB_t}{dA} &= \frac{dy_t}{dA} - \frac{dc}{dA} - \frac{d(k_t - k_{t-1})}{dA}, \qquad t = 1, 2, .., T-2 \\ \frac{dTB_t}{dA} &= \frac{dy_t}{dA} - \frac{dc_t}{dA} + \frac{dk}{dA} \\ &= f(\bar{k})\left(\frac{1}{1+r}\right)^T + (1+r)\frac{dk}{dA} > 0\end{aligned}$$

From period T onwards, there is a trade deficit, given by:

$$\frac{dTB_T}{dA} = -\frac{dc}{dA} = -f(\bar{k})\left[1 - \left(\frac{1}{1+r}\right)^T\right] < 0.$$

Let us now derive the path of the current account. Since the initial stock of net foreign assets, b_0, is given, the initial impact on the current account is the same as for the trade balance. Hence, equation (196) also gives us the change in the current account. To compute the entire path of the current account, we need to keep track of the path of net foreign assets, as the latter influences saving through the returns on net foreign assets. While straightforward, the algebra is somewhat tedious and we omit it here. Intuitively, as the trade balances moves into surplus in period 1, so does the current account balance. It then continues to improve over time because while the trade balances remains constant up to, and including, period $T-2$, returns on assets keep accumulating. In period $T-1$, the current account surplus is further fed by the disinvestment. In period T, the current account balances falls to zero as the economy becomes stationary thereafter.

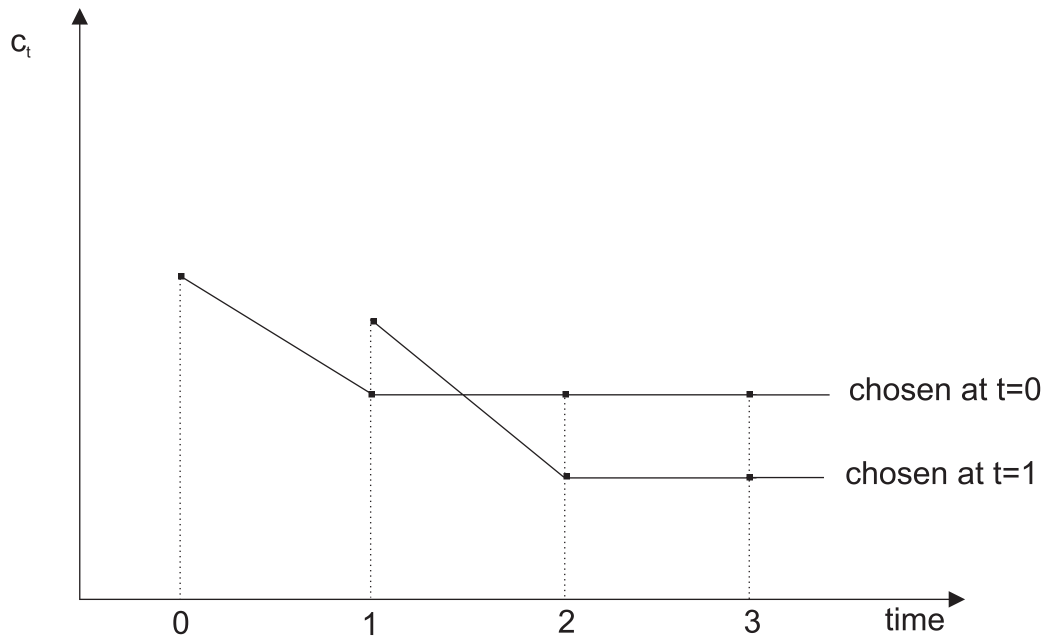
Finally, let us turn to saving. In period 0, saving increases:

$$\begin{aligned}\frac{dS_0}{dA} &= \frac{dy_0}{dA} - \frac{dc_0}{dA} \\ &= f(\bar{k}) - f(\bar{k})\left[1 - \left(\frac{1}{1+r}\right)^T\right] \\ &= f(\bar{k})\left(\frac{1}{1+r}\right)^T > 0\end{aligned}$$

If the change is permanent (i.e., if $T \longrightarrow \infty$), then the change in saving is zero. This is the particular case analyzed in subsection 4.1 above. The smaller is T, the larger is the fall is saving.

Figure 1. Time inconsistency of preferences

A. Hyperbolic consumer



B. Future consumption yields utility

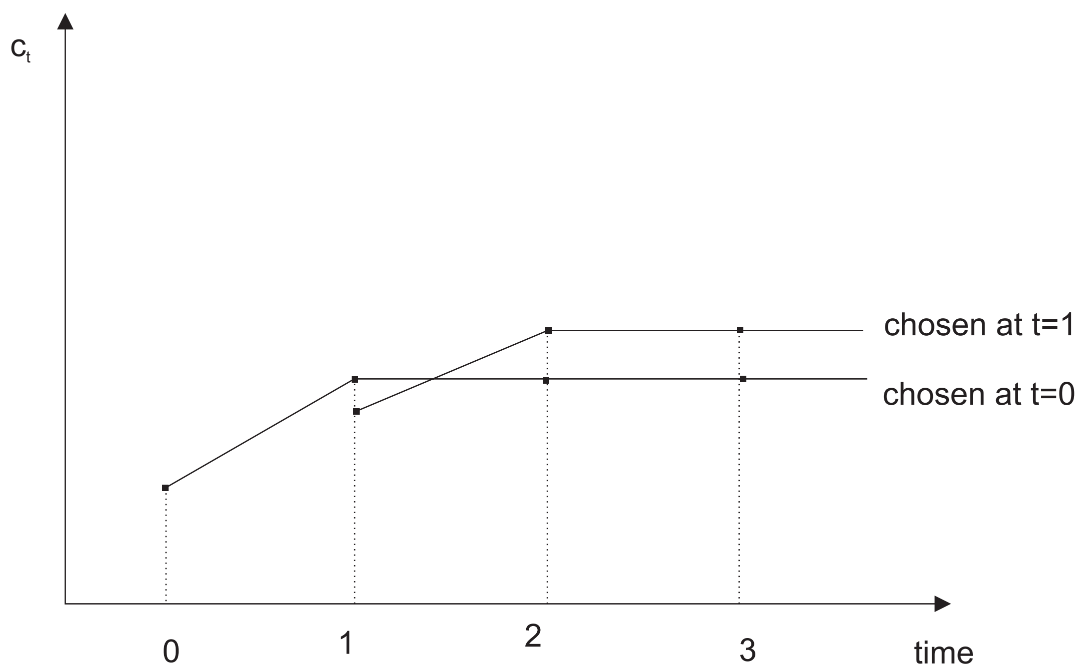


Figure 2. Temporary high productivity