Chapter 2
Capital market imperfections
Answer Key to Exercises[1]

Carlos A. Végh
University of Maryland and NBER
E-mail: vegh@econ.umd.edu

Current draft: October 10, 2010

1. **Reduced forms for consumption and welfare under uncertainty and incomplete markets**

   Consider the incomplete markets case analyzed in subsection 3.1 with the following two modifications. First, assume that preferences are given by the constant absolute risk aversion function:

$$u(c) = -\frac{1}{\alpha}e^{-\alpha c}.$$

   Second, instead of the Bernoulli distribution for period 2's output specified in (16), suppose that

$$y_2 = y_1 + \varepsilon, \qquad \varepsilon \sim N(0, \sigma^2).$$

   In this context:

   (a) Show that the coefficient of absolute risk aversion is equal to α. (Recall that the Arrow-Pratt measure of absolute risk aversion is given by $-u''/u'$.)

   (b) Following Kimball (1990), show that the coefficient of absolute prudence (defined as $-u'''/u''$) is equal to α.

[1] This answer key is part of a graduate textbook on "Open Economy Macroeconomics in Developing Countries", currently under preparation by the author (to be published by MIT Press) and should be cited accordingly. I am very grateful to Agustin Roitman for his help in preparing this answer key.

(c) Derive reduced-form solutions for c_1 and c_2. In particular, how does a higher σ^2 affect c_1 and c_2? [Hint: Recall that if $x \sim N(E\{x\}, \sigma_x^2)$, then $E\{e^x\} = e^{Ex+\sigma_x^2/2}$.]

(d) Compute the correlation coefficient between c_2 and y_2.

(e) Show that welfare is a decreasing function of σ^2.

In all cases, discuss the intuition behind the results.

Answer

(a) Since

$$\begin{aligned} u'(c) &= e^{-\alpha c} > 0, \\ u''(c) &= -\alpha e^{-\alpha c} < 0, \\ u'''(c) &= \alpha^2 e^{-\alpha c} > 0, \end{aligned}$$

it follows that

$$\text{Coefficient of absolute risk aversion} \equiv -\frac{u''(c)}{u'(c)} = \alpha.$$

(b) Use the derivations in a) to show that:

$$\text{Coefficient of absolute prudence} \equiv -\frac{u'''(c)}{u''(c)} = \alpha.$$

(c) Notice that since y_2 depends on the realization of the shock, c_2 will also depend on the realization of the shock. (In other words, c_2 is also a stochastic variable.) Since the intertemporal constraint must hold for every possible realization of y_2, we need to set up the maximization as

$$\underset{\{c_1, c_2(\varepsilon)\}}{\text{Max}} \; u(c_1) + \beta \int_{-\infty}^{\infty} u(c_2(\varepsilon)) f(\varepsilon) d\varepsilon + \int_{-\infty}^{\infty} \lambda(\varepsilon) \left[y_1 + \frac{y_2(\varepsilon)}{1+r} - c_1 - \frac{c_2(\varepsilon)}{1+r} \right] d\varepsilon,$$

where $f(\varepsilon)$ is the density function.[28]

[28] In other words, it would be *incorrect* to set up the maximization problem using the intertemporal constraint in expected values.

The first-order conditions are given by (assuming $\beta(1+r)=1$):

$$\begin{aligned} u'(c_1) &= \int_{-\infty}^{\infty} \lambda(\varepsilon) d\varepsilon, \qquad (109) \\ \int_{-\infty}^{\infty} u'(c_2(\varepsilon)) f(\varepsilon) d\varepsilon &= \int_{-\infty}^{\infty} \lambda(\varepsilon) d\varepsilon \end{aligned}$$

By definition, the second optimality condition can be written as:

$$E\{u'(c_2)\} = \int_{-\infty}^{\infty} \lambda(\varepsilon) d\varepsilon \qquad (110)$$

Combining (109) and (110), we obtain the stochastic Euler equation:

$$u'(c_1) = E\{u'(c_2)\}$$

Taking into account that preferences are given by (102), we can rewrite the Euler equation as:

$$e^{-\alpha c_1} = E\{e^{-\alpha c_2}\}. \qquad (111)$$

Using the flow constraints, we obtain:

$$c_2 = (1+r)(y_1 - c_1) + y_2.$$

Since $y_2 = y_1 + \varepsilon$, then

$$c_2 = (2+r)y_1 - (1+r)c_1 + \varepsilon.$$

Hence, c_2 is normally distributed with mean $(2+r)y_1 - (1+r)c_1$ and variance σ^2. It follows that $-\alpha c_2$ will also be normally distributed:

$$-\alpha c_2 \sim N(-\alpha E\{c_2\}, \alpha^2 \sigma^2).$$

Recall that if $x \sim N(E\{x\}, \sigma_x^2)$, then $E\{e^x\} = e^{Ex + \sigma_x^2/2}$. Hence, using the distribution of $-\alpha c_2$, it follows that

$$E\{e^{-\alpha c_2}\} = e^{-\alpha E\{c_2\} + \frac{\alpha^2}{2}\sigma^2}.$$

Using this last expression, we can rewrite the stochastic Euler equation (111)as:

$$e^{-\alpha c_1} = e^{-\alpha E\{c_2\} + \frac{\alpha^2}{2}\sigma^2},$$

which reduces to

$$E\{c_2\} = c_1 + \frac{\alpha}{2}\sigma^2. \tag{112}$$

Since the intertemporal constraint holds for every state of nature, it holds in expected value. Hence:

$$c_1 + \frac{E\{c_2\}}{1+r} = y_1 + \frac{E\{y_2\}}{1+r}. \tag{113}$$

Substituting (112) into (113), we obtain:

$$c_1 = y_1 - \frac{\alpha\sigma^2}{2(2+r)}.$$

Consumption in period 1 is a *decreasing* function of the variance of y_2 The reason is that – as we would have expected – the higher the variance of output in the second period, the higher are precautionary saving. Notice also that, for a given σ^2, the higher the coefficient of absolute risk aversion (α), the lower is consumption (i.e., the higher are precautionary saving).
Since $b_1 = y_1 - c_1$, it follows from the last expression that

$$b_1 = \frac{\alpha\sigma^2}{2(2+r)}.$$

Using $c_2 = (1+r)b_1 + y_2$, it follows that

$$c_2 = y_2 + \left(\frac{1+r}{2+r}\right)\frac{\alpha}{2}\sigma^2.$$

Consumption is in period is higher than y_2 because of the precautionary saving that consumers engage in. As a particular case, notice that if $\sigma^2 = 0$ (i.e., there is no uncertainty), we are back to the world of chapter 1 and there is full consumption smoothing (i.e., $c_1 = c_2$).

(d) To compute the correlation between c_2 and y_2, recall that, by definition,

$$Corr(c_2, y_2) = \frac{Cov(c_2, y_2)}{\sqrt{Var(c_2)Var(y_2)}}$$

The covariance is given by:

$$Cov(c_2, y_2) = E\underbrace{(c_2 - Ec_2)}_{\varepsilon}\underbrace{(y_2 - Ey_2)}_{\varepsilon}$$

As indicated below the equation, $y_2 - Ey_2 = \varepsilon$ by definition of the stochastic process for y_2. It can also be easily checked that $c_2 - Ec_2 = \varepsilon$.
Hence:

$$Cov(c_2, y_2) = E\varepsilon^2 = \sigma^2$$

Since $Var(c_2) = Var(y_2) = \sigma^2$, it follows that

$$Corr(c_2, y_2) = 1.$$

(e) To show that welfare falls with σ^2, notice that welfare is given by

$$W(\sigma^2) = -\frac{1}{\alpha}e^{-\alpha c_1} - \beta\frac{1}{\alpha}E\{e^{-\alpha c_2}\}.$$

Using the stochastic Euler equation (111), we can rewrite this as

$$W(\sigma^2) = -\frac{1}{\alpha}e^{-\alpha c_1}(1 + \beta).$$

Hence,

$$\frac{dW(\sigma^2)}{d\sigma^2} = (1 + \beta)e^{-\alpha c_1}\frac{dc_1}{d\sigma^2} < 0.$$

Intuitively, the higher is σ^2, the larger are the deviations from consumption smoothing due to the consumer's desire to engage in larger precautionary saving. As a result, welfare decreases.

2. A two-country world with complete markets

Consider a two country (domestic and foreign) version of our small open economy model with complete markets analyzed in Subsection 3.3. Suppose that preferences are logarithmic and that countries have the same discount factor. We will use star superscripts to denote the foreign country variables. Since this is a two-country model, the following world output constraints will hold:

$$\begin{aligned} c_1 + c_1^* &= y_1 + y_1^* \equiv y_1^W, \\ c_2^A + c_2^{A*} &= y_2^A + y_2^{A*} \equiv y_2^{AW}, \\ c_2^B + c_2^{B*} &= y_2^B + y_2^{B*} \equiv y_2^{BW}, \end{aligned}$$

where a superscript "W" denotes world quantities. Notice that we have re-labeled states "high" and "low" as "A" and "B," respectively. (The reason is that if, for instance, there is no aggregate uncertainty, then when domestic output is high, foreign output will be low and viceversa.)

In this context:

(a) Derive first-order conditions and show that the ratio of consumption across states of nature is the same at home and abroad.

(b) Derive equilibrium expressions for q^A and q^B.

(c) Derive a reduced-form solution for the world real interest rate.

(d) Show that consumption as a proportion of world output is constant across time and states of nature in both countries.

(e) Show that $corr(c_2, c_2^*) = 1$.

(f) Show that, if there is no world uncertainty (i.e., $y_2^{AW} = y_2^{BW}$), c_2 is uncorrelated with domestic output.

Answer

(a) The first-order conditions in the home and foreign country will be given by, respectively,

$$\begin{aligned}
\frac{1}{c_1} &= \lambda, && (114)\\
\frac{\beta p}{c_2^A} &= \lambda \frac{q^A}{1+r}, && (115)\\
\frac{\beta(1-p)}{c_2^B} &= \lambda \frac{q^B}{1+r}, && (116)\\
\frac{1}{c_1^*} &= \lambda^*, && (117)\\
\frac{\beta p}{c_2^{*A}} &= \lambda^* \frac{q^A}{1+r}, && (118)\\
\frac{\beta(1-p)}{c_2^{*B}} &= \lambda^* \frac{q^B}{1+r}. && (119)
\end{aligned}$$

Combining equations (115), (116), (118), and (119), we obtain:

$$\frac{c_2^A}{c_2^B} = \frac{c_2^{*A}}{c_2^{*B}},$$

which shows that, due to complete markets, the consumption ratios across states of nature is the same.

(b) Solve for c_2^A and c_2^{*A} from (115) and (118) and substitute into $c_2^A + c_2^{A*} = y_2^{AW}$ to obtain:

$$\frac{\beta p(1+r)}{q^A}\left(\frac{1}{\lambda} + \frac{1}{\lambda^*}\right) = y_2^{AW}.$$

But, given (114), (117), and the fact that $c_1 + c_1^* \equiv y_1^W$, we can rewrite this last expression as:

$$q^A = \beta p(1+r)\frac{y_1^W}{y_2^{AW}}. \qquad (120)$$

Proceeding analogously, we can show that

$$q^B = \beta(1-p)(1+r)\frac{y_1^W}{y_2^{BW}}. \qquad (121)$$

These are equilibrium expressions because, in this two-country world, r is an endogenous variable. As expected, the price of the contingent claims are inversely proportional to world output in the corresponding state of nature.

(c) By arbitrage, $q^A + q^B = 1$. Combining this fact with expressions (120) and (121) and solving for $1 + r$:

$$1 + r = \frac{1}{\beta y_1^W} \left(\frac{1}{\frac{p}{y_2^{AW}} + \frac{1-p}{y_2^{BW}}} \right).$$

Intuitively, a higher y_1^W calls for a lower real interest rate rate to induce households both at home and abroad to consume more. Notice that if there is no aggregate uncertainty in period 2 and the world output path is constant over time (i.e., if $y_2^{AW} = y_2^{BW} = y_1^W$), then

$$\beta (1 + r) = 1.$$

Incidentally, this shows how the assumption that we made for our small open economy in Chapter 1 ($\beta (1 + r) = 1$ in discrete time or $\beta = r$ in continuous time) can be derived endogenously in a two-country world with no uncertainty and constant output over time.

(d) Using (120), (121), and first-order conditions (114) through (119), it follows that

$$\begin{aligned} \frac{c_1}{y_1^W} &= \frac{c_2^A}{y_2^{AW}} = \frac{c_2^B}{y_2^{BW}}, \\ \frac{c_1^*}{y_1^W} &= \frac{c_2^{*A}}{y_2^{AW}} = \frac{c_2^{*B}}{y_2^{BW}}, \end{aligned} \tag{122}$$

which shows that, as a proportion of world output, consumption in each country is constant across both time and states of nature.

(e) Let us first compute $E\{c_2\}$ and $Var\{c_2\}$. Since $c_2^A = c_1 y_2^{AW}/y_1^W$ and $c_2^b = c_1 y_2^{BW}/y_1^W$, it follows that

$$\begin{aligned} E\{c_2\} &= \frac{c_1}{y_1^W} E\{y_2^W\}, \\ Var\{c_2\} &= \left(\frac{c_1}{y_1^W}\right)^2 Var\{y_2^W\}. \end{aligned}$$

By the same token:

$$\begin{aligned} E\{c_2^*\} &= \frac{c_1^*}{y_1^W} E\{y_2^W\}, \\ Var\{c_2^*\} &= \left(\frac{c_1^*}{y_1^W}\right)^2 Var\{y_2^W\}. \end{aligned}$$

Let us know compute the covariance between c_2 and c_2^*:

$$\begin{aligned} E\{(c_2 - E\{c_2\})(c_2^* - E\{c_2^*\})\} &\equiv E\{(c_2 - \frac{c_1}{y_1^W} E\{y_2^W\})(c_2^* - \frac{c_1^*}{y_1^W} E\{y_2^W\})\}, \\ &= E\{(c_1 \frac{y_2^W}{y_1^W} - \frac{c_1}{y_1^W} E\{y_2^W\})(c_1^* \frac{y_2^W}{y_1^W} - \frac{c_1^*}{y_1^W} E\{y_2^W\})\}, \\ &= E\{\frac{c_1}{y_1^W}(y_2^W - E\{y_2^W\}) \frac{c_1^*}{y_1^W}(y_2^W - E\{y_2^W\})\}, \\ &= \left(\frac{c_1}{y_1^W}\right)\left(\frac{c_1^*}{y_1^W}\right) E\{(y_2^W - E\{y_2^W\})^2\}, \\ &= \left(\frac{c_1}{y_1^W}\right)\left(\frac{c_1^*}{y_1^W}\right) Var\{y_2^W\}. \end{aligned}$$

The correlation is then given by:

$$\begin{aligned} Corr(c_2, c_2^*) &\equiv \frac{E\{(c_2 - E\{c_2\})(c_2^* - E\{c_2^*\})\}}{\sqrt{Var\{c_2\}}\sqrt{Var\{c_2^*\}}}, \\ &= \frac{\left(\frac{c_1}{y_1^W}\right)\left(\frac{c_1^*}{y_1^W}\right) Var\{y_2^W\}}{\sqrt{\left(\frac{c_1}{y_1^W}\right)^2 Var\{y_2^W\} \left(\frac{c_1^*}{y_1^W}\right)^2 Var\{y_2^W\}}}, \\ &= \frac{\left(\frac{c_1}{y_1^W}\right)\left(\frac{c_1^*}{y_1^W}\right) Var\{y_2^W\}}{\left(\frac{c_1}{y_1^W}\right)\left(\frac{c_1^*}{y_1^W}\right)\sqrt{Var\{y_2^W\} Var\{y_2^W\}}}, \\ &= 1. \end{aligned}$$

(f) If there is no world uncertainty in the second period ($y_2^{AW} = y_2^{BW}$), it follows from (122) that $c_2^A = c_2^B$. Since domestic consumption is constant across states of nature,

$$corr(c_2, y_2) = 0.$$

3. An ad-hoc upward sloping supply of funds

Let preferences be given by:

$$W = \log(c_1) + \beta \log(c_2), \tag{84}$$

where $\beta(\equiv 1/(1+\delta))$ is the discount factor and δ is the discount rate. Assume that $\beta(1+r) = 1$, where r is the world real interest rate.

The flow constraints are given by:

$$c_1 = d_{1,} \tag{85}$$
$$c_2 = y_2 - (1+r^s)d_1, \tag{86}$$

where d_1 is net external debt and $y_2 > 0$ is second period's output (notice that, for simplicity, we have assumed that output in the first period is zero).

The economy faces an upward sloping supply of funds of the form:

$$r^s = r + f(d_1), \qquad f(0) = 0, \; f'(d_1) > 0, \tag{87}$$

where r^s is the real interest rate charged to the country.

In this context:

(a) Solve the planner's problem (i.e., the social optimum). Show that the planner will choose not to smooth consumption over time.

(b) Solve the consumer's problem (i.e., the market solution). Show that, relative to the planner's solution, the market solution implies that consumption in the first period is too high relative to the second period.

(c) Consider a linear version of this model (i.e., linear preferences and linear supply of funds):

$$W = c_1 + \frac{c_2}{1+\delta}, \tag{88}$$
$$f(d_1) = \alpha d_1. \tag{89}$$

Assume $\delta > r$. In this context:

i. Derive a reduced-form solution for the equilibrium values of d_1 and r^s for both the planner's problem and the market problem.

ii. Provide a graphical illustration of how the equilibrium values of d_1 and r^s are determined and interpret the results intuitively. [Hint: Think of the country as a monopsonist in world capital markets and proceed as in the textbook analysis of a monopsony in factor markets. To this end, you may want to review the analysis of a monopsonist in your favorite undergraduate microeconomics textbook.]

iii. Show that by imposing a borrowing tax rate (which increases linearly with the amount of borrowing), the government can implement the planner's solution.

Answer

(a) The planner will take into account the effect that the country's borrowing has on the real interest rate charged to the country by international creditors. To solve for the planner's problem, substitute the flow constraints, (104) and (105), into lifetime utility, (103), so that the maximization problem becomes:

$$\underset{d_1}{Max} \log(y_1 + d_1) + \beta \log \{y_2 - [1 + r + f(d_1)]d_1\} .$$

Differentiating with respect to d_1, and using (104) and (105), yields:

$$\frac{1}{c_1} = \beta \frac{1}{c_2}[1 + r + f(d_1) + f'(d_1)d_1]. \tag{123}$$

The term $1+r+f(d_1)+f'(d_1)d_1$ can be interpreted as the marginal social cost of funds in the international credit markets. Since, by assumption, $\beta(1 + r) = 1$, we can rewrite this Euler equation as

$$\frac{1}{c_1} = \frac{1}{c_2}\frac{[1 + r + f(d_1) + f'(d_1)d_1]}{1 + r}. \tag{124}$$

Naturally, if $f(.) \equiv 0$, there would be no distortion in international capital markets and the planner would choose to smooth consumption over time by borrowing in the first period and repaying in the second. In the presence of the distortion, however, it follows from (124) that $c_1 < c_2$. (Notice that we know that, in equilibrium, $d_1 > 0$ because the country has no endowment in period 0 and $c_1 > 0$ because marginal utility goes to infinity as consumption goes to zero). In other words, due to the imperfection in international capital markets, the planner would choose not to smooth consumption.

(b) Unlike the planner, the consumer takes as given r^s. In other words, the consumer maximizes (103) subject to (104) and (105), taking r^s as given. By substituting (104) and (105) into (103), we can set the maximization problem as

$$\underset{d_1}{Max} \log(y_1 + d_1) + \beta \log(y_2 - (1 + r^s)d_1).$$

Differentiating with respect to d_1 – and using (104) and (105) – yields a standard-looking Euler equation:

$$\frac{1}{c_1} = \beta(1 + r^s)\frac{1}{c_2}. \tag{125}$$

To solve for the *equilibrium* path of consumption, we substitute (106) into (125) to obtain:

$$\frac{c_1}{c_2} = \frac{1}{\beta[1 + r + f(d_1)]}. \tag{126}$$

Since we know that $d_1 > 0$ it follows that $\beta[1 + r + f(d_1)] > 1$ and hence $c_1 < c_2$. To show that in this market solution the ratio c_1/c_2 will be higher than in the planner's equilibrium, notice from (123) that

$$\left.\frac{c_1}{c_2}\right|_{planner} = \frac{1}{\beta\left[1 + r + f(d_1) + f'(d_1)d_1\right]}. \tag{127}$$

Since $f'(d_1)d_1 > 0$, it follows from comparing (126) and (127) that c_1/c_2 will be higher in the market case than in the planner's case. In other words, left to its own devices, this economy will tend to overconsume in the first period (i.e., borrow too much) relative to the second period.

(c) Let us consider the case with linear preferences and linear supply of funds.

i. Consider first the planner's case. When preferences are linear (given by equation (107)) and the supply of funds is also linear (given by (108)), the planner's maximization is given by

$$\underset{d_1}{Max} d_1 + \frac{1}{1+\delta}[y_2 - (1 + r + \alpha d_1)d_1]$$

The Euler equation is then given by:

$$1 = \frac{1}{1+\delta}(1 + r + 2\alpha d_1) \tag{128}$$

Solving for d_1:

$$d_1|_{planner} = \frac{\delta - r}{2\alpha}. \tag{129}$$

The country borrows more the more impatient it is (i.e., the higher is δ) and the lower is α.
Substituting (129) into (108), we obtain the equilibrium real interest rate charged to the country:

$$r^s = r + \frac{\delta - r}{2} > r.$$

Notice how the equilibrium real interest rate charged by creditors does not depend on α. This is because while, for a given level of debt, a higher α increases r^s, the resulting fall in the quantity borrowed exactly cancels out this effect. In other words, αd_1 does not depend on a.
Consider now the market case. In this case, the consumer's problem becomes linear in d_1:

$$\underset{d_1}{Max} d_1 + \frac{1}{1+\delta}[y_2 - (1 + r^s)d_1].$$

The Kuhn-Tucker condition for d_1 is given by:[29]

$$1 - \frac{1+r^s}{1+\delta} \leq 0, \qquad d_1 \geq 0, \qquad \left(1 - \frac{1+r^s}{1+\delta}\right) d_1 = 0.$$

[29] Notice that we know that d_1 cannot be negative because, by assumption, $\delta > r$. The consumer would never choose to save (i.e., lend) in the first period.

Notice that we can rule out a solution in which $d_1 = 0$. To show this, suppose that $d_1 = 0$. Then the Kuhn-Tucker condition would imply that $1 - \frac{1+r^s}{1+\delta} \leq 0$. Since, in equilibrium, $r^s = r$, this condition implies that $\delta - r \leq 0$. But this contradicts our assumption that $\delta > r$. Hence, $d_1 > 0$. From the Kuhn-Tucker condition, it then follows that

$$1 - \frac{1+r^s}{1+\delta} = 0$$

Hence, in equilibrium,

$$r^s = \delta. \tag{130}$$

Intuitively, the linear preferences imply that consumers borrow up to the point at which r^s has become equal to the discount rate. In other words, as long as $r^s < \delta$, consumers will keep borrowing.

Since, in equilibrium, $r^s = r + \alpha d_1$, it follows that

$$1 = \frac{1}{1+\delta}(1 + r + \alpha d_1). \tag{131}$$

Solving for d_1,

$$d_1|_{market} = \frac{\delta - r}{\alpha}. \tag{132}$$

Comparing (129) and (132), we see that, in the market solution, borrowing is exactly twice as much as in the planner's case.

ii. See Figure 1. The country is a monopsonist in international capital markets because the amount it borrows affects the interest rate that it pays. The country's total cost of borrowing is $(r + \alpha d_1)d_1$. The curve $r + \alpha d_1$ in Figure 1 gives us the average cost of borrowing. The curve $r + 2\alpha d_1$ gives us the marginal cost of borrowing.

The planner internalizes the fact that the country is a monopsonist in world capital markets and takes the marginal cost curve as the relevant cost. The demand for funds in the planner's case is the downward sloping curve $\delta - \alpha d_1$. To see this, notice that we can rewrite the Euler equation (128) as (recalling that $r^s = r + \alpha d_1$)

$$r^s = \delta - \alpha d_1.$$

The intersection of these two curves (point A in Figure 1) provides the equilibrium value of d_1. The equilibrium value of r^s must be read off the average cost of borrowing curve, as in the standard analysis of a monopsonist in factor markets.
In the market case – and since the consumer does not internalize the effect of his/her borrowing on the real interest rate charged to the country – the relevant curve (or supply of funds) is the average cost curve, given by $r + \alpha d_1$ in Figure 1. The demand curve in this case is simply δ (recall equation (130)).
The intersection of the two curves (point B in Figure 1) gives us the equilibrium values of d_1 and r^s.
The figure makes clear that the market solution (point B) yields excessive borrowing relative to the socially optimal amount of borrowing given by point A.

iii. Suppose that the government imposes a *proportional tax rate* on borrowing, which takes the form $\tau(d_1) = \gamma d_1$ where γ is a positive parameter (to be optimally determined by the government). In other words, the tax rate increases with the amount of borrowing. We also assume that the tax is imposed in the second period. In this case, the flow constraint for the second period becomes $\tau = \frac{\delta - r}{2}\tau = \frac{\delta - r}{2}\tau = \frac{\delta - r}{2}$

$$c_2 = y_2 - (1 + r^s + \gamma d_1)d_1.$$

It can be checked that the Euler equation for the market problem becomes

$$1 = \frac{1}{1+\delta}(1 + r^s + 2\gamma d_1).$$

In other words, consumers face a marginal cost of funds given by $1 + r^s + 2\gamma d_1$. Recall that the marginal social cost of funds is given by $1 + r + 2\alpha d_1$. Hence, for consumers to face the social marginal cost of funds, it must be the case that

$$1 + r^s + 2\gamma d_1 = 1 + r + 2\alpha d_1.$$

Noticing that $r^s = r + \alpha d_1$ and solving for γ

$$\gamma = \frac{\alpha}{2}.$$

We conclude that if the government imposes a borrowing tax rate of the form $\tau(d_1) = (\alpha/2)d_1$, the social optimum is achieved (under the assumption, of course, that the proceeds of the tax are rebated in a lump-sum fashion). This is a formal illustration of the celebrated "Tobin tax", advocated by James Tobin (see Tobin (1978)).

Finally, note that the government could also impose a *flat tax rate* (denote it by θ). In this case, the flow constraint for the second period becomes:

$$c_2 = y_2 - (1 + r^s + \theta)d_1.$$

Proceeding in an analogous manner, we can obtain a reduced form for the optimal tax rate:

$$\theta = \frac{\delta - r}{2}.$$

Unlike the proportional tax rate, this flat tax rate depends on preferences (i.e., on the parameter δ, which is in principle unobservable). Hence, from a practical point of view, it is more attractive to think of a proportional tax rate (α could in principle be estimated). But, from a theoretical point of view, both tax rates would do the job.

4. Numerical example of incomplete markets model with CRRA preferences

Consider the incomplete markets model analyzed in Subsection 3.1 with CRRA preferences of the form

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta},$$

where θ is the coefficient of relative risk aversion. In this context:

(a) Show that the coefficient of relative risk aversion is equal to θ. (Recall that the Arrow-Pratt measure of relative risk aversion is given by $-cu''/u'$.)

(b) Following Kimball (1990), show that the coefficient of relative prudence (defined as $-cu'''/u''$) is equal to θ.

(c) Plot c_1, c_2^H, c_2^L, current account, and expected utility as a function of the coefficient of risk aversion. Explain the intuition behind the results

(d) Plot the same variables as a function of a mean-preserving spread in the output distribution. Explain the intuition behind the results.

Answer

(a) Since

$$\begin{aligned} u'(c) &= c^{-\theta} > 0, \\ u''(c) &= -\theta c^{-(1+\theta)} < 0, \\ u'''(c) &= \theta(1+\theta)c^{-(2+\theta)} > 0, \end{aligned}$$

it follows that

$$\text{Coefficient of relative risk aversion} \equiv -\frac{cu''(c)}{u'(c)} = \theta.$$

(b) Use the derivations in a) to show that:

$$\text{Coefficient of relative prudence} \equiv -\frac{cu'''(c)}{u''(c)} = 1 + \theta.$$

(c) See Figure 2. As expected, c^1 falls with θ (and hence the current account increases) because a higher θ implies a higher degree of prudence leading to higher precautionary saving. Second-period consumption is higher in both states of natures because of higher saving in period 1. Expected utility falls as higher prudence leads to further deviations from consumption smoothing.

(d) See Figure 3. As expected, c^1 increases (and hence the current account falls) since the higher uncertainty leads to more precautionary saving. Second-period consumption is higher in the good state but lower in the bad state (as the low realization dominates the higher saving). Since the consumer is risk-averse, expected utility falls.

(e) Numerical example of incomplete markets model with imprudent preferences

Consider the incomplete markets model analyzed in Subsection 3.1 with the following preferences:

$$u(c) = 10c - \frac{1}{5}c^3,$$

for $c < \sqrt{\frac{50}{3}}$.

In this context:

(a) Show that these preferences exhibit risk aversion and *imprudence.*

(b) Plot c_1, c_2^H, c_2^L, current account, and expected utility as a function of the coefficient of risk aversion. Explain the intuition behind the results.

Answer

(a) Since

$$\begin{aligned} u'(c) &= 10 - \frac{3}{5}c^2 > 0 \\ u''(c) &= -\frac{6}{5}c < 0 \\ u'''(c) &= -\frac{6}{5} < 0 \end{aligned}$$

these preferences exhibit risk aversion (i.e., $u''(c) < 0$) and imprudence (i.e., $u'''(c) < 0$).

1. (a) See Figure 4. As expected, c^1 increases (and hence current account falls) as higher uncertainty leads to more imprudence (i.e., less precautionary saving). Second-period consumption is higher in the good state but lower in the bad state (as the realization dominates the higher saving). Expected utility (welfare), however, falls because the consumer is still risk averse and dislikes more uncertainty.

Figure 1. Equilibrium in the upward sloping supply of funds model

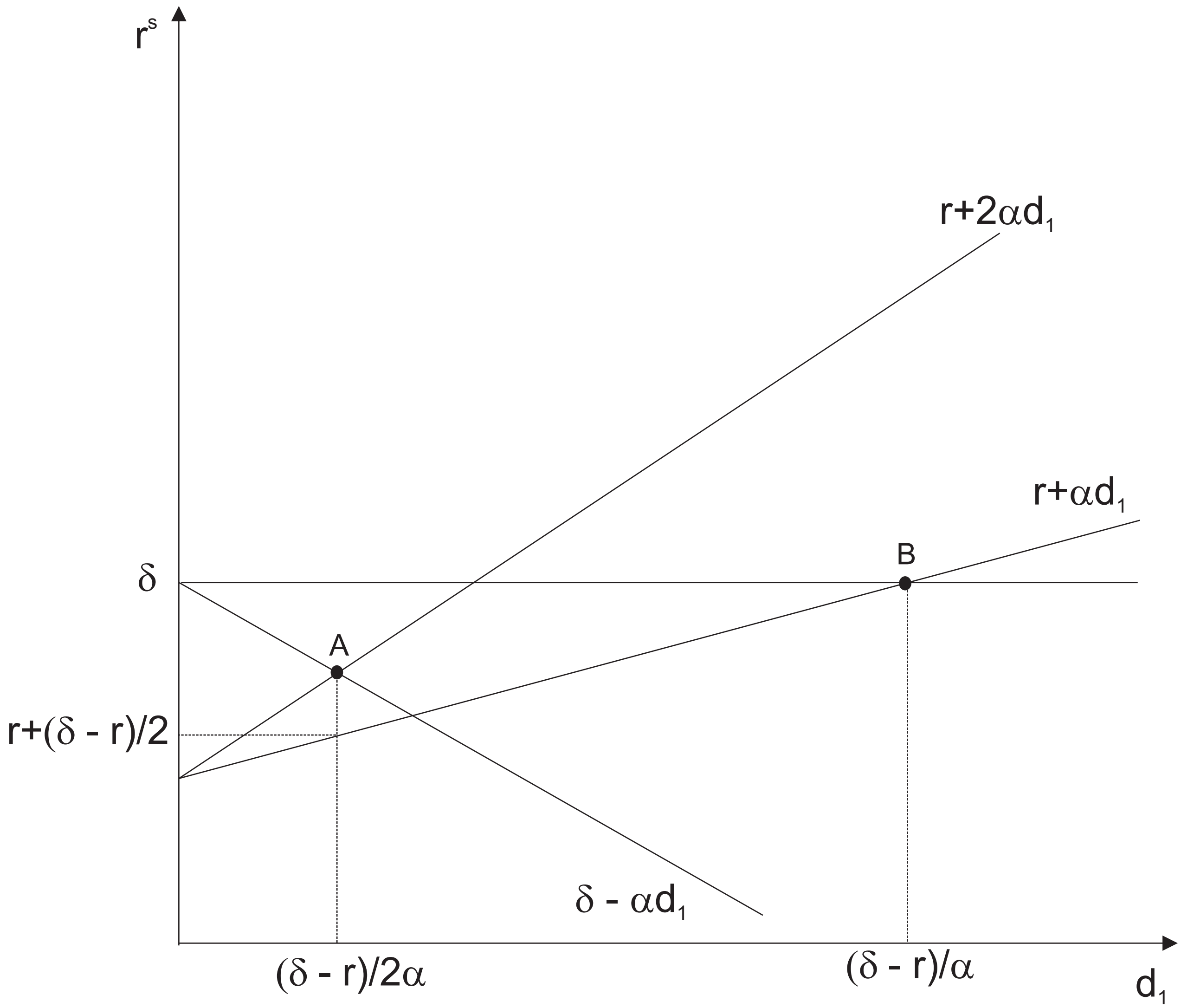

Figure 2. Plots of c_1, c_2^H, c_2^L, current account, and expected utility as a function of θ

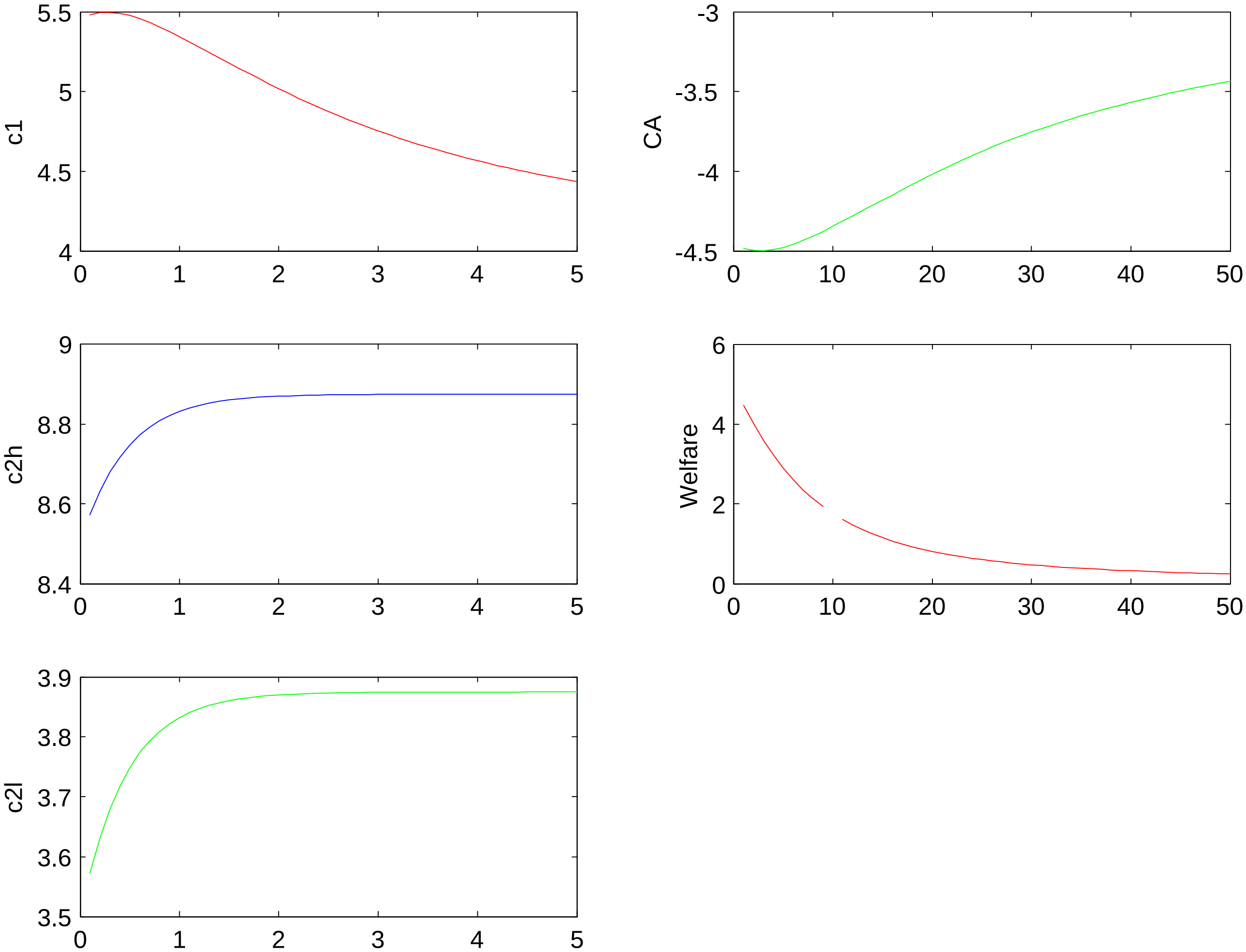

Figure 3. Plots of c_1, c_2^H, c_2^L, current account, and expected utility as a function of a mean-preserving spread in the output distribution

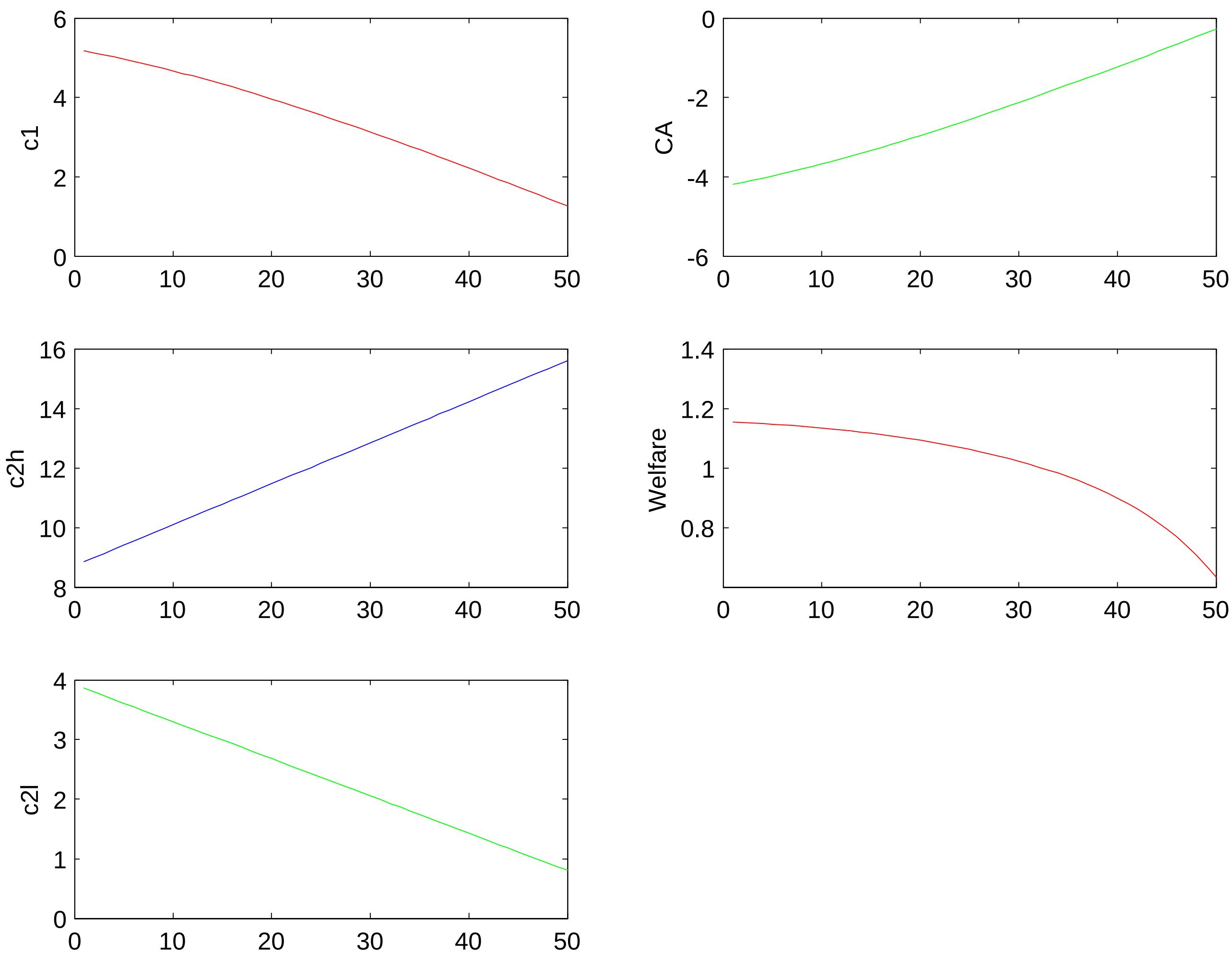

Figure 4. Case $u''' < 0$. Plots of c_1, c_2^H, c_2^L, current account, and expected utility as a function of a mean-preserving spread in the output distribution