1. Homotopy Groups

1.0 Introduction

The problems with which we shall be concerned here are those of the extension and classification of continuous maps. These may be stated as follows.

EXTENSION PROBLEM (E): Given spaces X, Y, a closed subset A of X, and a map $f:A \to Y$, does there exist a map $g:X \to Y$ such that $g|A = f$? If so, then g is called an extension of f.

CLASSIFICATION PROBLEM (C): Given spaces X, Y, a closed subset A of X, and maps f', f'' of X into Y such that $f'|A = f''|A$, does there exist a map $g:I \times X \to Y$ such that $g(0,x) = f'(x)$, $g(1,x) = f''(x)$ for $x \in X$, and $g(t,x) = f'(x) = f''(x)$ for $x \in A$? (I is the unit interval.) If so, then g is called a *homotopy* of f' to f'' relative to A. If A is the null set, then g is simply called a homotopy of f' to f''.

The classification problem is a special case of the extension problem. For let $X^* = I \times X$, $A^* = 0 \times X \cup 1 \times X \cup I \times A$, and define $f:A^* \to Y$ by $f(0,x) = f'(x)$, $f(1,x) = f''(x)$, $f(t,x) = f'(x) = f''(x)$ for $x \in A$. Then an extension of f over X^* is exactly a homotopy of f' to f'' relative to A.

On the other hand, under suitable hypotheses on the spaces involved, if f' and f'' are homotopic maps of A into Y, then f' can be extended over X if f'' can. In fact we have the homotopy extension theorem, which may be stated as follows.

THEOREM 1.0.1. *Let X,Y be spaces, $A \subset X, f:X \to Y, F:A \times I \to Y$ maps such that $f|A = F|A \times \{0\}$. Then if either of the two sets of conditions stated at the end of the theorem are satisfied, there exists a map $G:X \times I \to Y$ such that $G|A \times I = F$, $G|X \times \{0\} = f$. The possible conditions are*

I. *X separable regular, A closed*
Y an absolute neighborhood retract

or

II. *(X, A) finitely triangulable*
Y arbitrary

PROOF. Part I is standard. See, for instance, Hurewicz and Wallman [B.1, p. 86 (Borsuk's theorem)]. For Part II, we need the fact that a finitely triangulable space is an ANR. See, for instance, Lefschetz [B.2, p. 292].

Let $\tilde{Y} = A \times I \cup X \times \{0\}$, $\tilde{F}$ and $\tilde{f}$ the identity. By the foregoing remark, Y is an ANR. It follows from Part I that there exists a map

$$\tilde{G}:X \times I \to \tilde{Y}$$

such that $\tilde{G}|\tilde{Y}$ is the identity. Then the map $G:X \times I \to Y$ given by

$$\begin{array}{ll} G(p) = (f \circ \tilde{G})(p) & \tilde{G}(p) \in X \times \{0\} \\ G(p) = (F \circ \tilde{G})(p) & \tilde{G}(p) \in A \times I \end{array}$$

satisfies the conditions of the theorem.

COROLLARY 1.0.2. *Let $(K;L,K_0)$ be a finitely triangulable triad, $L_0 = K_0 \cap L$. Let*

$$f:(K,L) \to (X,A)$$

$$\phi:(K_0 \times I, L_0 \times I) \to (X,A)$$

be maps with $\phi(x,0) = f(x)$ for $x \in K_0$. Then there is a map $\psi:(K \times I, L \times I) \to (X,A)$ such that

$$\begin{cases} \psi(x,0) = f(x) & (x \in K) \\ \psi(x,t) = \phi(x,t) & (x \in K_0) \end{cases}$$

PROOF. The pair (L,L_0) being triangulable, there is a map $g:L \times I \to A$ such that

$$\begin{cases} g(x,0) = f(x) & (x \in L) \\ g(x,t) = \phi(x,t) & (x \in L_0) \end{cases}$$

Define $h:(K_0 \cup L) \times I \to X$ by

$$h(x,t) = \begin{cases} \phi(x,t) & (x \in K_0) \\ g(x,t) & (x \in L) \end{cases}$$

The pair $(K,K_0 \cup L)$ being triangulable, there is a map $\psi: K \times I \rightarrow X$ such that

$$\begin{cases} \psi(x,0) = f(x) & (x \in K) \\ \psi(x,t) = h(x,t) & (x \in K_0 \cup L) \end{cases}$$

Then ψ is the desired map.

We now give examples in which we demonstrate necessary conditions for the solubility of (C) and (E). In general, these conditions will be far from sufficient, as will be demonstrated later.

THEOREM 1.0.3. *Let E be an n-cell, S its boundary, and let $f: S \rightarrow Y$. Then a necessary condition that f be extendable to a map $g: E \rightarrow Y$ is that the induced homomorphism $f_*: H_{n-1}(S) \rightarrow H_{n-1}(Y)$ be identically zero.*

PROOF. Let $i: S \rightarrow E$ be the inclusion map. If g exists, then $f = g \circ i$, and hence $f_* = g_* \circ i_*$. But i_* is identically 0, since $H_{n-1}(E) = 0$.
Q.E.D.

REMARK. *It is easy to see that S may be replaced by any subset of E, and $n - 1$ by any integer greater than* 0.

THEOREM 1.0.4. *Under the same conditions as those of Theorem* 1.0.3, *let $f', f'': S \rightarrow Y$. Then, in order for f' and f'' to be homotopic, it is necessary that the induced maps f'_* and f''_* on the homology groups be equal.*

PROOF. Since f' and f'' are homotopic, the maps $f'_\#$ and $f''_\#$ induced by f' and f'', respectively, on the singular complex of S are chain-homotopic. But it is well known that chain-homotopic maps on a complex induce the same map on the homology groups. Q.E.D.

REMARK. *The same remark holds for Theorem* 1.0.4 *as for Theorem* 1.0.3.

At present, the foregoing problems have been solved only in a few special cases. If either X or Y is a cell, the problem of classifying the maps of X into Y is trivial. But if X is so simple a space as an n-sphere, the classification problem is far from solved, although great progress has been made in the last few years. The attempt to solve the latter problem ($X = S^n$) led Hurewicz to define the homotopy groups, which may be thought of as higher dimensional generalizations of the fundamental group. In this chapter we describe the basic properties of the homotopy groups; later we shall show how they are used in attacking the general extension and classification problems.

1.1 Function Spaces [7, 16]

Let X and Y be topological spaces.

DEFINITION. *$N(A,B) = \{f | f: X \to Y, f(A) \subset B\}$ for $A \subset X$, $B \subset Y$.*

DEFINITION. *Y^X = space of all maps (continuous functions) $f: X \to Y$, with the smallest topology containing all sets $N(C,U)$, C compact and contained in X, U open and contained in Y. This is called the compact-open topology.*

LEMMA 1.1.1. *Let X be Hausdorff; O_i open $\subset X$, $i = 1, \cdots, n$;*

$$C \text{ compact} \subset O = \bigcup_{i=1}^{n} O_i$$

Then $\exists\, C_i$ such that

$$C = \bigcup_{i=1}^{n} C_i$$

$$C_i \text{ closed} \subset O_i$$

$i = 1, \cdots, n.$

PROOF. Standard. See Lefschetz [B.2, p. 26 (33.4)(a)].

LEMMA 1.1.2. *Let A be a subbasis for the topology of Y. If X is Hausdorff, then the sets $N(C,U)$, C compact, $U \in A$, form a subbasis for the topology of Y^X.*

PROOF. It suffices to show that, for each f and $N(C,U)$, C compact, U open, $f \in N(C,U)$, there exist compact sets C_i and members U_i of A, $i = 1, \cdots, m$, with

$$f \in \bigcap_{i=1}^{m} N(C_i,U_i) \subset N(C,U)$$

Now since A is a subbasis, we certainly have $U = \bigcup_\alpha V_\alpha$, where each V_α is a finite intersection of elements of A. Now $f(C) \subset U$, so $C \subset f^{-1}(U) = f^{-1}(\bigcup_\alpha V_\alpha) = \bigcup_\alpha f^{-1}(V_\alpha)$. Since C is compact, it follows that

$$C \subset \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$$

with the V_{α_i} picked from among the V_α. From Lemma 1.1.1 it follows that there exist closed subsets C_i of C, $i = 1, \cdots, n$, with

$$C_i \subset f^{-1}(V_{\alpha_i}) \qquad \text{and} \qquad C = \bigcup_{i=1}^{n} C_i$$

Let $V_{\alpha_i} = U_{i,1} \cap \cdots \cap U_{i,k_i}$, $U_{i,j} \in A$. Then

$$f \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{k_i} N(C_i, U_{i,j}) \subset N(C,U)$$

For $f(C_i) \subset V_{\alpha_i} \subset U_{i,j}$, i.e., $f \in N(C_i,U_{i,j})$; and if $g \in N(C_i,U_{i,j})$, then $g(C_i) \subset U_{i,j}$, $g(C_i) \subset \bigcap U_{i,j} = V_{\alpha_i} \subset U$. But the C_i exhaust C; so $g(C) \subset U$, $g \in N(C,U)$. This completes the proof.

Let X,Y,Z be topological spaces. Then, if f takes X into Z^Y, we denote by f^* that function from $X \times Y$ into Z which takes (x,y) into $f(x)(y)$.

THEOREM 1.1.3. *a.* $f^* \in Z^{X \times Y} \Rightarrow f \in (Z^Y)^X$

b. $f \in (Z^Y)^X$, Y locally compact and Hausdorff

$f^* \in Z^{X \times Y}$

PROOF. a. Suppose $f^* \in Z^{X \times Y}$. It suffices to prove that if $x \in X$ and $f(X) \in N(C,U)$, with C compact and U open, then $\exists$ a neighborhood V of X such that $f(V) \subset N(C,U)$. Now if $y \in C$, we have $f^*(x,y) = f(x)(y) \in f(x)(C) \subset U$; hence $\exists$ neighborhoods V_y of x and W_y of y such that $f^*(V_y \times W_y) \subset U$. The sets W_y cover C, and therefore there exist $y_1, \cdots, y_n \in C$ with $C \subset W_{y_1} \cup \cdots \cup W_{y_n}$. Let $V = V_{y_1} \cap \cdots \cap V_{y_n}$. Then if $x' \in V$, $y \in C$, we have, for some i, $y \in W_{y_i}$. Also $x' \in V \subset V_{y_i}$ and therefore

$$f^*(x',y) \in f^*(V_{y_i} \times W_{y_i}) \subset U$$

Thus $f^*(V \times C) \subset U, f(x) \subset N(C,U)$. Q.E.D.

b. Suppose, on the contrary, that $f \in (Z^Y)^X$. Let $x \in X$, $y \in Y$, and let U be a neighborhood of $f^*(x,y)$. Since $f(x) \in Z^Y$ and Y is locally compact Hausdorff, there exists a compact neighborhood W of y such that $f(x)(W) \subset U$; i.e., $f(x) \in N(W,U)$. Since f is continuous and $N(W,U)$ is a neighborhood of $f(x)$, there exists a neighborhood V of x such that $f(V) \subset N(W,U)$. Then $V \times W$ is a neighborhood of (x,y) and $f^*(V \times W) \subset U$. Q.E.D.

THEOREM 1.1.4. *If X and Y are Hausdorff and Y is locally compact, then the correspondence $f \to f^*$ is a homeomorphism of $(Z^Y)^X$ onto $Z^{X \times Y}$.*

PROOF. By Theorem 1.1.3, $f \to f^*$ is a 1:1 correspondence ϕ of $(Z^Y)^X$ onto $Z^{X \times Y}$. Now $f \in N(A,N(B,U))$ if and only if $f^* \in N(A \times B,U)$. Hence ϕ maps subbasic open sets onto open sets and therefore ϕ^{-1} is continuous. Conversely, let $f \in (Z^Y)^X$ and let C be a compact subset of $X \times Y$, U an open subset of Z, such that $f^* \in N(C,U)$. Let A and

B be the projections of C into X and Y, respectively. If $(x,y) \in C$, then there exist neighborhoods $V_{x,y}$ of x relative to A, and $W_{x,y}$ of y relative to B, such that $f^*(V_{x,y} \times W_{x,y}) \subset U$. Now we may assume without loss of generality that $V_{x,y}$ and $W_{x,y}$ are compact; for A and B are compact Hausdorff spaces, and thus are regular. Thus even if $V_{x,y}$ and $W_{x,y}$ are not closed, they contain closed neighborhoods of x and y, respectively; and these will be compact, since closed subsets of compact spaces are compact. Now

$$C \subset \bigcup_{(x,y)\in C} (V_{x,y} \times W_{x,y})$$

Since C is compact, it follows that

$$C \subset \bigcup_{i=1}^{n} [V_{(x_i,y_i)} \times W_{(x_i,y_i)}], \qquad (x_i,y_i) \in C_i, \qquad i = 1, \cdots, n$$

Then if we can show

1. $f \in \bigcap_{i=1}^{n} N(V_{x_i,y_i}, N(W_{x_i,y_i}, U))$

and

2. $g \in \bigcap_{i=1}^{n} N(V_{x_i,y_i}, N(W_{x_i,y_i}, U)) \Rightarrow g^* \in N(C,U)$

then the proof is complete; for then the right side of Part 1 is a neighborhood of f, and by Part 2 is sent into the given neighborhood of $\phi(f)$ by ϕ, and so ϕ is continuous. If the given neighborhood were

$$\bigcap_{i=1}^{m} N(C_i, U_i)$$

then we could have obtained the corresponding neighborhood of f by intersecting the appropriate neighborhoods. Part 1 follows from the fact that

$$f^*(U_{x_i,y_i} \times W_{x_i,y_i}) \subset U, \qquad i = 1, \cdots, n$$

i.e.,

$$f^* \in N(V_{x_i,y_i} \times W_{x_i,y_i}, U), \qquad i = 1, \cdots, n$$

i.e.,

$$f \in N(V_{x_i,y_i}, N(W_{x_i,y_i}, U)) \qquad i = 1, \cdots, n$$

and Part 2 follows from

$$\begin{aligned} & g \in N(V_{x_i,y_i}, N(W_{x_i,y_i}, U)) && i = 1, \cdots, n \\ \Rightarrow\ & g^* \in N(V_{x_i,y_i} \times W_{x_i,y_i}, U) && i = 1, \cdots, n \\ \Rightarrow\ & g^*(V_{x_i,y_i} \times W_{x_i,y_i}) \subset U && i = 1, \cdots, n \\ \Rightarrow\ & g^*(\bigcup_i V_{x_i,y_i} \times W_{x_i,y_i}) \subset U \end{aligned}$$

$\Rightarrow g^*(C) \subset U \Rightarrow g^* \in N(C,U)$. Q.E.D

THEOREM 1.1.5. *$(f,x) \to f(x)$ is a map of $Y^X \times X$ into Y, if X is locally compact Hausdorff.*

PROOF. Let $\phi: Y^X \to Y^X$ be the identity map, which is certainly continuous. Then $\phi^*(f,x) = \phi(f)(x) = f(x)$. Q.E.D.

THEOREM 1.1.6. *Suppose B_i are closed subsets of a topological space X,* $i = 1, \cdots, n$, and that

$$\bigcup_{i=1}^{n} B_i = X$$

Suppose further that we are given n continuous mappings $f_i: B_i \to Y$, with $f_i|B_i \cap B_j = f_j|B_i \cap B_j$ for all i,j. Then if we define $f: X \to Y$ by $f\,|B_i = f_i$, we have that f is continuous.

PROOF. Standard.

THEOREM 1.1.7. *Let X and Y be spaces, Z a locally compact Hausdorff space, and let $\{A_1, \cdots, A_n\}$ be a closed covering of Z. Let $\theta_i: A_i \to X$ be maps. Let*

$$H = \{(f_1, \cdots, f_n) \in Y^X \times \cdots \times Y^X | f_i \circ \theta_i | A_i \cap A_j = f_j \circ \theta_j | A_i \cap A_j \quad i,j = 1, \cdots, n\}$$

Define $\phi: H \to Y^Z$ by

$$\phi(f_1, \cdots, f_n)(z) = f_i(\theta_i(z)), \qquad z \in A_i$$

Then ϕ is continuous.

PROOF. By Theorem 1.1.3, it suffices to prove that $\phi^*: H \times Z \to Y$ is continuous. For this, it suffices by Theorem 1.1.6 to prove that $\phi^*|H \times A_i$ is continuous. But this is the composite of

$$(f_1, \cdots, f_n, z) \to (f_i, z) \to (f_i, \theta_i(z)) \to f_i(\theta_i(z))$$

The first function is continuous because it is a projection; the second because θ_i is; and the third by Theorem 1.1.5.

PROBLEM 1.

Let X, Y, Z be spaces. Then

1. For each $f \in Z^Y$, $g \to f \circ g$ is a map of Y^X and Z^X.
2. For each $f \in Y^X$, $g \to g \circ f$ is a map of Z^Y into Z^X.
3. If Y is locally compact and Hausdorff, then $(f,g) \to g \circ f$ is a map of $Y^X \times Z^Y$ into Z^X.

1.2 Paths and the Fundamental Group

Let X be a space and let $I = \{t \in R | 0 \leq t \leq 1\}$. A *path* in X is a map $f: I \to X$; f is said to *start* at $f(0)$ and to *end* at $f(1)$. A *loop* in X is a path f such that $f(0) = f(1)$; the loop f is said to be *based at* the point $f(0) = f(1)$ of X.

Let f,g be paths in X such that $f(1) = g(0)$. We define a new path $f \cdot g$ by

$$(f \cdot g)(t) = \begin{cases} f(2t) & (0 \leq t \leq \frac{1}{2}) \\ g(2t - 1) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

Clearly $f \cdot g$ is a path starting at $f(0)$ and ending at $g(1)$. We also define a path $\hat{f}$ by

$$\hat{f}(t) = f(1 - t) \qquad (t \in I)$$

$\hat{f}$ is a path from $f(1)$ to $f(0)$.

THEOREM 1.2.1.
1. $(f,g) \to f \cdot g$ *is a map of* $F = \{(f,g) \in X^I \times X^I | f(1) = g(0)\}$ *into* X^I.
2. $f \to \hat{f}$ *is a map of* X^I *into* X^I.

PROOF. The Proof follows from Theorem 1.1.7, with $Y = X$, $X = I$, $Z = I$.

1. Here we set $A_1 = [0,\frac{1}{2}]$, $A_2 = [\frac{1}{2},1]$, $\theta_1(x) = 2x$, $\theta_2(x) = 2x - 1$.
2. Here we set $A = [0,1]$, $\theta(x) = 1 - x$.

For $x \in X$, let e_x be the constant map of I into the point x; e_x is a path from x to x. If f is a path from x to y, then $\hat{f}$ is a path from y to x. If f is a path from x to y and g a path from y to z, then $f \cdot g$ is a path from x to z. Let $C(X,x)$ be the set of points of X which can be joined to x by a path; $C(X,x)$ is called the *path component* of x in X. The path components form a decomposition of X, in virtue of the foregoing remarks. We say that X is *pathwise-connected* or 0-*connected* if and only if $C(X,x) = X$ for some (and therefore for all) $x \in X$. Define $x \equiv y$ if and only if $\exists$ a path in X from x to y.

We readily verify the following Theorem.

THEOREM 1.2.2.
1. $C(X \times Y,(x,y)) = C(X,x) \times C(Y,y)$.
2. *If* $f: X \to Y$ *is a map, then* $f(C(X,x)) \subset C(Y,f(x))$.

Let $F(X,x,y)$ be the space of all paths in X from x to y. If $f,g \in F(X,x,y)$ then $f \equiv g$ if and only if $g \in C(F(X,x,y),f)$. Thus $f \equiv g$ if and only if $\exists$ a map $h: I \times I \to X$ such that

$$\begin{cases} h(s,0) = f(s) \\ h(s,1) = g(s) \\ h(0,t) = x \\ h(1,t) = y \end{cases}$$

We have the following Corollary to the continuity of multiplication and inversion.

COROLLARY 1.2.3.
1. *If* $f \equiv f'$ *and* $g \equiv g'$, *then* $f \cdot g \equiv f' \cdot g'$.
2. *If* $f \equiv f'$ *then* $\bar{f} \equiv \bar{f}'$.

PROOF.
1. We have $f' \in C(F(X,x,y),f)$
$g' \in C(F(X,y,z),g)$
By Theorem 1.2.2 (1), we have

$$(f',g') \in C(F(X,x,y) \times F(X,y,z),(f,g))$$

Suppose we denote by ψ the map in Theorem 1.2.1 (1). Then $\psi' = \psi|F(X,x,y) \times F(X,y,z)$ is certainly continuous, and we have

$$\psi' : F(X,x,y) \times F(X,y,z) \to F(X,x,z)$$

Thus from Theorem 1.2.2 (2) it follows that

$$\begin{aligned} f' \cdot g' &= \psi(f',g') \\ &= \psi'(f',g') \in \psi'(C(F(X,x,y) \times F(X,y,z),(f,g))) \subset C(F(X,x,z) \end{aligned}$$

$\psi'(f,g)) = C(F(X,x,z),f \cdot g)$. Q.E.D.

2. This follows in a similar manner from Theorem 1.2.1 (2) and Theorem 1.2.2 (2). Theorem 1.2.2 (1) is not used, although Theorem 1.2.2 (2) was needed in the proof of Theorem 1.2.3 (1).

Let $\pi_1(X,x,y)$ be the set of path components of $F(X,x,y)$. We shall abbreviate $F(X,x,x)$ and $\pi_1(X,x,x)$ to $F(X,x)$ and $\pi_1(X,x)$. By the Corollary, we see that the operations on paths induce operations on the equivalence classes: If $\alpha \in \pi_1(X,x,y)$, $\beta \in \pi_1(X,y,z)$, then we may define without ambiguity

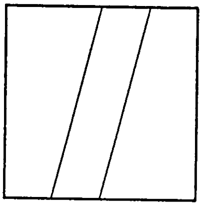
$$\alpha \cdot \beta = C(F(X,x,z),f \cdot g)$$

$$\alpha^{-1} = C(F(X,y,x),\bar{f})$$

for any $f \in \alpha$, $g \in \beta$.

To prove that the operations above have reasonable properties, we prove the following Theorem.

THEOREM 1.2.4.



1. $(f,g,h) \to (f\cdot g)\cdot h$ *and* $(f,g,h) \to f\cdot(g\cdot h)$ *are homotopic maps of a subset* G *of* $X^I \times X^I \times X^I$ *into* X^I.
2. $f \to f\cdot e_{f(1)}$ *is homotopic to the identity map of* X^I.
3. $f \to f\cdot f$ *and* $f \to e_{f(0)}$ *are homotopic maps of* X^I *into* X^I.

PROOF.

1. Let $G = \{(f,g,h)|f(1) = g(0)$ and $g(1) = h(0)\}$. Define a map $\phi: G \times I \to X^I$ by

$$\phi(f,g,h,t|(s) = \begin{cases} f\left(\dfrac{4s}{1+t}\right) & 0 \leq s \leq \dfrac{1+t}{4} \\ g(4s - t - 1) & \dfrac{1+t}{4} \leq s \leq \dfrac{2+t}{4} \\ h\left(\dfrac{4s - t - 2}{2 - t}\right) & \dfrac{2+t}{4} \leq s \leq 1 \end{cases}$$

We verify easily that $\phi(f,g,h,0) = (f\cdot g)\cdot h$ and $\phi(f,g,h,1) = f\cdot(g\cdot h)$.

To prove continuity let $\phi_1: G \to (X^I)^I$ be the function defined by ϕ (i.e., $\phi = \phi_1^*$), and let σ be the natural homeomorphism of $(X^I)^I$ onto $X^{I\times I}$. Then it is sufficient to prove $\sigma \circ \phi_1 \cdot G \to X^{I\times I}$ is a mapping.

We now apply Theorem 1.1.7, with $Y = X$, $Z = I \times I$, $Y = I$,

$$A_1 = \left\{(s,t) \in I \times I | 0 \leq s \leq \frac{1+t}{4}\right\}$$

$$A_2 = \left\{(s,t) \in I \times I \,\middle|\, \frac{1+t}{4} \leq s \leq \frac{2+t}{4}\right\}$$

$$A_3 = \left\{(s,t) \in I \times I \,\middle|\, \frac{2+t}{4} \leq s \leq 1\right\}$$

$$\theta_1 = \frac{4s}{1+t}$$

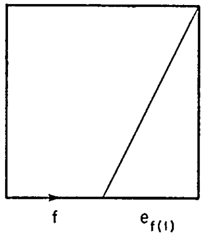
$$\theta_2 = 4s - t - 1$$

$$\theta_3 = \frac{4s - t - 2}{2 - t}$$

Since it is easily verified that the conditions of Theorem 1.1.7 are satisfied, it follows that $\sigma \circ \phi_1$ is continuous. Thence, since σ is a homeomorphism, it follows that ϕ_1 is continuous, and therefore ϕ. Q.E.D.

2. Define $\psi: X^I \times I \to X^I$ by

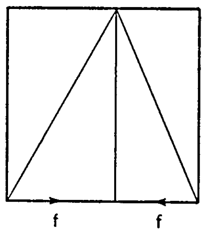
$$\psi^*(f,t)(s) = \begin{cases} f\left(\dfrac{2s}{1+t}\right) & 0 \leq s \leq \dfrac{1+t}{2} \\ f(1) & \dfrac{1+t}{2} \leq s \leq 1 \end{cases}$$



Clearly $\psi(f,0) = f \cdot e_{f(1)}$, $\psi(f,1) = f$. Continuity follows from Theorem 1.1.7 as previously shown.

3. Define $\chi: X^I \times I \to X^I$ by

$$\chi^*(f,t)(s) = \begin{cases} f(0) & 0 \leq s \leq \dfrac{t}{2} \\ f(2s - t) & \dfrac{t}{2} \leq s \leq \dfrac{1}{2} \\ f(2 - 2s - t) & \dfrac{1}{2} \leq s \leq \dfrac{2-t}{2} \\ f(0) & \dfrac{2-t}{2} \leq s \leq 1 \end{cases}$$



Clearly $\chi(f,0) = f \cdot \hat{f}$, $\chi(f,1) = e_{f(0)}$. Continuity again follows from Theorem 1.1.7.

COROLLARY 1.2.5. *The map $f \to e_{f(0)} \cdot f$ is homotopic to the identity. The maps $f \to \hat{f} \cdot f$ and $f \to e_{f(1)}$ are homotopic.*

PROOF. $e_{f(0)} \cdot f = \widehat{\hat{f} \cdot e_{f(0)}}$ and $\hat{f} \cdot f = \hat{f} \cdot \hat{\hat{f}}$.

COROLLARY 1.2.6. *The operations of multiplication and inversion of homotopy classes have the following properties.*

1. *Each $\alpha \in \pi_1(X,x,y)$ has a left identity ϵ_x and a right identity ϵ_y.*
2. *Each $\alpha \in \pi_1(X,x,y)$ has an inverse α^{-1} with $\alpha\alpha^{-1} = \epsilon_x$, $\alpha^{-1} \cdot \alpha = \epsilon_y$.*
3. *If $\alpha \in \pi_1(X,x,y)$, $\beta \in \pi_1(X,y,z)$, $\gamma \in \pi_1(X,z,w)$, then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.*

COROLLARY 1.2.7. *$\pi_1(X,x)$ is a group.*

COROLLARY 1.2.8. *If x and y can be joined by a path in X, then $\pi_1(X,x) \approx \pi_1(X,y)$.*

For let $\alpha \in \pi_1(X,x,y)$. For $\beta \in \pi_1(X,x)$ let $\phi(\beta) = \alpha^{-1}\beta\alpha$; for $\gamma \in \pi_1(X,y)$ let $\psi(\gamma) = \alpha\gamma\alpha^{-1}$. Then ϕ and ψ are homomorphisms inverse to each other.

1.3 Grouplike Spaces

Let X be a space, $e \in X$. We say that (X,e) is an H-space if and only if $\exists$ a map $\mu:(X \times X,(e,e)) \to (X,e)$ such that the maps $x \to \mu(x,e)$ and $x \to \mu(e,x)$ are homotopic relative e to the identity map $(X,e) \to (X,e)$. X is an IH-space if and only if X is an H-space and $\exists$ a map $\nu:(X,e) \to (X,e)$ such that the maps $x \to \mu(x,\nu(x))$ and $x \to \mu(\nu(x),x)$ are homotopic rel e to the constant map of X into e. X is an AIH-space if and only if X is an IH-space and the maps $(x,y,z) \to \mu(x,\mu(y,z))$ and $(x,y,z) \to \mu(\mu(x,y),z)$ are homotopic rel (e,e,e). (Note: We shall usually abbreviate $\mu(x,y)$ to $x \cdot y$, $\nu(x)$ to x^{-1}.)

REMARK 1.3.0. *A topological group is an AIH-space. If X is any space, $x \in X$, then the space $F'(X,x)$ is an AIH-space under $\mu(f \cdot g) = f \cdot g$, $\nu(f) = f$, in the sense of Section 2.*

If X is an H-space, we may use the multiplication in X to define a new multiplication of paths. Let f,g be paths in X, and let $f \# g$ be the path given by

$$(f \# g)(t) = f(t) \cdot g(t)$$

LEMMA 1.3.1. *Let X be an H-space. Then $(f,g) \to f \# g$ is a map of $X^I \times X^I$ into X^I.*

PROOF. It suffices by Theorem 1.1.3 to show that $(f,g,t) \to (f \# g)(t) = f(t) \cdot g(t)$ is continuous. This can be decomposed as follows:

$$(f,g,t) \to (f(t),g(t)) \to f(t) \cdot g(t)$$

The first is continuous because a map into a product space is continuous if and only if each of the projections is continuous; i.e., we must prove

a. $(f,g,t) \to f(t)$ is continuous

b. $(f,g,t) \to g(t)$ is continuous

The function in Part a is the composition of $(f,g,t) \to (f,t) \to f(t)$. The first is continuous because it is a projection; the second by Theorem 1.1.5. Similarly Part b is true. As for the mapping $(f(t),g(t)) \to f(t) \cdot g(t)$, this is continuous by the definition of H-space.

LEMMA 1.3.2. *Suppose X is an H-space, $f \in X^Y$; define map f',f'' by*

$$f'(t) = f(t) \cdot e$$
$$f''(t) = e \cdot f(t)$$

Then $f \to f'$ and $f \to f''$ are maps of X^I into X^I which are homotopic to the identity.

PROOF. Let $\phi: X \times I \to X$ be a map such that

$$\phi(x,0) = x$$
$$\phi(x,1) = x \cdot e$$
$$\phi(e,t) = e$$

To prove $f \to f'$ homotopic to the identity define $\Phi \cdot X^I \times I \to X^I$ by

$$\Phi(f,t)(s) = \phi(f(s),t)$$

Then $\Phi(f,0) = f, \phi(f,1) = f'$. To show Φ continuous it suffices to show $\Phi^*: X^I \times I \times I \to X$ is continuous. This can be broken up into

$$(f,t,s) \to (f(s),t) \to \phi(f(s),t)$$

and both steps are continuous. The proof for $f \to f''$ is similar.

THEOREM 1.3.3. *Let X be an H-space, $F = F(X,e)$. Then the maps*

$$(f \cdot g) \to f \cdot g$$
$$(f,g) \to g \cdot f$$
$$(f,g) \to f \# g$$

are homotopic maps of $F \times F$ into F.

PROOF. Define $\phi: F \times F \times I \to F$ by

$$\phi(f,g,t)(s) = \begin{cases} f(2s(1-t)) \cdot g(2st) & 0 \leq s \leq \frac{1}{2} \\ f(1-2t(1-s)) \cdot g(2(s+t-st)-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

There are several verifications which have to be made.

1. Consistency: if $s = \frac{1}{2}$, the first line gives $f(1-t)g(t)$ and the second gives $f(1-t)g(t)$.
2. $\phi(f,g,t) \in F$: $s = 0$ gives $f(0)g(0) = e \cdot e = e$; $s = 1$ gives $f(1)g(1) = e \cdot e = e$.
3. Continuity: routine (by now).

Now note that

$$\phi(f,g,0)(s) = \begin{cases} f(2s) \cdot g(0) = f(2s) \cdot e = f'(2s) & 0 \leq s \leq \frac{1}{2} \\ f(1)g(2s-1) = e \cdot g(2s-1) = g''(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

so that $\phi(f,g,0) = f' \cdot g''$. Also

$$\phi(f,g,\tfrac{1}{2})(s) = f(s) \cdot g(s)$$

so that $\phi(f,g,\frac{1}{2}) = f \# g$. Finally

$$\phi(f,g,1)(s) = \begin{cases} f(0)g(2s) = e \cdot g(2s) = g''(2s) & (0 \leq s \leq \frac{1}{2}) \\ f(2s-1)g(1) = f(2s-1) \cdot e = f'(2s-1) & (\frac{1}{2} \leq s \leq 1) \end{cases}$$

so that $\phi(f,g,1) = g''\cdot f'$. Let $\Phi: X^I \times I \to X^I$, $\tilde{\Phi}: X^I \times I \to X^I$ be maps such that

$$\Phi(f,0) = f \qquad \Phi(f,1) = f'$$
$$\tilde{\Phi}(f,0) = f \qquad \tilde{\Phi}(f,1) = f''$$

Then the mapping

$$\psi: F \times F \times I \to F$$

given by

$$\psi(f,g,t) = \Phi(f,t)\cdot\tilde{\Phi}(g,t)$$

provides a homotopy from $(f,g) \to f\cdot g$ to $(f,g) \to f'\cdot g''$. Similarly $(f,g) \to g\cdot f$ and $(f,g) \to g''\cdot f'$ are homotopic. But we have shown that $(f,g) \to f'\cdot g''$ and $(f,g) \to g''\cdot f'$ are both homotopic to $(f,g) \to f \# g$. From this and the transitivity of homotopy the result follows.

COROLLARY 1.3.4. *Let X be an H-space. Then if $f \in \alpha \in \pi_1(X,e)$ and $g \in \beta \in \pi_1(X,e)$, then $f \# g \in \alpha\cdot\beta$.*

COROLLARY 1.3.5. *Let X be an H-space. Then $\pi_1(X,e)$ is abelian.*

THEOREM 1.3.6. *Let X be an AIH-space, $F = F(X,e)$. Denote by $\tilde{f}$ the member of F given by $\tilde{f}(t) = (f(t))^{-1}$. Then F is an AIH-space under $\mu(f,g) = f \# g$, $\nu(f) = \tilde{f}$, $e = e_e'$, where e_e' = the constant map of I into e.*

PROOF. 1. $\mu: F \times F \to F$ is continuous.
This follows from Lemma 1.3.1.

2. $\mu(e_e',e_e') = e_e'$.
For $(e_e' \# e_e')(t) = e_e'(t)\cdot e_e'(t) = e\cdot e = e = e_e'(t)$. Thus $e_e' \# e_e' = e_e'$.
3. The maps $f \to \mu(e_e',f)$ and $f \to \mu(f,e_e')$ are homotopic relative e_e' to the identity map of $F \to F$.
For let $\phi: X \times I \to X$ with

$$\phi(x,0) = e\cdot x$$
$$\phi(x,1) = x$$
$$\phi(e,s) = e$$

The existence of ϕ is assured by the fact that X is an H-space. Now define

$$\Phi: F \times I \to F$$

by

$$\Phi(f,s)(t) = \phi(f(t),s)$$

Φ is certainly a mapping, and we have

$$
\begin{aligned}
\Phi(f,0)(t) &= \phi(f(t),0) \\
&= e\cdot f(t) \\
&= e'_e(t)\cdot f(t) \\
&= (e'_e \# f)(t) \\
\Phi(f,1)(t) &= \phi(f(t),1) \\
&= f(t) \\
\Phi(e'_e,s)(t) &= \phi(e'_e(t),s) \\
&= \phi(e,s) \\
&= e \\
&= e'_e(t)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Phi(f,0) &= e'_e \# f \\
\Phi(f,1) &= f \\
\Phi(e'_e,s) &= e'_e
\end{aligned}
$$

similarly for $f \to \mu(f,e'_e)$.

4. $\nu: F \to F$ is continuous.

The proof is similar to that of Lemma 1.3.1.

The rest of the conditions may be proved in a manner similar to that of Part 3.

DEFINITION. *Let (X,e_X), (Y,e_Y) be H-spaces, $f:(X,e_X) \to (Y,e_Y)$. Then f is said to be an H-space homomorphism if and only if $\mu(f(X_1),f(X_2)) = f(\mu(X_1,X_2))$.*

THEOREM 1.3.7. *Let (X,e_X) be an AIH-space, $\pi_0(X,e_X)$ the set of its path components, $C(x)$ the path component of x in X. Then $\pi_0(X)$ is a group under the operation $C(x) + C(y) = C(\mu(x,y))$, where $C(x)$ is the path component of x.*

PROOF. Evident. The identity is $C(e_X)$.

THEOREM 1.3.8. *Let (X,e_X), (Y,e_Y) be AIH-spaces, $f:(X,e_X) \to (Y,e_Y)$ an H-space homomorphism. For $x \in X$, define $f_*(C(x)) = C(f(x))$. Then $f_*: \pi_0(X,e_X) \to \pi_0(Y,e_Y)$ is a homomorphism.*

PROOF. We first show that f_* is well defined. Let $x' \in C(x)$. Then clearly $f(x') \in C(f(x))$, since f is continuous. So f_* is indeed well defined. To show it is a homomorphism it is only necessary to show

a. $f_*(C(e_X)) = C(e_Y)$

b. $f_*(C(x) + C(x')) = f_*(C(x)) + f_*(C(x'))$

Part a follows from the hypothesis, while Part b may be shown as follows:

$$f_*(C(x) + C(x')) = f_*(C\mu(x,x')) = C(f(\mu(x,x'))) = C(\mu(f(x),f(x')))$$
$$= C(f(x)) + C(f(x')) = f_*(C(x)) + f_*(C(x'))$$

THEOREM 1.3.9. *Let (X,e_X), (Y,e_Y) be AIH-spaces, $f{:}(X,e_X) \to (Y,e_Y)$. Then according as f is a homeomorphism onto or onto, $f_*{:}\pi_0(X,e_X) \to \pi_0(Y,e_Y)$ is an isomorphism onto or onto.*

PROOF. If f is a homeomorphism onto it cannot take two path components into one. So f_* must have kernel zero. On the other hand, let f be onto. Then every path component of Y must be represented in $f(X)$, so that f_* is onto. This completes the proof. Note that in order for f_* to be an isomorphism, it is not sufficient that f be 1:1 onto or even a homeomorphism into. Counterexamples are easy to construct.

1.4 Homotopy Groups [14]

Denote by I^n, as usual, the set of real n-tuples all of whose coordinates are in the interval $0 \leq t \leq 1$ for $n > 0$, and $I^0 = \{0\}$. Define

$$\dot{I}^n = \{(t_1,\cdots,t_n) \in I^n | \prod_{j=1}^{n} (t_j(1 - t_j)) = 0\}$$

(i.e., we simply demand that at least one of the coordinates be either one or zero for $n > 0$) and $\dot{I}^0 = \phi$, the null set. The boundary of I^n in Euclidean n-space is $\dot{I}^n$. Clearly $I^n = I^{n-1} \times I$. Furthermore, we have the following Remark.

REMARK 1.4.1. $\dot{I}^n = \dot{I}^{n-1} \times I \cup I^{n-1} \times \dot{I}$.

PROOF. Obvious.

DEFINITION. *Let X be a space, $x \in X$. Then*

1. $F^n(X,x) = \{f \in X^{I^n} | f(\dot{I}^n) = \{x\}\}, \qquad n > 0$
   $F^0(X,x) = X$
2. $e_x^n =$ *the constant map of I^n into x,* $\qquad n > 0$
   $e_x^0 = x$
3. $\pi_n(X,x) =$ *the set of path components of $F^n(X,x)$,* $\qquad n \geq 0.$

LEMMA 1.4.2. *If $n \geq 1$, then $F^n(X,x)$ is homeomorphic with*

$$F^1(F^{n-1}(X,x),e_x^{n-1})$$

under a mapping which takes e_x^n onto $e^1_{e_x^{n-1}}$.

PROOF. This is a consequence of the homeomorphism between $X^{I\times I^{n-1}}$ and $(X^{I^{n-1}})^I$. For, if

$$\phi^*: I \times I^{n-1} \to X, \qquad \phi^*(\dot{I}^n) = (x)$$

then from Lemma 1.4.1, we have $\phi^*(\dot{I} \times I^{n-1}) = (x)$, that is,

$$\phi(\dot{I})(I^{n-1}) = (x)$$

that is,

$$\phi(\dot{I}) = e_x^{n-1} \tag{1}$$

On the other hand, again from Lemma 1.4.1, we have

$$\phi^*(I \times \dot{I}^{n-1}) = (x)$$

that is

$$\phi(I)(\dot{I}^{n-1}) = (x)$$

that is

$$\phi(I) \in F^{n-1}(X,x) \tag{2}$$

From Expressions 1 and 2, we obtain that the image of $F^n(X,x)$ is in $F^1(F^{n-1}(X,x),e_x^{n-1})$. Similarly we may prove that the preimage of $F^1(F^{n-1}(X,x),e_x^{n-1})$ is in $F^n(X,x)$. It remains only to prove that e_x^n goes into $e^1_{e_x^{n-1}}$. But this is easy to verify. This completes the proof.

COROLLARY 1.4.3. *For $n \geq 1$, $F^n(X,x)$ is an AIH-space with $\mu(f,g) = f + g$ and $\nu(f) = -f$ given by*

$$(f+g)(t_1,\cdots,t_n) = \begin{cases} f(t_1,\cdots,t_{n-1},2t_n) & t_n \leq \frac{1}{2} \\ g(t_1,\cdots,t_{n-1},2t_n - 1) & t_n \geq \frac{1}{2} \end{cases}$$

$$(-f)(t_1,\cdots,t_n) = f(t_1,\cdots,t_{n-1},1 - t_n)$$

PROOF. By induction on n. For $n = 1$ it is a consequence of Remark 1.3.0. (In fact, it *is* Remark 1.3.0.) For $n > 1$ it follows immediately from the induction hypothesis, Theorem 1.3.6 and Lemma 1.4.2.

COROLLARY 1.4.4. *For $n \geq 1$, the homeomorphism of Lemma 1.4.2 induces a 1:1 correspondence between $\pi_n(X,x)$ and $\pi_1(F^{n-1}(X,x),e_x^{n-1})$.*

DEFINITION. *We make $\pi_n(X,x)$ into a group by demanding that the correspondence of Corollary 1.4.4 be an isomorphism. $\pi_n(X,x)$ is called the nth homotopy group of (X,x).*

THEOREM 1.4.5. *Let*

$$f \in \alpha \in \pi_n(X,x)$$

$$g \in \beta \in \pi_n(X,x)$$

Then $f + g \in \alpha + \beta$. That is, $\pi_n(X,x) = \pi_0(F^n(X,x),e_x^n)$, as groups.

PROOF. Let $f^* \in F^1(F^{n-1}(X,x),e_x^{n-1})$ correspond to f under the homeomorphism of Lemma 1.4.2 and $\alpha^* \in \pi_1(F^{n-1}(X,x),e_x^{n-1})$ to α under the correspondence of Corollary 1.4.4. Then $f^* \in \alpha^*$, $g^* \in \beta^*$. By Corollary 1.3.4, we have $f^* \# g^* \in \alpha^* + \beta^*$. So by the Definition of $\pi_n(X,x)$, we have $f^* \# g^* \in (\alpha + \beta)^*$. Now

$$(f^* \# g^*)(t_1)(t_2,\cdots,t_n) = (f^*(t_1) + g^*(t_1))(t_2,\cdots,t_n)$$

(here the $+$ is in the *AIH*-space $F^{n-1}(X,x)$)

$$= \begin{cases} f^*(t_1)(t_2,\cdots,2t_n) & t_n \leq \frac{1}{2} \\ g^*(t_1)(t_2,\cdots,2t_n - 1) & t \geq \frac{1}{2} \end{cases}$$

$$= \begin{cases} f(t_1,\cdots,2t_n) & t_n \leq \frac{1}{2} \\ g(t_1,\cdots,2t_n - 1) & t_n \geq \frac{1}{2} \end{cases}$$

$$= (f + g)(t_1,\cdots,t_n)$$

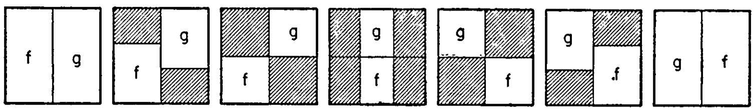
(here the $+$ is in the *AIH*-space $F^n(X,x)$)

$$= (f + g)^*(t_1)(t_2,\cdots,t_n)$$

It follows that $f^* \# g^* = (f + g)^*$. Therefore $(f + g)^* \in (\alpha + \beta)^*$. Since $*$ is 1:1, the result follows.

THEOREM 1.4.6. *$\pi_n(X,x)$ is abelian for $n \geq 2$.*

PROOF. Follows from the definition and Theorem 1.3.5. A more direct proof that $\pi_n(X,x)$ is abelian is suggested by the picture:



It is left as an exercise to the reader to write down formulas for the homotopies suggested by the picture.

THEOREM 1.4.7. *$F^{n+k}(X,x)$ is homeomorphic with $F^k(F^n(X,x),e_x^n)$ under a homeomorphism which sends e_x^{n+k} into $e_{e_x^n}^{e^k}$.*

PROOF. For $k = 1$ this is Lemma 1.4.2. The theorem now follows by induction from the fact that the spaces

$$F^k(F^n(X,x),e_x^n)$$
$$F^1(F^{k-1}(F^n(X,x),e_x^n),e_{e_x^n}^{k-1})$$
$$F^1(F^{n+k-1}(X,x),e_x^{n+k-1})$$
$$F^{n+k}(X,x)$$

are homeomorphic, the base points behaving correctly.

LEMMA 1.4.8. *Let $f:(X,x) \to (Y,y)$. Then*

$$\bar{f}:(F^n(X,x),e_x^n) \to (F^n(Y,y),e_y^n)$$

is an H-space homomorphism, where

$$\bar{f}(g)(t_1,\cdots,t_n) = f(g(t_1,\cdots,t_n))$$

PROOF. Continuity follows from Problem 1(1). The homomorphic property is immediate.

COROLLARY 1.4.9. *$\bar{f}_*:\pi_n(X,x) \to \pi_n(Y,y)$ is a homomorphism.*

PROOF. Theorems 1.3.8 and 1.4.5. In the future we shorten $\bar{f}_*$ to f_*.

DEFINITION. *$f \equiv f'$ means f homotopic to f'.*

LEMMA 1.4.10. *Let X,Y be spaces, $f,f':X \to Y, f \equiv f'$. Then $C(f(x)) = C(f'(x))$.*

PROOF. Obvious.

THEOREM 1.4.11. *Let X,Y be spaces, $f,f':(X,x) \to (Y,y), f \equiv f'$ relative x. Then $f_* = f'_*$.*

PROOF. Follows from Problem 1 and Lemma 1.4.8.

THEOREM 1.4.12. *Let X,Y,Z be spaces,*

$$f:(X,x) \to (Y,y), \qquad h:(Y,y) \to (Z,z)$$

Then $(h \circ f)_* = h_* \circ f_*$.

PROOF.
$$\begin{aligned}(h \circ f)_*(C(g)) &= C(h \circ f \circ g)\\ &= h_*C(f \circ g)\\ &= h_* \circ f_*(C(g))\end{aligned}$$

THEOREM 1.4.13. *Let f be the identity map of a space onto itself. Then f_* is the identity.*

PROOF. Utterly trivial. Notice that Theorems 1.4.n, $n = 11,12,13$, show that $\pi_n(X,x)$ satisfy the first two Eilenberg-Steenrod axioms for homology theory, and the fifth.

1.5 The Operations of π_1 on π_n [2, 21]

DEFINITION. $G^n(X) = \bigcup_{x \in X} F^n(X,x)$

$G^n(X)$ *is a subspace* of X^{I^n}

DEFINITION. *The function* π *on* $G^n(X)$ *into* X *is that function for which* $\pi(f) = f(0,\cdots,0)$.

THEOREM 1.5.1. π *is continuous.*

PROOF. $\pi = (g \circ h)|G^n(X)$, where

$$h: X^{I^n} \to X^{I^n} \times I^n$$

$$g: X^{I^n} \times I^n \to X$$

are given by $h(f) = (f,(0,\cdots,0))$
and

$$g(f,t) = f(t)$$

Obviously h is continuous and g is by Theorem 1.1.5.

DEFINITION. *Let* $f,g \in G^n(X)$, $p \in F^1(X,a,b)$. *Then* $f \underset{p}{\equiv} g$ *(*f *is freely homotopic to* g *via* p*) if and only if there exists a path* q *in* $G^n(X)$ *from* f *to* g *such that* $\pi \circ q = p$. *Alternatively: There exists a map* $q^*: I^{n+1} \to X$ *such that*

$$q^*(0,t) = f(t),\ q^*(1,t) = g(t), \qquad t \in I^n$$

$$q^*(s,t) = p(s), \qquad t \in \dot{I}^n$$

DEFINITION. *Let* $t \in I^n$. *Then* $|t| = \max\limits_{i=1,\ldots,n} |2t_i - 1|$.

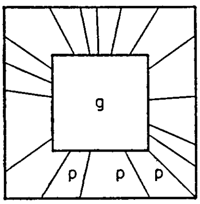
LEMMA 1.5.2. $|t| = 0$ *if and only if* $t = (\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2})$

$|t| = 1$ *if and only if* $t \in \dot{I}^n$

DEFINITION. *Let* $g \in F^n(X,b)$

$p \in F^1(X,a,b)$

Then $$t_p(g)(t_1,\cdots,t_n) = \begin{cases} g(2t_1 - \frac{1}{2},\cdots,2t_n - \frac{1}{2}) & |t| \leq \frac{1}{2} \\ p(2 - 2|t|) & |t| \geq \frac{1}{2} \end{cases}$$



(An idea of what the map looks like is suggested by the accompanying picture.)

LEMMA 1.5.3. $(g,p) \to t_p(g)$ *is continuous from*

$$\bigcup_{a,b \in X} (F^n(X,b) \times F^1(X,a,b)) \text{ to } G^n(X)$$

PROOF. Follows from Theorem 1.1.7. Here $Y = X$, $X = Z = I^n$. $A_1 = \{t \in I^n \big| |t| \leq \frac{1}{2}\}$, $\{A_2 = t \in I^n \big| |t| \geq \frac{1}{2}\}$

$$\theta_1(t_1,\cdots,t_n) = (2t_1 - 1,\cdots,2t_n - 1)$$

$$\theta_2(t_1,\cdots,t_n) = (t_1,\cdots,t_n)$$

It follows that $(g,\tilde{p}) \rightarrow t_p(g)$ is continuous, where $\tilde{p}(t_1,\cdots,t_n) = p(2 - 2|t|)$. It remains to prove that $p \rightarrow \tilde{p}$ is continuous from X^I to X^{I^n}. This follows from Problem 1, Part 2, once it is shown that $(t_1,\cdots,t_n) \rightarrow 2 - 2|t|$ is continuous, which follows from the continuity of $(t_1,\cdots,t_n) \rightarrow |t|$. Q.E.D.

THEOREM 1.5.4. *If* $g \equiv g'$ in $F^n(X,b)$ *and* $p \equiv p'$ in $F^1(X,a,b)$, then $t_p(g) \equiv t_{p'}(g')$ *in* $F^n(X,a)$.

PROOF. Trivial; merely rewrite the homotopies.

LEMMA 1.5.5. $f \underset{p}{\equiv} g$ *and* $g \underset{q}{\equiv} h \Rightarrow f \underset{pq}{\equiv} h$

LEMMA 1.5.6. $f \underset{p}{\equiv} g \Rightarrow g \underset{\tilde{p}}{\equiv} f$

LEMMA 1.5.7. $f \underset{p}{\equiv} f'$, $p \equiv p'$ *in* $F^1(X,f(\dot{I}^n),f'(\dot{I}^n))$

$$\Rightarrow f \underset{p'}{\equiv} f'$$

PROOF. Define a mapping

$$h_r : I^n \rightarrow I^n \qquad r \geq \tfrac{1}{2}$$

by $h_r(t_1,\cdots,t_n) = \begin{cases} (\bar{h}(t_1),\cdots,\bar{h}(t_n)), & |t| \leq r \\ (\bar{\bar{h}}(t_1),\cdots,\bar{\bar{h}}(t_n)), & |t| \geq r \end{cases}$

where

$$\bar{h}(t_i) = \frac{2(t_i + r) - 1}{4r}$$

$$\bar{\bar{h}}(t_i) = \frac{t_i}{2(1 - r)} \qquad t_i \leq \frac{1 - r}{2}$$

$$\bar{\bar{h}}(t_i) = \frac{t_i + 1 - 2r}{2(1 - r)} \qquad t_i \geq \frac{1 + r}{2}$$

Obviously h_r is continuous for each r in the range indicated; $h_{1/2}$ is the identity.

Define $t_p^r(g) = t_p(g) \circ h_r$. Then $t_p^{1/2}(g) = t_p(g)$, $t_p^1(g) = g$, and it may easily be shown that the mapping $(r,g,p) \rightarrow t_p^r(g)$ is continuous. Let q

be a path from f to f' such that $\pi \circ q = p$, and let k be a path from p to p' in $F^1(X,f(\dot{I}^n),f'(\dot{I}^n))$, $k_s(t) = k(t)(s)$. Now let q' be the path from f to f' given by $q'(s) = t_{k_s}^{|s-\frac{1}{2}|+\frac{1}{2}}(q(s))$. It is easy to show that q' is indeed continuous and satisfies the initial conditions, and thus the result is proved.

LEMMA 1.5.8. $f \underset{p}{\equiv} g, f' \underset{p}{\equiv} g \Rightarrow f \equiv f'$ *in* $F^n(X,f(0,\cdots,0))$.

PROOF. By Lemma 1.5.6, $g \underset{\hat{p}}{\equiv} f'$. By Lemma 1.5.5, $f \underset{p\hat{p}}{\equiv} f'$. But $p\hat{p} \equiv e_{p(0)}$. [For suppose that we consider the composition G of the string of mappings

$$I \times I \rightarrow (X^I \times I) \times I \rightarrow X^I \times I \rightarrow X$$

where the first is the mapping $h(s,t) = ((p,s),t)$, the second is induced by the mapping of Theorem 1.2.5 (3), and the third is that of Theorem 1.1.5. Then G provides a homotopy from $p\hat{p}$ to $e_{p(0)}$.] The result now follows from Lemma 1.5.7 and the trivial fact that $f \equiv f'$ and $f_{e_{f(0,\cdots,0)}} \equiv f'$ are the same statement.

LEMMA 1.5.9. $t_p(g) \underset{p}{\equiv} g$.

PROOF. Let $p_s(t) = p(s + t - st)$. Then the path $q(s) = t_{p_s}^{(s+1)/2}(g)$ defines the desired homotopy.

COROLLARY 1.5.10. $f \underset{p}{\equiv} g$ *if and only if* $f \equiv t_p(g)$ *in* $F^n(X,f(\dot{I}^n))$.

LEMMA 1.5.11. $f \underset{p}{\equiv} g, f' \underset{p}{\equiv} g' \Rightarrow f + f' \underset{p}{\equiv} g + g'$.

PROOF. Let ϕ and ϕ' be paths in $G^n(X)$ between f and g and f' and g', respectively, $\pi \circ \phi = \pi \circ \phi' = p$. Then define a path ψ between $f + f'$ and $g + g'$ by $\psi(t) = \phi(t) + \phi'(t)$.

COROLLARY 1.5.12. $t_p(g + g') \equiv t_p(g) + t_p(g')$ *in* $F^n(X,p(0))$.

PROOF. $t_p(g + g') \underset{p}{\equiv} g + g'$. But $t_p(g) \underset{p}{\equiv} g$, $t_p(g') \underset{p}{\equiv} g'$. So $t_p(g) + t_p(g') \underset{p}{\equiv} g + g'$. So $t_p(g + g') \underset{p}{\equiv} t_p(g) + t_p(g')$.

LEMMA 1.5.13. $t_{e_b}(g) \equiv g$ *in* $F^n(X,b)$.

LEMMA 1.5.14. $t_{pq}(h) \equiv t_p(t_q(h))$.

PROOF.

$$t_{pq}(h) \underset{pq}{\equiv} h \tag{2}$$

$$t_q(h) \underset{q}{\equiv} h \tag{3}$$

$$t_p(t_q(h)) \underset{p}{\equiv} t_q(h) \tag{4}$$

from Formulas 3 and 4 and Lemma 1.5.5, we have

$$t_p(t_q(h)) \underset{pq}{\equiv} h \tag{5}$$

From Formulas 2 and 5 and Lemma 1.5.7, the result follows.

Let $g \in \alpha \in \pi_n(X,b)$, $p \in \xi \in \pi_1(X,a,b)$. Then $t_p(g)$ is in $F^n(X,a)$; but by Lemma 1.5.4, its path component in $F^n(X,a)$ depends only on the component of g in $F^n(X,b)$, and that of p in $F^1(X,a,b)$, i.e., on α and ξ. So we may state the following Definition.

DEFINITION. *Let $p \in \xi \in \pi_1(X,a,b)$, $g \in \alpha \in \pi_n(X,b)$. Then $\theta_\xi(\alpha)$ is the component of $t_p(g)$ in $\pi_n(X,a)$.*

LEMMA 1.5.15. $\theta_\xi(\alpha + \beta) = \theta_\xi(\alpha) + \theta_\xi(\beta)$

$$\theta_{\epsilon_b}(\alpha) = \alpha \qquad \epsilon_b = C(e_b)$$

$$\theta_{\xi\eta}(\alpha) = \theta_\xi(\theta_\eta(\alpha))$$

PROOF. Lemmas 1.5.n, $n = 12, 13, 14$.

COROLLARY 1.5.16. *θ_ξ is a homomorphism from $\pi_n(X,b)$ to $\pi_n(X,a)$.*

DEFINITION. *Let X be a 0-connected space. A bundle* **G** *of groups in X consists of the following:*

1. *A function which assigns to each $x \in X$ a group G_x.*
2. *A function which assigns to each $\xi \in \pi_1(X,x,y)$ a homomorphism $\gamma_\xi: G_y \to G_x$, satisfying the following requirements.*
3. *If $\xi \in \pi_1(X,x,y)$, $\eta \in \pi_1(X,y,z)$, then $\gamma_{\xi\eta} = \gamma_\xi \circ \gamma_\eta$.*
4. *If $x \in X$, then $\gamma_{e_x} =$ identity.*

It follows that each γ_ξ is an isomorphism onto, and that the groups G_x are all isomorphic. We frequently write $\mathbf{G} = \{G_x,\gamma_\xi\}$. *The bundle* **G** *is said to be* simple *if and only if for every $x,y \in X$, ξ, $\eta \in \pi_1(X,x,y)$, we have $\gamma_\xi = \gamma_\eta$.*

THEOREM 1.5.17. *If X is 0-connected, then the system*

$$\pi_n(X) = \{\pi_n(X,x),\theta_\xi\}$$

is a bundle of groups in X.

PROOF. 2, Corollary 1.5.16; 3 and 4, Lemma 1.5.15.

DEFINITION. *The 0-connected space X is said to be n-simple if and only if the bundle $\pi_n(X)$ is simple.*

DEFINITION. *Let $f: Y \to X$ be a map, and let $\mathbf{G} = \{G_x,\gamma_\xi\}$ be a bundle of groups in X. We define a new bundle $f^*\mathbf{G} = \{H_y,\delta_\eta\}$ by*

$$\begin{cases} H_y = G_{f(y)} \\ \delta_\eta = \gamma_{f*(\eta)} \end{cases}$$

Let $\mathbf{G} = \{G_x,\gamma_\xi\}$, $\mathbf{H} = \{H_x,\delta_\xi\}$ be bundles of groups in X. A homomorphism $\phi:\mathbf{G} \to \mathbf{H}$ is a function which assigns to each $x \in X$ a homomorphism $\phi_x: G_x \to H_x$ satisfying the commutativity relation $\delta_\xi \circ \phi_y = \phi_x \circ \gamma_\xi$ for all $\xi \in \pi_1(X,x,y)$

$$\begin{tikzcd} G_y \arrow[r, "\phi_y"] \arrow[d, "\gamma_\xi"] & H_y \arrow[d, "\delta_\xi"] \\ G_x \arrow[r, "\phi_x"] & H_x \end{tikzcd}$$

If, for each x,

$$\phi_x \text{ is } \begin{Bmatrix} \text{an isomorphism into} \\ \text{onto} \end{Bmatrix}$$

we say that

$$\phi \text{ is } \begin{Bmatrix} \text{an isomorphism into} \\ \text{onto} \end{Bmatrix}$$

If, for each x, $G_x \subset H_x$, and ϕ_x is the inclusion map, we say that $\mathbf{G}$ is a subbundle *of* $\mathbf{H}$. *If each G_x is a normal subgroup of H_x, we say that $\mathbf{G}$ is a* normal subbundle *of* $\mathbf{H}$ *and define the* factor bundle $\mathbf{G}/\mathbf{H}$ *with groups G_x/H_x and homomorphisms γ_ξ^* induced by γ_ξ.*

DEFINITION. *$\Omega_n(X,x_0)$ is the subgroup of $\pi_n(X,x_0)$ generated by all elements of the form $\alpha - \theta_\xi(\alpha)$ with $\alpha \in \pi_n(X,x_0)$, $\xi \in \pi_1(X,x_0)$.*

THEOREM 1.5.18. *If $\xi \in \pi_1(X,x,y)$, then $\theta_\xi(\Omega_n(X,y)) \subset \Omega_n(X,x)$. The system $\{\Omega_n(X,x),\theta_\xi|\Omega_n(X,y)\} = \Omega_n(X)$ is a normal subbundle of $\pi_n(X)$. The bundle $\pi_n(X)/\Omega_n(X)$, denoted by $\pi_n^*(X)$, is simple.*

PROOF. We first prove the following Lemma.

LEMMA 1.5.19. *$\Omega_n(X,x)$ is a normal subgroup of $\pi_n(X,x)$. In particular, $\Omega_1(X,x)$ is the commutator subgroup of $\pi_1(X,x)$.*

PROOF. Since $\pi_n(X,x)$ is abelian for $n > 1$, we may confine our attention to the second statement. By returning to the definition of $t_p(g)$, we see at once that it is homotopic to $(p \cdot g) \cdot \hat{p}$. It follows that $\theta_\xi(\alpha) = \xi\alpha\xi^{-1}$. But $\Omega_1(X,x)$ is generated by the $\alpha\theta_\xi(\alpha)^{-1} = \alpha\xi^{-1}\alpha^{-1}\xi =$

$\alpha\xi^{-1}\alpha^{-1}(\xi^{-1})^{-1}$, and these are precisely the generators of the commutator subgroup. Q.E.D.

Now let $\eta \in \pi_1(X,y)$, $\alpha \in \pi_n(X,y)$. Then

$$\theta_\xi(\alpha - \theta_\eta(\alpha)) = \theta_\xi(\alpha) - \theta_{\xi\eta}(\alpha) = \theta_\xi(\alpha) - \theta_{\xi\eta\xi^{-1}}(\theta_\xi(\alpha))$$

Since $\xi_\eta\xi^{-1} \in \pi_1(X,x)$, the element just described belongs to $\Omega_n(X,x)$. Hence $\theta_\xi(\Omega_n(X,y)) \subset \Omega_n(X,x)$. To prove simplicity of the factor bundle, it suffices to show $\xi,\eta \in \pi_1(X,x,y)$, $\alpha \in \pi_n(X,y) \Rightarrow \theta_\xi(\alpha) - \theta_\eta(\alpha) \in \Omega_n(X,x)$. But

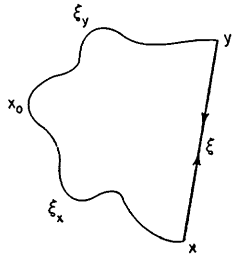
$$\theta_\xi(\alpha) - \theta_\eta(\alpha) = \theta_\xi(\alpha) - \theta_{\eta\xi^{-1}}(\theta_\xi(\alpha))$$

and this element is in $\Omega_n(X,x)$ since $\eta\xi^{-1} \in \pi_1(X,x)$.

We now show that the notion of "bundle of groups" is equivalent to the simpler notion "group with operators in $\pi_1(X)$." In fact, if $\mathbf{G} = \{G_x,\gamma_\xi\}$ is a bundle of groups in X, and $x_0 \in X$, then the homomorphism $\xi \to \gamma_\xi$, $(\xi \in \pi_1(X,x_0))$, defines $\pi_1(X,x_0)$ as group of operators on $G = G_{x_0}$. Conversely, if G is a group on which $\pi_1(X,x_0)$ operates, we define a bundle $\mathbf{G}$ as follows. For each $x \in X$, choose $\xi_x \in \pi_1(X,x_0,x)$ with $\xi_{x_0} = \epsilon_{x_0}$. Then let

$$\bar{G}_x = G$$

$$\bar{\gamma}_\xi(g) = (\xi_x\xi\xi_y^{-1})\cdot g$$



Then $\bar{\mathbf{G}} = \{\bar{G}_x,\bar{\gamma}_\xi\}$ is a bundle of groups in X. If the group G with operators in $\pi_1(X,x_0)$ is derived from a bundle $\mathbf{G}$, then $\mathbf{G} \approx \bar{\mathbf{G}}$ under the isomorphisms $\xi_x : G_x \to G_{x_0} = G$. Conversely, the group with operators derived from $\bar{\mathbf{G}}$ is G; for if $\xi \in \pi_1(X,x_0)$, then

$$\bar{\gamma}_\xi(g) = (\xi_x\xi\xi_x^{-1})\cdot g = \xi\cdot g$$

Thus the foregoing correspondence between bundles and groups with operators in π_1 is 1:1.

The notion of bundle of groups is useful in homology theory, as we shall see later. It is actually a special case of Cartan's notion of "faisceau."

REMARK 1.5.20. *Let $f\colon Y \to X$. Then f_* maps $\pi_n(Y)$ homomorphically into $f^*\pi_n(X)$. That is to say: For $\xi \in \pi_1(Y,x,y)$, $\alpha \in \pi_n(Y,y)$, we have $f_*(\theta_\xi(\alpha)) = \theta_{f*(\xi)}(f_*(\alpha))$.*

1.6 Relative Homotopy Groups [12]

DEFINITION. $J^{n-1} = I \times \dot{I}^{n-1} \cup 0 \times I^{n-1}$. *Let* $x \in A \subset X$. *Then* $F^n(X,A,x)$ = *space of all maps of* $(I^n,\dot{I}^n,J^{n-1}) \to (X,A,x)$, *i.e., those maps of* I^n *into* X *which take* $\dot{I}^n$ *into* A *and* J^{n-1} *into* x.

$\pi_n(X,A,x)$ = *set of path components of* $F^n(X,A,x)$.

LEMMA 1.6.1. $F^n(X,A,x)$ *is homeomorphic with* $F^1(F^{n-1}(X,A,x),e_x^{n-1})$.

PROOF. Similar to that of Lemma 1.4.2.

COROLLARY 1.6.2. *If* $n \geq 2$, $F^n(X,A,x)$ *is an AIH-space with addition the same as for* $F^n(X,x)$.

PROOF. The proof is similar to that of Corollary 1.4.3, insofar as the induction goes. It remains only to establish the truth of the theorem for $n = 2$, and this may be done in standard fashion.

However, it is instructive to pause at this point and examine why the theorem succeeds in this case, whereas it fails for $n = 1$. To this end, we must examine what members of $F^n(X,A,x)$ really are. They are simply maps of I^n into X, in which all faces but one, namely the face $I^{n-1} \times \{1\}$, go into x; while $I^{n-1} \times \{1\}$ goes into A. We denote by J^{n-1} the remainder of the faces, namely the set $\overline{I^n - I^{n-1} \times \{1\}}$. Now when $n > 1$, the sets J^{n-1} and $I^{n-1} \times \{1\}$ intersect, and their intersection is precisely the boundary of $I^{n-1} \times \{1\}$. (We know that they must intersect, because of the connectedness of $\dot{I}^n$ for $n > 1$.) But in the case of $n = 1$, $\dot{I}^{n-1} \times \{1\} = \{1\}$ and $J^{n-1} = \{0\}$ do not intersect. This is made possible by the disconnectedness of $\dot{I}^1$. Now when we add two maps of I^n, we are essentially "gluing" them along the hyperplane $x_n = 1$ of the first map, and $x_n = 0$ of the second. When $n > 1$, this is all right, because these hyperplanes are in J^{n-1}. But when $n = 1$, there are not enough dimensions to force the point 1 to be in J^{n-1}, and so it does not have to go into x, but may go into any point of A. Obviously no gluing can be accomplished if the two parts to be attached are not even brought together.

COROLLARY 1.6.3. $\pi_n(X,A,x)$ *is a group for* $n \geq 2$.

PROOF. Similar to that of Corollary 1.4.4.

COROLLARY 1.6.4. $\pi_n(X,A,x)$ *is abelian for* $n \geq 3$.

PROOF. By definition,

$$\pi_n(X,A,x) \approx \pi_1(F^{n-1}(X,A,x),e_x^{n-1})$$

$F^{n-1}(X,A,x)$ is an H space when $n - 1 \geq 2$, i.e., when $n \geq 3$. Now use Theorem 1.3.5.

COROLLARY 1.6.5. $\pi_n(X,A,x) = \pi_0(F^n(X,A,x),e_x^n)$, *as groups.*

PROOF. Similar to that of Theorem 1.4.5.

COROLLARY 1.6.6. $F^{n+k}(X,A,x)$ *is homeomorphic with*

$$F^n(F^k(X,x),F^k(A,x),e_x^k)$$

under a homeomorphism which is an H-space homomorphism.

PROOF. Similar to that of Lemma 1.4.2, using the natural homeomorphism between $X^{I^{n+k}}$ and $(X^{I^k})^{I^n}$. The fact that the resulting homeomorphism is an H-space homomorphism may be verified by simply examining the additions in the two spaces.

COROLLARY 1.6.7. $\pi_{k+n}(X,A,x) \approx \pi_n(F^k(X,x),F^k(A,x),e_x^k)$.

PROOF. Corollary 1.6.5, Theorem 1.3.9, and, of course, Corollary 1.6.6.

LEMMA 1.6.8. $A = \{x\} \Rightarrow \pi_n(X,A,x) = \pi_n(X,x)$.

DEFINITION. *Let* $n > 1$. *Then the boundary function* $\bar{\partial}$ *from* $F^n(X,A,x)$ *to* $F^{n-1}(A,x)$ *is given by* $(\bar{\partial}f)(t_1,\cdots,t_n) = f(1,t_2,\cdots,t_n)$.

LEMMA 1.6.9. ∂ *is an H-space homomorphism*; (*it is therefore called the boundary mapping*). *Furthermore, if* $\partial_* = \bar{\partial}_*, f:(X,A,x) \to (Y,B,y)$, *then* $\partial_* f_* = (f|A)_*\partial_*$, *and thus the third Eilenberg-Steenrod axiom for homology theory is satisfied by homotopy groups.*

PROOF. We must first check that ∂ is continuous. To this end, consider the mapping $d:X^{I^n} \to X^{I^{n-1}}$, which is induced by the projection $p:I^n \to I^{n-1}$ given by $p(t_1,\cdots,t_n) = p(1,t_2,\cdots,t_n)$. Then the continuity of d follows from that of p by Problem 1, Part 2. But $\bar{\partial} = d|F^n(X,A,x)$. The homomorphism property is readily verified. Commutativity follows from $\bar{\partial}(f \circ g)(t_1,\cdots,t_n) = (f \circ g)(1,t_2,\cdots,t_n) = f(g(1,t_2,\cdots,t_n)) = f((\bar{\partial}g)(t_1,\cdots,t_n))$, where $g \in F^n(X,A,x)$.

From now on through the end of Section 1.6, and occasionally thereafter, we abbreviate our symbols for homotopy groups and H-spaces by omitting explicit mention of the base point x. The resulting symbols are not to be confused with those for the corresponding bundles.

LEMMA 1.6.10. *Let $i{:}A \to X$ and $j{:}(X,x) \to (X,A)$ be inclusion maps, and write $X^k = F^k(X)$, $A^k = F^k(A)$. Then the diagram*

$$\begin{tikzcd}
\pi_{n+1}(A^k) \arrow[r, "i_*"] \arrow[d] & \pi_{n+1}(X^k) \arrow[r, "j_*"] \arrow[d] & \pi_{n+1}(X^k,A^k) \arrow[r, "\partial_*"] \arrow[d] & \pi_n(A^k) \arrow[r, "i_*"] \arrow[d] & \pi_n'X^k) \arrow[d] \\
\pi_{n+k+1}(A) \arrow[r, "i_*"] & \pi_{n+k+1}(X) \arrow[r, "j_*"] & \pi_{n+k+1}(X,A) \arrow[r, "\partial_*"] & \pi_{n+k}(A) \arrow[r, "i_*"] & \pi_{n+k}(X)
\end{tikzcd}$$

is commutative, where the vertical homomorphisms are the isomorphisms onto of Corollary 1.6.7. *The lemma holds even when $n = 0$, i_*, j_*,* and ∂_* *being defined in the obvious fashion.*

PROOF. Immediate, upon examination of the homomorphisms involved and the mappings which induce them.

LEMMA 1.6.11. *Suppose that the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "f_6"] & A_2 \arrow[r, "f_2"] \arrow[d, "f_3"] & A_3 \arrow[r] \arrow[d, "f_7"] & \cdots \\
\cdots \arrow[r] & B_1 \arrow[r, "f_4"] & B_2 \arrow[r, "f_5"] & B_3 \arrow[r] & \cdots
\end{tikzcd}$$

is commutative and that the vertical functions are one-one onto. Then if the top sequence is exact, so is the bottom sequence.

PROOF. Before we begin the proof, we ask the reader to recall that by commutativity we mean $f_3 \circ f_1 = f_4 \circ f_6$, $f_7 \circ f_2 = f_5 \circ f_3$, etc. We also note that the sets A_i and B_i in question by no means have to be groups, or even to possess any structure whatsoever, except that they have to have a distinguished zero element, such that all the functions take the zero of one group into the zero of the next. The definition of exactness remains unchanged,

The proof is as follows. We have

Kernel f_5	$= f_3$ Kernel $f_5 \circ f_3$	(because f_3 is onto)
	$= f_3$ Kernel $f_7 \circ f_2$	(by commutativity)
	$= f_3$ Kernel f_2	(because since f_7 is 1-1, the only element which goes into 0 is 0)
	$= f_3$ Image f_1	(by exactness)
	$=$ Image $f_3 \circ f_1$	
	$=$ Image $f_4 \circ f_6$	(by commutativity)
	$= f_4$ Image f_6	
	$= f_4\,(B_1)$	(because f_6 is onto)
	$=$ Image f_4	

This completes the proof.

LEMMA 1.6.12. *The sequence*

$$\pi_1(A) \xrightarrow{i_*} \pi_1(X) \xrightarrow{j_*} \pi_1(X,A) \xrightarrow{\partial_*} \pi_0(A) \xrightarrow{i_*} \pi_0(X)$$

is internally exact.

PROOF.

ⓐ Kernel $j_* =$ Image i_*

For,

$$\begin{aligned}\text{Kernel } j_* &= \{C(f) \in \pi_1(X) | j_*C(f) = C(e_x)\}\\ &= \{C(f) \in \pi_1(X) | C(jf) = C(e_x)\}\end{aligned}$$

[the components on the left side being taken in the space $F'(X,A)$] $=$ $\{C(f) \in \pi_1(X)|C(f) = C(e_x)\}$ $=$ $\{C(f) \in \pi_1(X)| \exists\ F{:}I \times I \rightarrow X$ such that $F(t,1) = x$, $F(t,0) = f(t)$, $F(0,t) = x$, $F(1,t) \in A\}$ [the latter conditions because the homotopy must take place in $F^1(X,A)$]. Now let $g(t) = F(1,1-t)$. Then $g(t) \in A$, $g(1) = F(1,0) = f(0) = x$, $g(0) =$ $F(1,1) = x$. Define

$$G(t,s) = \begin{cases} F(t,2s(1-t) & s \leq \frac{1}{2} \\ F(1-2(1-s)(1-t),1-t) & s \geq \frac{1}{2}\end{cases}$$

Then $G(t,s)$ provides a homotopy in X between $g(t)$ and $f(t)$, so $C(f) = C(g)$ in X. But $g \in F^1(A)$, so

$$\begin{aligned}C(f) = C(g) &= C(ig) \text{ in } X\\ &= i_*C(g) \text{ in } A\end{aligned}$$

So $C(f) \in$ Image i_*. It follows that Kernel $j_* \subset$ Image i_*. The opposite inequality is proved in a similar fashion.

ⓑ Kernel $\partial_* =$ Image j_*

For the image of j_* consists of the set of path components of $F^1(X,A)$ which contain a loop of X; whereas the kernel of ∂_* is the set of path components of $F^1(X,A)$ which contain paths whose end points are in the identity of $\pi_0(A)$, i.e., may be joined to x by a path in A. So Image $j_* \subset$ Kernel ∂_* is obvious; and for the opposite inequality, it is merely necessary to show that if

$$\begin{array}{lll} f(0) = x & f(1) = a \in A & \\ g(0) = a & g(1) = x & g(t) \in A \end{array}$$

then $\exists\ F{:}I \times I \rightarrow X$ such that

$$\begin{aligned}F(0,t) &= f(t) * e_a\\ F(t,0) &= x\\ F(t,1) &\in A\\ F(1,1) &= x\end{aligned}$$

F is provided by $F(s,t) = f(t) * g(st)$, and shows that every path in $F^1(X,A)$ whose end point can be joined to x by a path in A is homotopic in (X,A) to a loop of X. It follows that Kernel $\partial_* \subset$ Image j_*

ⓒ Kernel $i_* =$ Image ∂_*

Let $\alpha \in \pi_0(A)$; so α is a path component of A. Then for $\alpha \in$ Kernel i_*, it is necessary and sufficient that the elements of α be elements of $C(x)$ in X. On the other hand, for $\alpha \in$ Image ∂_* it is necessary and sufficient that the elements of α be end points of paths which start at x; that is, the elements of α must be points of $C(x)$. This completes the proof.

THEOREM 1.6.13. *The sequence*

$$\cdots \to \pi_{n+1}(X,A) \xrightarrow{\partial_*} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \to \cdots$$

is exact.

PROOF. The proof follows immediately from Lemmas 1.6.n; $n = 10$, 11, 12. In Lemma 1.6.12, we simply put $X = X^k$, $A = A^k$, then apply the other two.

THEOREM 1.6.14. *Let $B \subset A \subset X$, and let $k:(A,x) \subset (A,B)$, $\tilde{i}:(A,B) \subset (X,B)$, $\tilde{j}:(X,B) \subset (X,A)$ and let $\tilde{\partial}_* = k_* \circ \partial_*$. Then the sequence*

$$\to \pi_{n+1}(X,A) \xrightarrow{\tilde{\partial}_*} \pi_n(A,B) \xrightarrow{\tilde{i}_*} \pi_n(X,B) \xrightarrow{\tilde{j}_*} \pi_n(X,A) \to$$

is exact.

PROOF. See Eilenberg and Steenrod [B.3, p. 25, Theorem 10.2]. The theorem is there proved for homology; however, only his axioms 1, 2, 3, and 4 are used [see B.3, pp. 10, 11], all of which hold for homotopy as well, as we have proved. The proof thus goes through in exactly the same way.

PROBLEM 2. Let X be an $lc^{\frac{1}{2}}$, pathwise connected space, with $\tilde{X}$ a covering space of X. Then $\pi_n(\tilde{X}) \approx \pi_n(X)$, $n \geq 2$.

PROBLEM 3. Let $y \in \dot{I}^{n+1}$, and assume $\pi_n(X,x) = 0$. Then every map of $(\dot{I}^{n+1},y)$ into (X,x) can be extended to a map of I^{n+1} into X.

PROBLEM 4. Let $\pi_n(X,A) = 0$. Then every map of $(J^n,\{1\} \times \dot{I}^n,y)$ into (X,A,x) has an extension $f:I^{n+1} \to X$ with $f(\{1\} \times I^n) \subset A$.

1.7 The Bundle $\pi_n(X,A)$

The definition of this bundle corresponds to that of $\pi_n(X)$, and many of the lemmas in this case have proofs similar to those with the same number in Section 1.5. When this is so, the proof will be omitted.

Note, however, that these lemmas are not consequences of those of Section 1.5. For the bundle $\pi_n(X,A)$ is a bundle on A, not on X; and if we let $A = (x)$, so that $\pi_n(X,A,x) = \pi_n(X,x)$, then we have that the bundle $\pi_n(X,A)$ is a trivial bundle on a single point. The two cases then have to be stated separately.

Let

$$G^n(X,A) = \bigcup_{x \in A} F^n(X,A,x)$$

Define $\pi: G^n(X,A) \to A$ by $\pi(f) = x$ if $f \in F^n(X,A,x)$. Then we have the following theorem.

THEOREM 1.7.1. *π is continuous.*

DEFINITION. *Let $f \in F^n(X,A,x)$, $g \in F^n(X,A,b)$, and $p \in F^1(X,a,b)$. Then we say that $f \underset{p}{\equiv} g$ if and only if there exists a path P in $G^n(X,A)$ from f to g such that $\pi \circ P = p$. (Alternatively, there exists a map $P^*: I^{n+1} \to X$, such that $P^*(0,u) = f(u)$, $P^*(1,u) = g(u)$, $P^*(t,u) \in A$ if $n \in I^{n-1} \times \{1\}$, $P^*(t,u) = p(t)$ if $u \in J^{n-1}$.)*

DEFINITION. *Let $t \in I^n$. Then*

$$||t|| = \max(|2t_1 - 1|, \cdots, |2t_{n-1} - 1|, 1 - t_n)$$

LEMMA 1.7.2. $||t|| = 0$ *if and only if* $t = (\frac{1}{2}, \cdots, \frac{1}{2}, 1)$

$||t|| = 1$ *if and only if* $t \in J^{n-1}$

DEFINITION. *Let $g \in F^n(X,A,y)$, $p \in F^1(A,x,y)$.*
Then

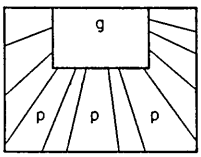
$$\bar{t}_p(g)(t_1, \cdots, t_n) = \begin{cases} g(2t_1 - \frac{1}{2}, \cdots, 2t_{n-1} - \frac{1}{2}, 2t_n - 1), & 0 \leq ||t|| \leq \frac{1}{2} \\ p(2 - 2||t||), & \frac{1}{2} \leq ||t|| \leq 1 \end{cases}$$

An idea of what the map looks like is suggested by the picture.

LEMMA 1.7.3. *$(g,p) \to \bar{t}_p(g)$ is continuous from*

$$\bigcup_{x,y \in A} F^n(X,A,y) \times F^1(A,x,y)$$

to $G^n(X,A)$.



THEOREM 1.7.4. *If $g \equiv g'$ in $F^n(X,A,y)$ and $p \equiv p'$ in $F^1(A,x,y)$, then $\bar{t}_p(g) \equiv \bar{t}_{p'}(g')$ in $F^n(X,A,x)$.*

LEMMA 1.7.5. $f \underset{p}{\equiv} g$, $g \underset{q}{\equiv} h \Rightarrow f \underset{pq}{\equiv} h$.

LEMMA 1.7.6. $f \underset{p}{\equiv} g \Rightarrow g \underset{\bar{p}}{\equiv} f$.

LEMMA 1.7.7. $f \underset{p}{\equiv} f', p \equiv p'$ *in* $F^1(A,x,y) \Rightarrow f \underset{p'}{\equiv} f'$.

LEMMA 1.7.8. $f \underset{p}{\equiv} g, f' \underset{p}{\equiv} g \Rightarrow f \equiv f'$ *in* $F^n(X,A,x)$.

LEMMA 1.7.9. $\bar{l}_p(g) \underset{p}{\equiv} g$.

COROLLARY 1.7.10. *$f \underset{p}{\equiv} g$ if and only if $f \equiv \bar{l}_p(g)$ in $F^n(X,A,x)$.*

LEMMA 1.7.11. $f \underset{p}{\equiv} g, f' \underset{p}{\equiv} g' \Rightarrow f + f' \underset{p}{\equiv} g + g'$.

COROLLARY 1.7.12. $\bar{l}_p(g + g') \equiv \bar{l}_p(g) + \bar{l}_p(g')$ *in* $F^n(X,A,x)$.

LEMMA 1.7.13. $\bar{l}_{e_x}(f) \equiv f$ *in* $F^n(X,A,x)$.

LEMMA 1.7.14. $\bar{l}_{pq}(h) \equiv \bar{l}_p(\bar{l}_q(h))$ *in* $F^n(X,A,x)$.

Thus if $\xi \in \pi_1(A,x,y)$, $\alpha \in \pi_n(X,A,y)$, then we may define $\bar{\theta}_\xi(\alpha) \in \pi_n(X,A,x)$ to be the path component of $\bar{l}_p(g)$ for any $p \in \xi$, $g \in \alpha$. Then we have the following Lemma.

LEMMA 1.7.15.
$$\begin{aligned} \bar{\theta}_\xi(\alpha + \alpha') &= \bar{\theta}_\xi(\alpha) + \bar{\theta}_\xi(\alpha') \\ \bar{\theta}_{\xi\eta}(\alpha) &= \bar{\theta}_\xi(\bar{\theta}_\eta(\alpha)) \\ \bar{\theta}_{\epsilon_y}(\alpha) &= \alpha \end{aligned}$$

COROLLARY 1.7.16. *$\bar{\theta}_\xi$ is a homomorphism from $\pi_n(X,A,y)$ to $\pi_n(X,A,x)$.*

THEOREM 1.7.17. *If A is 0-connected, then the system $\pi_n(X,A) = \{\pi_n(X,A,x),\bar{\theta}_\xi\}$ is a bundle of groups in A.*

DEFINITION. *If A is 0-connected, the pair (X,A) is said* ***to be n-simple if*** *and only if the bundle $\pi_n(X,A)$ is simple.*

PROBLEM 5. Let $\alpha,\beta \in \pi_2(X,A,x)$; then
$$\bar{\theta}_{\partial_*(\alpha)}(\beta) = \alpha + \beta - \alpha$$

(Note that this group is not necessarily commutative, so that we do not necessarily have $\alpha + \beta - \alpha = \beta$.)

COROLLARY 1.7.18. *The kernel of*

$$\partial_* : \pi_2(X,A,x) \to \pi_1(A,x)$$

is contained in the center of $\pi_2(X,A,x)$.

PROOF. Let $\alpha \in$ Kernel ∂_*. Then for $\beta \in \pi_2(X,A)$ we have $\alpha + \beta - \alpha = \bar{\theta}_{\partial_*(\alpha)}(\beta) = \theta_0(\beta) = \beta$. So $\alpha + \beta = \beta + \alpha$, and it follows that $\alpha \in$ center of $\pi_2(X,A,x)$.

DEFINITION. $\bar{\Omega}_n(X,A,x_0)$ *is the subgroup of* $\pi_n(X,A,x_0)$ *generated by all elements of the form* $\alpha - \bar{\theta}_\xi(\alpha)$, $\alpha \in \pi_n(X,A,x_0)$, $\xi \in \pi_1(A,x_0)$.

THEOREM 1.7.19. *If* $\xi \in \pi_1(A,x,y)$, *then* $\bar{\theta}_\xi(\bar{\Omega}_n(X,A,y)) \subset \Omega_n(X,A,x)$. *The system* $\{\bar{\Omega}_n(X,A,x),\theta_\xi|\bar{\Omega}_n(X,A,y)\} = \bar{\Omega}_n(X,A)$ *is a normal subbundle of* $\pi_n(X,A)$. *The bundle* $\pi_n(X,A)/\bar{\Omega}_n(X,A)$, *denoted by* $\pi_n^*(X,A)$, *is simple.*

PROOF. We first prove the following Lemma.

LEMMA 1.7.20. $\bar{\Omega}_n(X,A,x)$ *is a normal subgroup of* $\pi_n(X,A,x)$. *In particular,* $\bar{\Omega}_2(X,A,x)$ *contains the commutator subgroup of* $\pi_2(X,A,x)$.

PROOF. Since $\pi_n(X,A,x)$ is abelian for $n > 2$, we may confine our attention to the case of $n = 2$. We then have, for $\beta \in \pi_2(X,A,x)$, $\beta + (\alpha - \bar{\theta}_\xi(\alpha)) - \beta$

$$\begin{aligned}
&= \beta + \alpha - \beta + (\beta - \bar{\theta}_\xi(\alpha) - \beta) \\
&= \bar{\theta}_{\partial^*(\beta)}(\alpha) - \theta_{\partial^*(\beta)}(\bar{\theta}_\xi(\alpha)) \qquad \text{(by Problem 6)} \\
&= \bar{\theta}_{\partial^*(\beta)}(\alpha) - \bar{\theta}_{\partial^*(\beta)\cdot\xi}(\alpha) \qquad \text{(by Lemma 1.7.15)} \\
&= \bar{\theta}_{\partial^*(\beta)}(\alpha) - \bar{\theta}_{\partial^*(\beta)\xi[\partial^*(\beta)]^{-1}\partial^*(\beta)}(\alpha) \\
&= \bar{\theta}_{\partial^*(\beta)}(\alpha) - \bar{\theta}_{\partial^*(\beta)\xi[\partial^*(\beta)]^{-1}}(\bar{\theta}_{\partial^*(\beta)}(\alpha)) \in \bar{\Omega}_2(X,A,x)
\end{aligned}$$

This proves normality.

For the second statement, let $\alpha + \beta - \alpha - \beta$ be a generator of the commutator subgroup. Then

$$\begin{aligned}
\alpha + \beta - \alpha - \beta &= \bar{\theta}_{\partial^*(\alpha)}(\beta) - \beta \\
&= [\beta - \bar{\theta}_{\partial^*(\alpha)}(\beta)]^{-1} \in \bar{\Omega}_2(X,A,x_0)
\end{aligned}$$

This completes the proof of the lemma. The remainder of the proof of Theorem 1.7.19 is similar to the corresponding part of that of Theorem 1.5.18.

REMARK 1.7.21. *Let $f:(Y,B) \to (X,A)$. Then f_* maps $\pi_n(Y,B)$ homomorphically into $f^*\pi_n(X,A)$. That is to say: For $\xi \in \pi_1(B,x,y)$, $\alpha \in \pi_n(Y,B,y)$, we have*

$$f_*(\bar{\theta}_\xi(\alpha)) = \bar{\theta}_{f_*(\xi)}(f_*(\alpha))$$

DEFINITION. *Let $\xi \in \pi_1(A,x)$, $\alpha \in \pi_1(X,x)$, $i:(A,x) \to (X,x)$ be the inclusion map. Then $\bar{\theta}_\xi(\alpha) = \theta_{i_*\xi}(\alpha)$.*

THEOREM 1.7.22. *The diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(X,A,x_0) \arrow[r, "\partial_*"] \arrow[d, "\bar{\theta}_\xi"] & \pi_n(A,x_0) \arrow[r, "i_*"] \arrow[d, "\theta_\xi"] & \pi_n(X,x_0) \arrow[r, "j_*"] \arrow[d, "\theta'_\xi"] & \pi_n(X,A,x_0) \arrow[r] \arrow[d, "\bar{\theta}_\xi"] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}(X,A,x_0) \arrow[r, "\partial_*"] & \pi_n(A,x_0) \arrow[r, "i_*"] & \pi_n(X,x_0) \arrow[r, "j_*"] & \pi_n(X,A,x_0) \arrow[r] & \cdots
\end{tikzcd}$$

is commutative.

PROOF. 1. Clearly $\bar{\partial} t_p(g) = t_p(\bar{\partial} g)$.
Therefore $\partial_* \circ \bar{\theta}_\xi = \theta_\xi \circ \partial_*$.
2. By Remark 1.5.20, we have

$$i_*(\theta_\xi(\alpha)) = \theta_{i_*\xi}(i_*(\alpha)) = \theta'_\xi(i_*(\alpha))$$

3. It is clearly sufficient to show that $t_p(g)$ and $\bar{t}_p(g)$ can be joined by a path in $F^n(X,A,x_0)$. By Corollary 1.7.10, it is sufficient to show that $t_p(g) \underset{p}{\equiv} g$ in the sense of Section 1.7, that is, that there exists a path P in $G^n(X,A)$ joining $t_p(g)$ to g, with $\pi \circ P = p$. Now there does exist such a path in $G^n(X)$. But clearly $G^n(X) \subset G^n(X,A)$. This completes the proof.