CHAPTER I

Electron Ballistics

Electronics includes in a broad sense all electrical phenomena, for all electric conduction involves electrons. The common interpretation of the term at present, however, is expressed by a standard definition[1] of electronics, which is "that field of science and engineering which deals with electron devices and their utilization." Here an electron device[1] is "a device in which conduction by electrons takes place through a vacuum, gas, or semi-conductor." Electronics has become increasingly important because of its growing application to the problems of the electrical industry. During the early years of the industry, electronic conduction—except for the arc lamp—usually took the form of annoying and somewhat puzzling accidents, such as puncture of insulation, flashover of insulators, and corona leakage current. Recently, however, despite the fact that electronic conduction still has many puzzling aspects, scientists and engineers have found an increasing number of ways in which it can be harnessed, guided, and controlled for useful purposes. Electronics consequently is now as important to the engineer concerned with rolling of steel rails or the propulsion of battleships as to the engineer concerned with the communication of intelligence.

The occurrence of electronic conduction is widespread, and its nature diverse. Sometimes it is unconfined, as in lightning or some arcs; at other times confined, as in the electron tube or the neon sign. Sometimes it is visible, as in the arc light; at other times invisible, as in the vacuum tube. Sometimes the conduction is undesirable and uncontrolled, as in the example of lightning striking a transmission line, or in corona formation on the line. At other times the conduction is intentional, and may be controlled by minute electrical forces, as in the electron tube.

The field of application of electronic phenomena already covers a very wide range of power. The asymmetric nonlinear property common to many types of electronic conduction finds application not only in the radio detector tube, where the power handled is extremely small, but also in the railway mercury-arc rectifier that handles the power to move trains over mountains. The property of certain types of electronic conduction that makes possible the control of a large flow of energy by the expenditure of a relatively small amount of power

[1] "Standards on Electron Tubes: Definitions of Terms, 1950," *I.R.E. Proc.*, *38* (1950), 433.

finds use over ranges of current varying all the way from that involved in the electrometer vacuum tube capable of measuring currents of 10^{-15} ampere to the enormous bursts of current amounting to thousands of amperes required for modern electric spot-welders and nuclear particle accelerators.

Electronic conduction through a vacuum or gas takes place by virtue of the fact that under certain conditions charged particles, known as electrons and ions, are liberated from electrodes and produced in the gas in the conducting path; and that in the presence of an impressed electric field these charged particles experience a force that causes them to move and constitute an electric current. Thus electronic conduction in a vacuum or gas embraces the following important physical processes:

(a) the liberation of charged particles from electrodes,
(b) the motion of the particles through the space between the electrodes,
(c) the production of charged particles in the space between the electrodes, and
(d) the control of the flow of the particles by the electric field caused by electrodes interposed in the space, or by the magnetic field produced by an external means.

Practical circuit elements that embody possible combinations of the foregoing processes are almost always nonlinear, and effective utilization of such elements in circuits requires an analysis suited to their nonlinearity.

In the application of electronic devices the engineer must have a knowledge of their characteristics and limitations. As in most electrical equipment, the electrical aspects of the design of these devices are often not the limiting ones; chemical, thermal, mechanical, and physical phenomena often govern their rating. A thorough understanding of the physical principles underlying the behavior of a device is therefore necessary in order that intelligent application be made of it. Accordingly, the first part of this book is devoted to a discussion of the physical aspects of electronic conduction, the second part is a description of the electrical characteristics of typical electron tubes, and the third part is a treatment of the fundamental methods of circuit analysis and the basic engineering considerations important in the application of electronic devices.

1. Charge and mass of elementary particles

Over a period of years a number of elementary particles of importance in electronic conduction have been identified, and the charge and mass of each have been measured. A few of the more frequently encountered particles, all of which are constituents of the atom, together with their charges and masses, are listed in Table I.

TABLE I*

Name	*Charge*	*Mass*
Electron	$-Q_e$	m_e
Positron	$+Q_e$	m_e
Neutron	0	$1{,}838m_e$
Proton	$+Q_e$	$1{,}837m_e$

* The symbols e and m are generally used in the literature for the charge and mass of the electron; however, in this book the symbols shown are used to be consistent with those introduced in *Electric Circuits.*

Note that Q_e is the symbol for the *magnitude* of the charge of an electron—it is a positive number and does *not* include the negative sign associated with the negative charge. The negative sign is indicated separately in all the following analytical work where Q_e appears.

In Table I:

$$Q_e = (1.60203 \pm 0.00034) \times 10^{-19} \text{ coulomb} \quad [1]$$

$$m_e = (9.1066 \pm 0.0032) \times 10^{-31} \text{ kilogram} \quad [2]$$

Electronic constants in mks units[2]

Of the elementary particles, the electron is basic in the field of electronics, and the charge and mass of the others are expressed in terms of its charge and mass. The neutron and the proton are particles which have the next higher quantity of mass ordinarily observed. Mesons, which are short-lived charged particles found in cosmic-ray and other nuclear studies, have values of mass intermediate between those of the electron and the neutron. Neutrinos, which are postulated to satisfy the requirements of nuclear theory, have neither charge nor mass. Neither mesons nor neutrinos have engineering significance at present. The ratio of charge to mass for the electron appears in many of the theoretical expressions for the motion of charged particles in electric and magnetic fields; hence there are numerous ways of measuring it experimentally. Precise measurements[2] give for the ratio the value

$$Q_e/m_e = (1.7592 \pm 0.0005) \times 10^{11} \text{ coulombs per kilogram} \quad [3]$$

[2] These values are taken from R. T. Birge, "A New Table of Values of the General Physical Constants," *Rev. Mod. Phys.*, *13* (October, 1941), Table *a*, p. 234, and Table *c*, pp. 236–237, with permission.

Use of nuclear resonance as a measuring tool has resulted in a further improvement[3] by a factor of about three in the precision of measurement of this ratio.

All charged particles of importance in engineering have essentially multiples of the charge of the electron or proton. Particles having the mass of a molecule and the charge of an electron or proton are known as *positive or negative ions*, depending on the sign of their charge. Occasionally particles are encountered which have the mass of the molecule and small multiples of the electron's charge. These are called multiple-charged ions. Ions, which are discussed in Ch. III, generally result from collision processes in gases.

The value for the mass m_e given above is for the electron moving with speeds small compared with the speed of light. This value of mass is ordinarily called the *rest mass*, although no experimental measurements of mass have yet been made on an electron at rest. Experiment shows that the apparent mass of the electron increases with its speed. The theory of relativity,[4] which is based on the hypothetical law that "it is of necessity impossible to determine absolute motion of bodies by any experiment whatsoever," predicts that the speed of light is an asymptotic value unattainable by any material body. In other words, the mass of an electron approaches infinity as its speed approaches the speed of light.

The dependence of the mass of any particle on its speed is given by the expression

$$m = \frac{m_0}{\sqrt{1 - (v/c)^2}}, \qquad [4]$$

where

m is the mass of the particle in motion,

m_0 is the mass of the particle at rest,

v is the speed of the particle,

c is the speed of light*—$(2.99776 \pm 0.00004) \times 10^8$ meters per second.

In general, force is given by the time rate of change of momentum; that is,

$$F = \frac{d(mv)}{dt}. \qquad [5]$$

[3] H. A. Thomas, R. L. Driscoll, and J. A. Hipple, "Determination of e/m from Recent Experiments in Nuclear Resonance," *Phys. Rev.*, *75* (1949), 922.

[4] *The Encyclopædia Britannica* (14th ed.; New York: Encyclopædia Britannica, Inc., 1938), 89–99.

* See footnote 2 on page 3.

The right-hand side of Eq. 5 reduces to the simple product of mass and acceleration only when the mass is constant. From Eq. 4 it follows that the mass is not increased by so much as 1 per cent until the speed of the particle reaches about 15 per cent of the speed of light. It is evident from subsequent considerations that this speed is not generally reached except in devices with impressed voltages that exceed 6,000 volts. Hence assumption that the mass is constant at the rest value is reasonable in computing the force on charged particles in devices having impressed voltages lower than this value.

Because of the electric and magnetic fields surrounding a moving electron, the mass exhibited in its inertia may be entirely electromagnetic.[5] On this assumption, the radius of the equivalent charged sphere, which has no mass in the ordinary sense, may be calculated to be about 2×10^{-15} meter. This is to be compared with the radius of a molecule, which ranges around 10^{-10} meter.[6]

It is found experimentally that a beam of moving electrons may be diffracted by a metallic crystal in a manner similar to the diffraction of light-waves by a grating.[7] This wave-like behavior of electrons shows that the particle concept is not complete. The wavelength experimentally found to be associated with a moving electron is

$$\lambda = \frac{h}{m_e v}, \qquad [6]$$

where

h is the Planck radiation constant*—$(6.624 \pm 0.002) \times 10^{-34}$ joule second,

m_e is the mass of the electron,

v is the speed of the electron.

One of the valuable features of electronic devices is the rapidity with which they act; it is possible to start, stop, or vary a current with them in as short a time as a small fraction of a microsecond. This rapidity of action results from the extreme agility of the electron—a property associated with the fact that the electron has the large ratio of charge to mass stated in Eq. 3. Although both quantities are small, their ratio is very large—much larger than that of any other charged body dealt with in engineering. The size of this ratio can

[5] H. A. Lorentz, *The Theory of Electrons* (Leipzig: B. G. Teubner, 1909).

[6] R. A. Millikan, *Electrons (+ and −), Protons, Photons, Neutrons, and Cosmic Rays* (2nd ed.; Chicago: The University of Chicago Press, 1947), 184, 188.

[7] C. Davisson and L. H. Germer, "Diffraction of Electrons by a Crystal of Nickel," *Phys. Rev.*, *30* (1927), 705–740; C. Davisson, "Electron Waves," *J.F.I.*, *208* (1929), 595–604.

* See footnote 2 on page 3.

perhaps be grasped from a computation of the force of repulsion between one kilogram of electrons located at each of the poles of the earth. Their separation is about 7,900 miles, or 1.27×10^7 meters. Their charge is given by Eq. 3, and, by Coulomb's law, the force of repulsion between them is

$$= \frac{Q_1 Q_2}{4\pi\varepsilon_v d^2} = \frac{(1.76 \times 10^{11})^2}{4\pi \times 8.85 \times 10^{-12} \times (1.27 \times 10^7)^2} = 1.73 \times 10^{18} \text{ newtons} \quad [7]$$

$$= 1.95 \times 10^{14} \text{ tons.} \quad [8]$$

Clearly, the electron's charge-to-mass ratio is enormous to produce such a large force at such a great distance. Consequently, the electric force on an electron in an electrostatic field can overcome the inertia of the electron and produce high velocities in a very short time.

With knowledge of the charge and mass of the particles involved in electronics, it is possible to proceed with the analysis of the motion of particles in electrostatic and magnetostatic fields given in the following articles of this chapter. The source of the charged particles is reserved for consideration in subsequent chapters. At this point it is sufficient to know that electrons are given off by a heated metallic surface, and that electrons and ions are produced in a gas when the process of ionization occurs.

2. The elements of the operation of electron tubes

An *electron tube*[8] is "an electron device in which conduction by electrons takes place through a vacuum or gaseous medium within a gas-tight envelope." Ordinarily it consists of two or more metallic electrodes enclosed in an evacuated glass or metal chamber. The electrodes are insulated from one another. If the chamber is evacuated until the remaining gas molecules have no effect—chemically or electrically—on the operation of the tube, it is called a *vacuum tube*. Other tubes contain gas introduced after the evacuation process has been carried out. These are called *gas tubes* when the amount of gas is sufficient to have an appreciable effect on their electrical characteristics. One of the electrodes, called the *cathode*, serves as a source of electrons by virtue of one or more of the several electron-emission processes discussed in Ch. II. Another electrode, called the *anode* (or *plate*), is usually maintained electrically positive with respect to the cathode. The resulting electric field in the tube exerts a force on the electrons and causes them to move toward the anode, thereby setting up an

[8] "Standards on Electron Tubes: Definitions of Terms, 1950," *I.R.E. Proc.*, *38* (1950), 433.

electric current in the interelectrode space. A simple circuit involving such a tube is shown in Fig. 1. In the external circuit, the electrons flow from the anode through the voltage source to the cathode; by convention, the electric current is in the opposite direction.

When a device of this general character has only two electrodes, one of which serves as a source of electrons, it is usually termed a *diode*. By appropriate control of the voltages of other electrodes which may be inserted in the chamber, the electric field between the cathode and the anode can be modified. The flow of the electrons is thereby

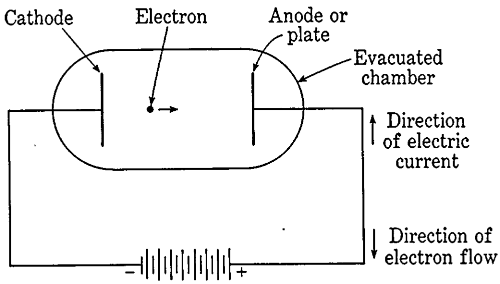


Fig. 1. A simple electron tube.

changed, and the current in the external circuit can be controlled. These *control electrodes* are often called *grids* because of the form they had in early tubes. Tubes with one control electrode are called *triodes*; those with two control electrodes, *tetrodes*; and so on, in accordance with the total number of active electrodes.

In some electron tubes of the vacuum type, the number of charged particles traversing the interelectrode region is so small that the electric field established in this region by the charge on the particles is negligible in comparison with the field established by the charges on the electrodes. This condition is expressed by the statement that the space charge of the charged particles in motion is negligible. In devices for which this condition is true, the electrostatic force on any single particle may be considered to result wholly from the field that exists in the absence of all the particles. For example, the motion of the single electrons in the vacuum phototube or in the electron beam in a cathode-ray tube may often be computed with sufficient accuracy on the assumption that space-charge effects are negligible.

The following articles of this chapter deal with the behavior of particles in only those devices in which space-charge effects are negligible. The behavior of particles in devices in which space-charge effects are of appreciable importance, and some of the fundamental properties of these devices, are discussed in Ch. III. The paths of the charged particles in electrostatic and magnetostatic fields discussed in this chapter are similar in many respects to the trajectories of projectiles in the gravitational field of the earth—hence the chapter is titled *electron ballistics*.

3. Motion of charged particles in electrostatic fields in vacuum

Because the charged particles of interest in electron tubes are so small in comparison with the dimensions of the tubes in which they move, the forces that act upon them may be calculated as though the particles were concentrated at points. Thus the force exerted on such a particle by an electrostatic field is given by

$$\mathbf{F} = Q\mathcal{E}, \tag{9}$$

where

$\mathbf{F}$ is the force acting upon the charged particle,

Q is the charge carried by the particle,

$\mathcal{E}$ is the electric field intensity at the location of the particle.

The quantities $\mathbf{F}$ and $\mathcal{E}$ in this relation are vectors.* Equation 9 specifies (a) that the magnitude of the force is the product of the magnitude of the field intensity and the charge and (b) that the direction of the force is that of the field if the charge is positive and is opposite to the direction of the field if the charge is negative.

If E is the potential at each point in the tube, taken with respect to any arbitrary zero of potential, the gradient[9] of E, written **grad** E, is a vector oriented in the direction in which E increases most rapidly and whose magnitude is the rate of change of E with distance in this direction. Since

$$\mathcal{E} = -\,\mathbf{grad}\,E, \tag{10}$$

Eq. 9 may be written in the form

$$\mathbf{F} = -\,Q\,\mathbf{grad}\,E. \tag{11}$$

If the particle is free to move, it is accelerated according to the equation

$$\mathbf{F} = m\mathbf{a}, \tag{12}$$

or

$$\mathbf{F} = m\frac{d^2\mathbf{l}}{dt^2}, \tag{13}$$

* Quantities that are vectors in space are printed in **boldface script or roman (upright)** type.

[9] N. H. Frank, *Introduction to Electricity and Optics* (2nd ed.; New York: McGraw-Hill Book Company, Inc., 1950), 1–14.

where

- **F** is the total force acting on the particle,
- m is the mass of the particle,
- **a** is the acceleration of the particle,
- **l** is the displacement of the particle from an arbitrary origin,
- d**l** is the differential displacement of the particle, and is along the path the particle traverses,
- t is the time measured from an arbitrary reference instant.

The quantities **F**, **a**, and **l** are vectors, and Eqs. 12 and 13 relate both magnitudes and directions, just as do Eqs. 9, 10, and 11.

The only forces experienced by a particle moving in an evacuated tube are those caused by the fields of force, such as electric, magnetic or gravitational fields, that may be present. In this article it is supposed that no field other than an electrostatic one is present; therefore, Eqs. 9, 11, and 13 may be combined as

$$Q\boldsymbol{\mathcal{E}} = m\frac{d^2\mathbf{l}}{dt^2}, \tag{14}$$

or

$$\frac{d^2\mathbf{l}}{dt^2} = \frac{Q}{m}\boldsymbol{\mathcal{E}}, \tag{15}$$

and

$$\frac{d^2\mathbf{l}}{dt^2} = -\frac{Q}{m}\,\text{grad}\,E. \tag{16}$$

Equations 15 and 16 do not involve any co-ordinate system. They may be expressed, however, in terms of any desired co-ordinate system. If, for example, a set of rectangular co-ordinate axes is chosen, the equations may be used to express the relations among the components of the vectors along these axes. Thus, if $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$ are the components of $\boldsymbol{\mathcal{E}}$ along the x, y, and z axes, respectively, Eq. 15 becomes

$$\frac{d^2x}{dt^2} = \frac{Q}{m}\mathcal{E}_x, \qquad ▸[17]$$

$$\frac{d^2y}{dt^2} = \frac{Q}{m}\mathcal{E}_y, \qquad ▸[18]$$

$$\frac{d^2z}{dt^2} = \frac{Q}{m}\mathcal{E}_z. \qquad ▸[19]$$

Since the components of **grad** E along the co-ordinate axes are the rates of change of E with distance along these axes,

$$x \text{ component of } \mathbf{grad}\, E = \frac{\partial E}{\partial x}, \tag{20}$$

$$y \text{ component of } \mathbf{grad}\, E = \frac{\partial E}{\partial y}, \tag{21}$$

$$z \text{ component of } \mathbf{grad}\, E = \frac{\partial E}{\partial z}, \tag{22}$$

and

$$\frac{d^2x}{dt^2} = -\frac{Q}{m}\frac{\partial E}{\partial x}, \tag{23}$$

$$\frac{d^2y}{dt^2} = -\frac{Q}{m}\frac{\partial E}{\partial y}, \tag{24}$$

$$\frac{d^2z}{dt^2} = -\frac{Q}{m}\frac{\partial E}{\partial z}. \tag{25}$$

If the initial velocity and position of a charged particle and the potential distribution in the tube are known, it is possible to determine completely the motion of charged particles in electrostatic fields, provided the differential equations just derived can be solved. However, unless the field is uniform, at least one of the field components varies with the co-ordinates, and the equations are nonlinear. In addition, if the speed of the particle is a large fraction of the speed of light, the mass becomes a function of the speed of the particle, and the equations are again nonlinear. The solution of the nonlinear equations may often require the use of graphical, numerical, or mechanical methods.

An alternative and powerful attack on the problem of the motion of a charged particle in electrostatic fields, which yields much information about the motion, is the use of the principle of conservation of energy to derive a relation between the potential and the speed of a charged particle at any point. Let the particle travel from the point P_1 to the point P_2. The differential displacement of the particle along its path is $d\mathbf{l}$. Since the kinetic energy acquired by the particle equals the work done on the particle by the field,[10]

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{l}, \tag{26}$$

[10] For an explanation of the dot-product notation used in the integral of Eq. 26, see a textbook such as N. H. Frank, *Introduction to Electricity and Optics* (2nd ed.; New York: McGraw-Hill Book Company, Inc., 1950), 107.

where

v_2 is the speed of the particle at P_2,

v_1 is the speed of the particle at P_1.

In Eq. 26 and the equations derived from it, the assumption is made that the speed of the particle never exceeds a small fraction of the speed of light. If the speed of the particle is large enough, the mass becomes a function of the speed, in accordance with Eq. 4, and the kinetic energy is no longer given by $\frac{1}{2}mv^2$. The value of **F** from Eq. 9 may be substituted to give the result

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = Q\int_{P_1}^{P_2} \boldsymbol{\mathcal{E}} \cdot d\mathbf{l} = -[QE_2 - QE_1], \qquad [27]$$

where

E_2 is the potential at P_2,

E_1 is the potential at P_1.

Equation 27 may be written

$$\tfrac{1}{2}mv_1^2 + QE_1 = \tfrac{1}{2}mv_2^2 + QE_2\,, \qquad [28]$$

which states that the sum of the kinetic energy and the potential energy of the particle does not change during the motion. Equation 28 could have been written directly, since it is a statement of the principle of conservation of energy for a charged particle in an electrostatic field.

An alternative form of Eq. 28 is

$$v_2 = \sqrt{v_1^2 - 2\frac{Q}{m}(E_2 - E_1)}\,, \qquad ▶[29]$$

and this form may be used to find the speed of a particle at any point on its path if the speed at any one point is known. In particular, if the point P_1 is taken as the point at which the particle starts from rest, and if the potential of this point is chosen as the reference for potential, then v_1 and E_1 are zero. Since the point P_2 may be any point on the path of the particle, the subscripts may be dropped from the symbols relating to it to give the very useful relation

$$v = \sqrt{-2\frac{Q}{m}E} \qquad ▶[30]$$

for a particle that starts from rest at a point where the potential is zero.

From Eq. 30 it appears that the speed and the kinetic energy of a particle moving in an electrostatic field depend only upon the total potential through which the particle moves, and not upon the manner in which the potential varies along the entire path. It should be noted, however, that the direction of the particle velocity and the time required for the particle to move a given distance do depend upon the distribution of the electric field. These quantities cannot be determined without the use of information in addition to that contained in Eq. 30.

In the remainder of this article the differential equations and the relation between potential and velocity are used to determine the motion of charged particles in certain configurations of electrostatic fields which have plane symmetry and are of particular interest in electron tubes. It is fortunate that approximate plane or cylindrical symmetry exists in many practical electronic devices, because the symmetry makes determination of the electronic motion in them relatively easy.

3a. *Uniform Field; Zero Initial Velocity.* In Fig. 2a, k and p are the cathode and plate, respectively, of a simple electron tube. These electrodes are assumed to lie in parallel planes separated by a distance d, which is very small relative to the dimensions of the electrodes, so that they may be treated as infinite parallel planes. If a constant voltage is applied across these electrodes, the potential gradient and the field between the plates are constant in time and uniform in space, and are directed perpendicularly to the plates. The potential of the plate with respect to the cathode* is called e_b. Alternatively, e_b is the voltage rise from the cathode to the plate, or the voltage drop from the plate to the cathode. If e_b is positive, the vector $\mathcal{E}$ is directed from the plate to the cathode, and the potential gradient from the cathode to the plate; physically, the force on a positively charged particle between the electrodes is in a direction to move it toward the cathode. Because of its negative charge, however, an electron tends to move from cathode to anode.

The rectangular co-ordinate axes in Fig. 2a are drawn so that the origin is located in the plane of the cathode, and the x axis is perpendicular to the electrode surfaces. The potential distribution in the tube can therefore be described by the graph in Fig. 2b. Suppose that a charged particle is set free at the origin of co-ordinates in the surface of the cathode with zero initial velocity, and the equations of its motion are to be found. Under these conditions, $\mathcal{E}_x$ equals $-(e_b/d)$; and

* In general, in this book, constant voltages and currents are denoted by capital letters and variable voltages and currents by lower-case letters. The lower-case e_b is used here in preparation for a future use in which e_b becomes a variable.

since the field is along the x axis only, $\mathcal{E}_y$ and $\mathcal{E}_z$ are both zero. Thus either Eqs. 17, 18, and 19 or Eqs. 23, 24, and 25 become

$$\frac{d^2x}{dt^2} = -\frac{Q}{m}\frac{e_b}{d}, \tag{31}$$

$$\frac{d^2y}{dt^2} = 0, \tag{32}$$

$$\frac{d^2z}{dt^2} = 0. \tag{33}$$

In addition, x, y, z, dx/dt, dy/dt, and dz/dt are all zero when t is zero. Since the particle starts at rest and is not accelerated in the y or z

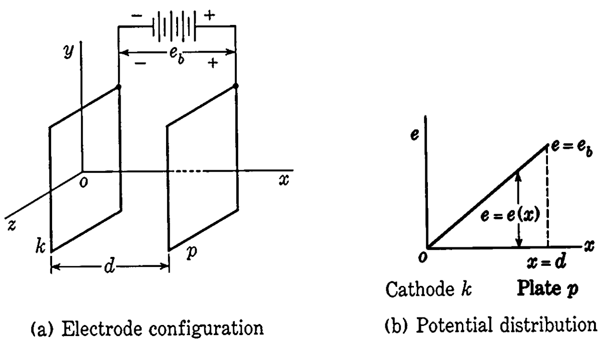


Fig. 2. Potential distribution between infinite, parallel-plane electrodes.

direction, its motion is confined to the direction along the x axis. By integration of Eq. 31 and use of the initial conditions to evaluate the constants of integration, the equations that describe the motion of the particle are found to be

$$v = -\frac{Q}{m}\frac{e_b}{d}t, \tag{34}$$

$$x = -\frac{1}{2}\frac{Q}{m}\frac{e_b}{d}t^2, \tag{35}$$

where v is the speed of the particle evaluated at any point on its path. In this integration it is assumed that the speed of the particle is never more than a small fraction of the speed of light, so that the mass of the particle may be considered a constant.

Equations 34 and 35 may be solved simultaneously to eliminate t and give the relation

$$v = \sqrt{-2\frac{Q}{m}e}, \qquad [36]$$

where

$$e = \frac{e_b}{d}x \qquad [37]$$

and is the potential at any point in the tube with respect to the cathode. It should be noticed that Eq. 36 agrees with Eq. 30 and might, in fact, have been obtained directly from that equation. Furthermore, the derivation of Eq. 30 is more general than that of Eq. 36 because it does not involve the assumption that the electric field is uniform. The speed with which the particle strikes the plate is

$$v_p = \sqrt{-2\frac{Q}{m}e_b}, \qquad [38]$$

and the kinetic energy of the particle as it reaches the plate is

$$\tfrac{1}{2}mv_p^2 = -Qe_b, \qquad [39]$$

which is the decrease in the potential energy of the particle that occurs as the particle moves across the tube. Thus, the increase of kinetic energy of the particle equals the decrease of potential energy and, in accordance with the principle of conservation of energy, the total energy of the particle is constant.

The time of transit from the cathode to the plate is obtained through solving Eq. 35 for t when x equals d, and is

$$t_{kp} = \frac{2d}{\sqrt{-2\frac{Q}{m}e_b}} = \frac{2d}{v_p}. \qquad ▸[40]$$

Alternatively, dividing the total distance traveled by the average speed will give this time of transit. Since the acceleration in the uniform field is constant and the particle starts from rest, the average speed is just half the final speed, and thus the same value for the transit time is obtained.

If e_b and Q are positive, negative values are obtained for v and x in Eqs. 34 and 35, and an imaginary value is obtained for v in Eq. 36. These apparently absurd results are explained by the physical fact that a positively charged particle will not leave a negative electrode to approach a positive electrode. If, however, the charged particle is an electron or other negatively charged particle, Q is itself negative; and if e_b is positive, v is real, x is positive, and the equations describe correctly the movement of the electron toward the plate.

3b. *Uniform Field; Initial Velocity in the Direction of the Field.* In some electronic devices the charged particles cannot be considered to start from rest at the cathode. Consideration of the initial velocity at the cathode is required, for example, in an analysis of the motion of an electron in a vacuum diode, as discussed in Ch. II, or in a multi-electrode device in which an electron set free at one electrode acquires a velocity by moving through a potential difference between one pair of electrodes and then enters the electric field between two other electrodes.

If a particle starts from the origin at the cathode in Fig. 2 with an initial velocity that is in the direction of the field and has a magnitude v_k, then, x, y, z, dy/dt, and dz/dt are zero when t is zero, but dx/dt equals v_k when t is zero. The equations of motion obtained by integration of Eqs. 23, 24, and 25 are then

$$\frac{d^2x}{dt^2} = -\frac{Q}{m}\frac{e_b}{d}, \qquad [41]$$

$$\frac{dx}{dt} = -\frac{Q}{m}\frac{e_b}{d}t + v_k, \qquad [42]$$

$$x = -\frac{1}{2}\frac{Q}{m}\frac{e_b}{d}t^2 + v_k t, \qquad [43]$$

$$y = 0, \qquad [44]$$

$$z = 0. \qquad [45]$$

The speed with which the particle strikes the plate, v_p, may be determined through application of the principle of conservation of energy. Accordingly, the kinetic energy of the particle as it strikes the plate equals the sum of the initial kinetic energy and that acquired from the field. Thus,

$$\tfrac{1}{2}mv_p^2 = -Qe_b + \tfrac{1}{2}mv_k^2, \qquad [46]$$

whence

$$v_p = \sqrt{-\frac{2Qe_b}{m} + v_k^2}. \qquad [47]$$

The time of transit from cathode to plate, t_{kp}, is the ratio of the distance to the average speed. Since the acceleration is uniform, the average speed is the average of the initial and final speeds. Hence,

$$t_{kp} = \frac{2d}{v_k + v_p}. \qquad [48]$$

3c. *Uniform Field; Any Initial Velocity.* If in the tube of Fig. 2 the initial velocity of the particle is considered to have an arbitrary direction, and if for further generality the origin of co-ordinates is supposed to have any location, the general equations for a charged particle moving in a uniform field aligned with the x axis are obtained. Under these generalized boundary conditions Eqs. 17, 18, and 19 yield by integration

$$\frac{dx}{dt} = \frac{Q}{m}\mathcal{E}t + C_1 , \tag{49}$$

$$\frac{dy}{dt} = C_2 , \tag{50}$$

$$\frac{dz}{dt} = C_3 , \tag{51}$$

and

$$x = \frac{1}{2}\frac{Q}{m}\mathcal{E}t^2 + C_1t + C_4 , \tag{52}$$

$$y = C_2t + C_5 , \tag{53}$$

$$z = C_3t + C_6 . \tag{54}$$

The constants of integration, C_1 through C_6, have values that depend upon two sets of boundary conditions, of which each set is expressible in terms of the three co-ordinates; for example, the boundary conditions are often the initial velocity and initial position of the particle.

Note that the motion of a charged particle in a uniform electrostatic field is strictly analogous to the motion of a material particle in a uniform gravitational field, since the acceleration imparted to the particle by either field is constant. The path of a charged particle, like the trajectory of a projectile, is in general a parabola. If the initial velocity is in the direction of the field, or is zero, the parabola degenerates into a straight line. This fact is seen at once if Eqs. 52, 53, and 54 are recognized as the parametric equations of a parabola. The equations of the parabola may be placed in a more commonly encountered form if the co-ordinate axes are so chosen that the x–y plane is in the direction determined by the field and the initial velocity, and the particle starts from a point in this plane. It is easy to show that the motion is then entirely in the x–y plane. The constants C_3 and C_6 become zero, and the motion is described by the displacements x and y. If the value of t given by Eq. 53 is substituted into Eq. 52, there results an expression of the form

$$x = C_7y^2 + C_8y + C_9 , \tag{55}$$

where C_7, C_8, and C_9 are constants. This form is recognized as the equation of a parabola.

4. Units for numerical computations; the electron volt

In the numerical solution of any practical problem involving the motion of charged particles in electrostatic and magnetostatic fields, the question of the units to be used always arises. Since the motion of the particle is essentially a problem in mechanics, it is desirable to use a system of units in which measurement of the motion, or the forces that give rise to the motion, is convenient, as well as one in which the electrical units are of convenient size. The international meter-kilogram-second (mks) system provides such units, and all the derived equations in this book hold when numerical values substituted into the equations for the quantities involved are expressed in the mks rationalized system or any other self-consistent rationalized system of units. However, with minor alterations* the equations also hold for any self-consistent unrationalized system, such as the cgs absolute electromagnetic (aem) system or the cgs absolute electrostatic (aes) system. These systems have been, and still are, used frequently in the literature of electronics, and familiarity with the commonly encountered conversion factors for converting from one system to another is desirable. A table of conversion factors is given in Appendix B.[11]

For illustration, consider the expression

$$F = \mathcal{E}Q = ma. \qquad [56]$$

In the three previously mentioned systems of units this equation may be written in the following three ways:

For the mks system, [57]

$$[F]_{newtons} = [\mathcal{E}]_{volts\ per\ meter}[Q]_{coulombs} = [m]_{kilograms}[a]_{meters\ per\ sec\ per\ sec}.$$

For the aes system, [58]

$$[F]_{dynes} = [\mathcal{E}]_{statvolts\ per\ cm}[Q]_{statcoulombs} = [m]_{grams}[a]_{cm\ per\ sec\ per\ sec}.$$

* See Appendix B for an explanation of the distinction between rationalized and unrationalized units. Substitution of $\varepsilon_v/(4\pi)$ for ε_v and $D/(4\pi)$ for D are the only changes necessary to convert the particular equations included in this book into the form for which unrationalized units are applicable. Only equations that involve ε_v and D explicitly need be changed. All others are suitable for either rationalized or unrationalized units.

[11] See also E. E. Staff, M.I.T., *Electric Circuits* (Cambridge, Massachusetts: The Technology Press of M.I.T.; New York: John Wiley & Sons, Inc., 1940), 754–756.

For the aem system, [59]

$$[F]_{dynes} = [\mathcal{E}]_{abvolts\ per\ cm}\,[Q]_{abcoulombs} = [m]_{grams}\,[a]_{cm\ per\ sec\ per\ sec}.$$

In Eqs. 57, 58, and 59, the units appended to the brackets around the terms are the three sets of units in which the quantities within the brackets may be expressed.

Frequently the quantities involved in a problem are known in a mixed system of units, and often the result is desired in another system of units. For example, electrical measuring instruments usually indicate potential differences in volts and currents in amperes, which are in the mks system. However, in electronic work some quantities, such as force and length, are often expressed in the cgs system, where the unit of force is the dyne and the unit of length is the centimeter. When the data are known in a hybrid system, conversion factors must be used before a useful numerical result can be obtained.

To illustrate the use of conversion factors, assume that the field intensity $\mathcal{E}$ is given in volts per centimeter, and the charge Q is given in coulombs. It may be desired to know the force F in dynes. Any convenient form of Eq. 56 involving units in which quantities are known or desired may be chosen as a starting point; for instance, Eq. 58 may be chosen as one containing the dyne. Thus

$$[F]_{dynes} = [\mathcal{E}]_{statvolts\ per\ cm}\,[Q]_{statcoulombs}. \qquad [60]$$

Any change in the units used for a quantity within a bracket, such as F, $\mathcal{E}$, or Q, can be made, provided the proper conversion factor is also included so that the units of the whole quantity within the bracket are not changed. For example,

$$[F]_{dynes} = \left[\mathcal{E}_{volts\ per\ cm} \times \frac{1}{300}\right]_{statvolts\ per\ cm} [Q_{coulombs} \times 3 \times 10^9]_{statcoulombs}. \qquad [61]$$

The names of the units outside the brackets can be dropped when new units are chosen for the quantities within the brackets, and the conversion factors can then be combined, giving the desired equation,

$$[F]_{dynes} = [\mathcal{E}]_{volts\ per\ cm}\,[Q]_{coulombs} \times 10^7. \qquad [62]$$

While at first sight this method may seem lengthy, it is sound and is often useful, especially when a number of quantities are involved and several of them must be converted to different units.

Publications in the field of electronics frequently use these mixed systems of units; hence, conversion factors of 10^7, 10^8, and 10^9 are often encountered in the equations. Consistent systems of units, however, are used for derived equations in this volume, except as specifically noted.

When the value for the ratio of the charge to the mass of the electron given in Eq. 3 is substituted in Eq. 30, the speed of an initially stationary electron is given in terms of the potential difference through which it moves, as

$$v = 5.94 \times 10^5 \sqrt{E} \text{ meters per second,} \qquad \blacktriangleright[63]$$

where E is in volts. This equation holds only for speeds small compared with that of light. The energy of the electron is then

$$\tfrac{1}{2}m_e v^2 = EQ_e \qquad [64]$$

$$= 1.60 \times 10^{-19} \times E \text{ joule.} \qquad [65]$$

Since for reasonable values of voltage the energy of the electron is extremely small, and since its energy is directly proportional to the potential difference through which it moves, it is convenient and customary to adopt as a unit of *energy* the *electron volt*, abbreviated to *ev*, which is the kinetic energy that an initially stationary electron acquires by moving through a potential difference of one volt. Thus

$$1 \text{ ev} = 1.60 \times 10^{-19} \text{ joule.} \qquad \blacktriangleright[66]$$

The electron volt serves as a convenient unit of energy for calculations involving the charge of an electron just as the joule, which might be called the coulomb volt of energy, serves for calculations involving coulombs of charge, or involving amperes.

5. Deflection of the Electron Beam in a Cathode-Ray Tube

One application of the analysis of the behavior of charged particles in an electrostatic field is in some types of electron-beam tubes. As a class such tubes include devices for many different purposes, as summarized in Fig. 3. They find extensive application, especially in television,[12] radar,[13] and in experimental studies of time-varying phenomena.[14] The oscilloscope tube in particular is indispensable for the

[12] V. K. Zworykin and G. A. Morton, *Television* (New York: John Wiley & Sons, Inc., 1940); D. G. Fink, *Television Engineering* (2nd ed.; New York: McGraw-Hill Book Company, Inc., 1952); Scott Helt, *Practical Television Engineering* (New York: Murray Hill Books, Inc., 1950).

[13] L. N. Ridenour, Editor, *Radar System Engineering,* Massachusetts Institute of Technology Radiation Laboratory Series, Vol. 1 (New York: McGraw-Hill Book Company, Inc., 1947); D. G. Fink, *Radar Engineering* (New York: McGraw-Hill Book Company, Inc., 1947).

[14] J. H. Ruiter, Jr., *Modern Oscilloscopes and Their Uses* (New York: Murray Hill Books, Inc., 1949); J. F. Ryder and S. D. Uslan, *Encyclopedia on Cathode-Ray Oscilloscopes and Their Uses* (New York: John F. Ryder Publisher, Inc., 1950).

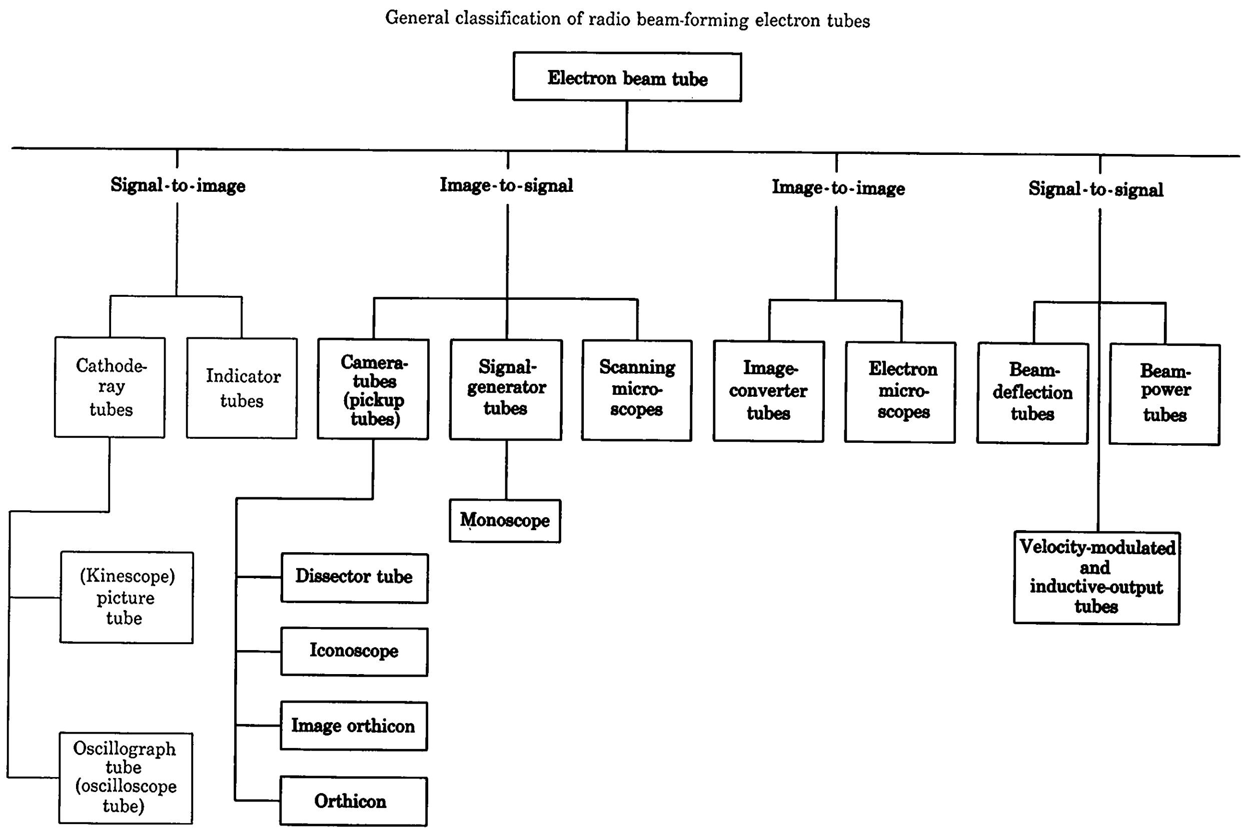


Fig. 3. General classification of electron-beam tubes. (This chart is taken from "Standards on Electron Tubes: Definitions of Terms, 1950," *I.R.E. Proc.*, *38* (1950), 428, with permission.)

study of cyclic and repetitive transient phenomena in the audible- and low radio-frequency range. High-speed oscilloscopes are available for the study of nonrepetitive transient phenomena having durations as short as a few millimicroseconds.

A photograph of a typical cathode-ray oscilloscope tube and its internal structure is shown in Fig. 4. The essential elements in this

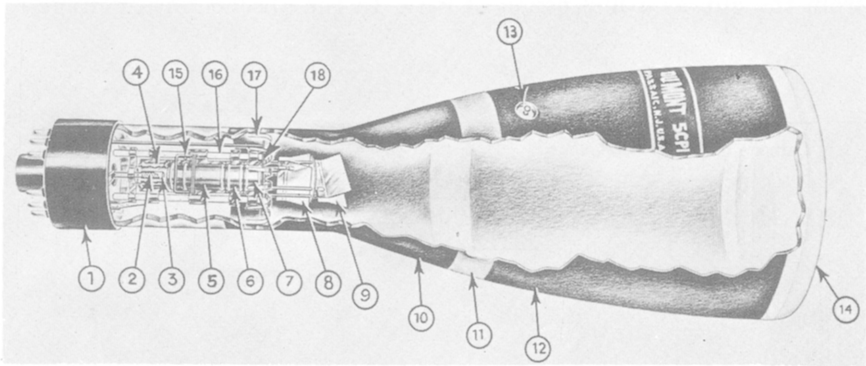


Fig. 4. Typical cathode-ray tube with electrostatic focusing and deflection. (*Courtesy Allen B. DuMont Laboratories, Inc.*)

1. Base
2. Heater
3. Cathode
4. Control grid (G)
5. Pre-accelerating electrode (connected internally to A_2)
6. Focusing electrode (A_1)
7. Accelerating electrode (A_2)
8. Deflection plate pair (D_3D_4)
9. Deflection plate pair (D_1D_2)
10. Conductive coating (connected internally to A_2)
11. Intensifier gap
12. Intensifier electrode (A_3)
13. A_3 terminal
14. Fluorescent screen
15. Getter
16. Ceramic gun supports
17. Mount support spider
18. Deflection plate structure support

tube are as illustrated in Fig. 5. They comprise: (a) a source of electrons, usually a heated cathode; (b) an arrangement of electrodes termed an *electron gun*, which serves to attract the electrons from the cathode, to focus them into a fine pencil or beam of rays, and to project them from the cathode down the major axis of the tube (hence the name *cathode ray*); (c) an arrangement of electrodes called deflecting plates, or of coils, as is discussed in Art. 7b, located beyond the gun and used to deflect the electron beam; and (d) a target or screen placed in a plane substantially perpendicular to the axis of the gun and coated with a phosphor such as willemite, calcium tungstate, or zinc silicate, which becomes luminescent when struck by the electrons. The whole assembly is enclosed in a glass or metal container having a glass window, and the container is evacuated to a pressure of about 10^{-9} atmosphere.

Electrons emitted by the hot cathode k are accelerated toward the final anode p of the electron gun under the influence of the field established by the anode-to-cathode voltage E_b. In the very simple electron gun of Fig. 5 this anode is shown as a disc with a hole in its center. Although some of the electrons strike an electrode in the gun and return to the cathode through the source of E_b, many of them emerge from the gun as a fine pencil of rays. In the absence of a voltage e_d

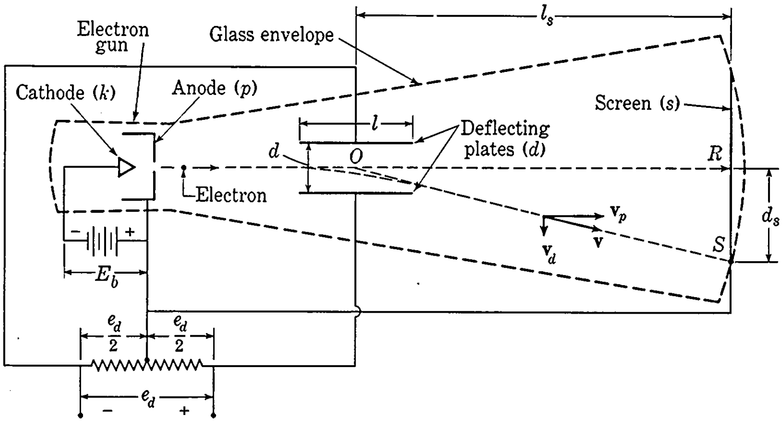


Fig. 5. Electrostatic deflection in a cathode-ray tube.

between the deflecting plates d, the region beyond the gun is essentially field free by virtue of the shielding effect of the metal container or of a conducting film on the inside of the glass container. The electrons then pass down the axis of the tube through the field-free space and strike the screen s at the point R with substantially the velocity they had on leaving the gun. (Secondary-emission phenomena invalidate this statement when E_b is below about 100 volts or above about 5,000 volts, the exact values depending on the screen material.[15] See Art. 12, Ch. II, for further discussion of this limitation.)

Since the initial velocity of the electrons as they leave the cathode corresponds at the most to one or two electron volts (see Art. 5, Ch. V), the velocity v_p of the electrons that emerge from the gun is practically that corresponding to the change in potential energy E_bQ_e. Thus, from Eq. 30,

$$v_p = \sqrt{2 \frac{Q_e}{m_e} E_b}\,. \qquad [67]$$

[15] W. B. Nottingham, "Electrical and Luminescent Properties of Phosphors under Electron Bombardment," *J. App. Phys.*, *10* (1939), 73–82.

When a voltage is applied between the deflecting plates, the electrons acquire a velocity component v_d perpendicular to the axis of the tube as they pass through the field in the region between the plates. Instead of striking the screen at the point R, they now strike it at some other point, say S. On the assumption that the fringing of the field at the edges of the deflecting plates can be neglected and that the plates are parallel to the axis of the tube, the field between the plates is uniform and perpendicular to the axis of the tube, and a relation can be found for the deflection d_s of the spot in terms of the anode-to-cathode voltage E_b, the deflecting voltage e_d, and the dimensions of the tube and electrodes.

The axial velocity component v_p of the electron is unchanged by the deflecting field, because the field acts in a direction perpendicular to the direction of that component. The time required for the electron to pass through the deflecting plates is therefore

$$t_d = l/v_p \, , \qquad [68]$$

where l is the length of the deflecting plates. During this time the electron experiences a constant sidewise acceleration given by $(Q_e/m_e)(e_d/d)$, where d is the separation of the deflecting plates. Since this acceleration is constant, the electron acquires a component of velocity v_d perpendicular to the axis of the tube given by

$$v_d = \frac{Q_e}{m_e} \frac{e_d}{v_p} \frac{l}{d} \, . \qquad [69]$$

Because of the uniform acceleration, the electron describes a parabolic path, and it emerges with the speed

$$v = \sqrt{v_p^2 + v_d^2} \, . \qquad [70]$$

After the electron leaves the region between the plates, its path is a straight line, since it is assumed then to be in a field-free space. If the straight-line path is projected backward, it can be shown to pass through the point O at the center of the plates. Then, by similar triangles,

$$\frac{d_s}{l_s} = \frac{v_d}{v_p} \, , \qquad [71]$$

where l_s is the distance from the center of the plates to the screen. Substitution of Eq. 69 in Eq. 71 gives

$$d_s = l_s \frac{Q_e}{m_e} \frac{e_d}{v_p^2} \frac{l}{d} \, , \qquad [72]$$

and elimination of v_p by means of Eq. 67 gives

$$d_s = \frac{1}{2} l_s \frac{l}{d} \frac{e_d}{E_b}. \qquad [73]$$

The sensitivity of the tube with respect to the deflection voltage is therefore

$$\text{Electrostatic deflection sensitivity} = \frac{d_s}{e_d} = \frac{1}{2} \frac{l}{d} \frac{l_s}{E_b}. \qquad ▶[74]$$

Since only the ratios of lengths and voltages are involved in Eq. 74, the equation holds for any units of length or voltage, provided corresponding quantities are measured in the same units. The sensitivity is evidently decreased as the accelerating voltage E_b is increased.

Often another set of deflecting plates is provided, these plates being so located along the axis of the tube that they deflect the spot as a function of a second deflecting voltage in a direction perpendicular to the deflection caused by the first set of plates. The path of the spot on the screen is then a function of the two deflecting voltages. If an unknown transient voltage is impressed on one set of plates, and a voltage of known waveform is impressed on the other, the path of the spot is a curve giving the unknown voltage as a function of the known. For example, one voltage may be made directly proportional to time, whereupon the path of the spot delineates the waveform of the second voltage as a function of time, and the tube may serve as an oscilloscope.

For the tube to be useful, the path of the spot must be visible or of such a nature that it can be photographed. Hence the anode-to-cathode voltage E_b must be large in order to transmit sufficient energy to the phosphor to make its luminescence visible or sufficiently bright to be photographed. On the other hand, the deflection sensitivity of the beam decreases as E_b is increased, and a compromise between sensitivity and luminosity must therefore be made. As a rule, the observation or recording of transient phenomena of short duration requires a large anode-to-cathode voltage and a consequent sacrifice of sensitivity.

In the foregoing analysis, it is assumed that the deflecting voltage e_d is constant; yet the real utility of the cathode-ray tube is in the study of phenomena involving a time variation of e_d. If the speed of the electrons is so great that the deflecting field does not change appreciably while each electron moves through it, the deflecting field is essentially an electrostatic field for each electron in the beam. However, although the electron is a very agile particle and the component of velocity v_p is large, the time variation of e_d is sometimes so rapid that, during transit through the region between the deflecting plates, the electron experiences a time variation of the deflecting field that is not negligible. In such circumstances the position of the spot on the screen is not a direct indication of the instantaneous phenomena, because the component of velocity v_d acquired by the electron is not the result of a single value of the deflecting voltage.

A guide to the conditions for which the electron velocity is important may be obtained from a numerical example. Suppose the accelerating voltage in the electron gun, E_b, is 1,000 volts and the length l of the deflecting plates is 0.02 meter. The speed of the electrons as determined from Eqs. 3 and 67 is

$$v_p = \sqrt{2 \times 1.76 \times 10^{11} \times 10^3} = 1.87 \times 10^7 \text{ meters per second,} \quad [75]$$

and the time required for the electron to travel the distance l through the deflecting plates is

$$t_d = \frac{0.02}{1.87 \times 10^7} = 1.07 \times 10^{-9} \text{ second.} \quad [76]$$

Although this time may at first appear to be small, it is not small compared with the duration of 10^{-7} or 10^{-8} second observed for many transient electrical phenomena, or the period of a radio-frequency wave used for deflection. Hence it follows that the anode-to-cathode voltage E_b must be large not only to insure a bright spot on the screen but also to insure accuracy in the display of phenomena that occur in such a short time, for a large velocity v_p and a correspondingly short time of transit through the deflecting plates are then required.[16]

The beam in a cathode-ray tube may be deflected by a magnetic field instead of an electric field. An analysis of magnetic deflection is given in Art. 7b.

6. Electron optics

A second application of the analysis of the behavior of charged particles in electrostatic fields is that of *electron optics.* The requirement of a cathode-ray oscilloscope tube as well as certain other electronic devices, such as electron microscopes, is that the surface concentration of electron current leaving a given plane in the device be reproduced on some other plane with a surface magnification greater or smaller than unity. Usually the magnification desired is less than unity in oscilloscope and television tubes but is greater than unity in electron

[16] F. M. Gager, "Cathode-Ray Electron Ballistics," *Communications, 18* (1938), 10; H. E. Hollmann, "Theoretical and Experimental Investigations of Electron Motions in Alternating Fields with the Aid of Ballistic Models," *I.R.E. Proc., 29* (1941), 70–79.

microscopes.[17] In oscilloscope and television tubes, the magnification is desired small in order that the electron concentration on the screen be intense and the luminescent spot be small and bright. A small, bright spot makes possible greater detail in the figure traced out on the screen and hence permits greater resolution of the data in electrical transient studies. In the electron microscope, the magnification is desired large for the same reason that it is desired large in optical microscopes.

Considerable attention has been given this problem during recent years, with the result that a new branch of science called electron optics[18] has appeared and has become of great importance in the design of electron microscopes and cathode-ray tubes. The term electron optics comes from the striking analogy that exists between the behavior of light when it passes through refracting media and the behavior of electrons when they pass through electrostatic or magnetostatic fields.

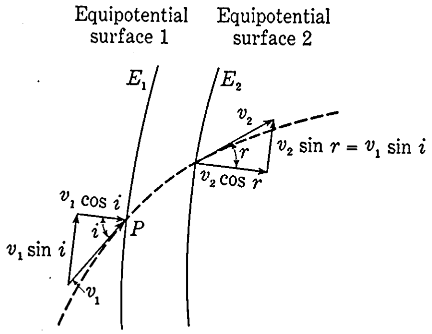


Fig. 6. Electron trajectory illustrating the optical analogy.

As an illustration of this analogy, consider that an infinitesimal region separates the two equipotential surfaces indicated in Fig. 6. Let the potential of surface 1 be E_1 and the potential of surface 2 be E_2; and let the datum of the potentials be the point where an electron that passes through point P on surface 1 had zero velocity. When this electron reaches point P, its speed is

$$v_1 = \sqrt{2 \frac{Q_e}{m_e} E_1}. \qquad [77]$$

If the angle between the path of the electron at P and the normal to

[17] R. P. Johnson, "Simple Electron Microscopes," *J. App. Phys.*, *9* (1938), 508–516; V. K. Zworykin, "Electron Optical Systems and Their Applications," *I.E.E.J.*, *79* (1936), 1–10; L. Marton, M. C. Banca, and J. F. Bender, "A New Electron Microscope," *RCA Rev.*, *5* (1940), 232–243.

[18] Several books that include the subject are: E. Brüche and O. Scherzer, *Geometrische Elektronenoptik* (Berlin: Julius Springer, 1934); I. G. Maloff and D. W. Epstein, *Electron Optics in Television* (New York: McGraw-Hill Book Company, Inc., 1938); L. M. Meyers, *Electron Optics* (New York: D. Van Nostrand Company, Inc., 1939); V. K. Zworykin and G. A. Morton, *Television* (New York: John Wiley & Sons, Inc., 1940); V. K. Zworykin, G. A. Morton, E. G. Ramberg, J. Hillier, and A. W. Vance, *Electron Optics and the Electron Microscope* (New York: John Wiley & Sons, Inc., 1945).

the equipotential surface is denoted by i, the velocity component of the electron perpendicular to the equipotential surface is $v_1 \cos i$. This velocity component is in the direction of the electric field at P. Similarly, the velocity component tangential to the equipotential surface and perpendicular to the electric field is $v_1 \sin i$ as indicated in the figure.

When the electron reaches the second equipotential surface its speed is

$$v_2 = \sqrt{2 \frac{Q_e}{m_e} E_2}. \qquad [78]$$

If the path of the electron where it passes through the second equipotential surface makes an angle denoted by r with the normal to the surface, the velocity component in a direction perpendicular to the surface is $v_2 \cos r$ and that tangential to the surface is $v_2 \sin r$. Although in general the equipotential surfaces are curved, they may be considered to be parallel planes in the region traversed by the electron if they are sufficiently close together. Thus, while the electron is passing through the infinitesimal region that separates the two surfaces, the velocity component $v_1 \sin i$ tangential to the surfaces does not change, because it is perpendicular to the direction of the force exerted on the electron by the electric field. Hence,

$$v_1 \sin i = v_2 \sin r, \qquad [79]$$

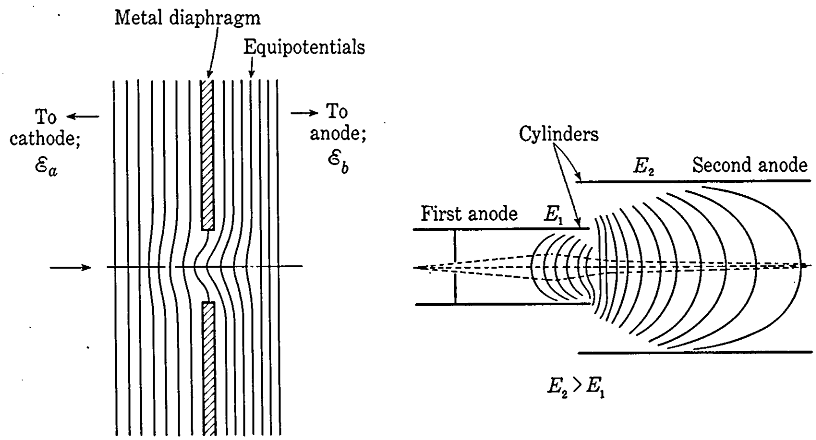
and

$$\frac{v_1}{v_2} = \frac{\sin r}{\sin i} = \frac{\sqrt{E_1}}{\sqrt{E_2}}. \qquad [80]$$

The angles i and r are analogous to the angles of incidence and refraction in optics, and Eq. 80 is equivalent to Snell's law if it is considered that $\sqrt{E_1}$ and $\sqrt{E_2}$ are analogous to the refractive indices η_1 and η_2 of the first and second media encountered by a light ray.

A complication arises when an attempt is made to formulate an analogy between light optics and electron optics in an actual problem. Instead of the uniform media with well-defined boundaries that are used in optical systems, the electric fields established by the charges on the electrodes in an electron tube present a continuously variable refracting medium for electrons. In most electron-optical systems, the field, even though it is a complicated function of the radius and the position along the axis, possesses approximate cylindrical symmetry. The distribution of such a field is not often readily calculable, but it can often be found by an experimental method involving models,[19] and a point-by-point calculation is then practical for the determination of the electron's path. Typical potential distributions for a space-charge-free aperture lens and a double-cylinder lens are shown in Figs. 7a and 7b.

A second complication sometimes of importance is that the electric field encountered by the particles consists not only of that set up by the charges on the electrodes but also of that resulting from the



(a) Aperture lens (b) Cylindrical lens*

Fig. 7. Electron lenses.*

charges of the particles in the interelectrode space. These modify the total field to a degree depending on the amount and distribution of the charge density in the space. This space-charge effect is frequently of appreciable importance near the cathode of an electron tube but is usually negligible in other regions.

The point-by-point solution for the path of an electron in an electron tube involves, first, a determination of the electric field intensity $\mathcal{E}$ throughout the region and, second, a determination of the motion of

[19] P. H. J. A. Kleynen, "The Motion of an Electron in a Two-Dimensional Electrostatic Field," *Philips Tech. Rev., 2* (1937), 338–345; E. D. McArthur, "Experimental Determination of Potential Distribution," *Electronics, 4* (June, 1932), 192–194; H. Salinger, "Tracing Electron Paths in Electric Fields," *Electronics, 10* (October, 1937), 50–54; D. Gabor, "Mechanical Tracer for Electron Trajectories," *Nature, 139* (1937), 373; V. K. Zworykin and J. A. Rajchman, "The Electrostatic Electron Multiplier," *I.R.E. Proc., 27* (1939), 558–566.

* This diagram is adapted from D. W. Epstein, "Electron Optical System of Two Cylinders as Applied to Cathode-Ray Tubes," *I.R.E. Proc., 24* (1936), Fig. 2, p. 1099, with permission.

an electron in this field. During the time interval Δt_1 in which the electron traverses a portion of its path Δl_1 from, say, point P_1 where l equals l_1 to point P_2 where l equals l_2, the change in its velocity component in the direction of the field is

$$\Delta v_1 = -\frac{Q_e}{m_e} \mathcal{E}_1 \, \Delta t_1 \, , \qquad [81]$$

where $\mathcal{E}_1$ is the electric field intensity encountered between P_1 and P_2, and is assumed to be constant over the interval. If the velocity of the electron at P_1 is resolved into the components v_n perpendicular and v_t parallel to the field, v_n does not change during the time interval Δt_1, and the speed of the electron at P_2 is

$$v_2 = \sqrt{v_n^2 + (v_t + \Delta v_1)^2} \, . \qquad [82]$$

During the interval Δt_1, the distance the electron moves in the direction perpendicular to the electric field is $v_n \, \Delta t_1$, and that in the direction of the field is $v_t \, \Delta t_1 + \frac{1}{2} \frac{-Q_e}{m_e} \mathcal{E}_1 \, \Delta t_1^2$. Thus the total distance moved is

$$\Delta l_1 = \sqrt{\left(v_t \, \Delta t_1 + \tfrac{1}{2} \frac{-Q_e}{m_e} \mathcal{E}_1 \, \Delta t_1^2\right)^2 + (v_n \, \Delta t_1)^2} \, . \qquad [83]$$

The position of P_2 relative to P_1 and the velocity at P_2 are thereby computed. During the next succeeding time interval Δt_2, the increment of path Δl_2 traversed by the electron may be computed by a repetition of the process; the velocity v_2 is resolved into components along and perpendicular to the electric field intensity $\mathcal{E}_2$ encountered between P_2 and P_3 and assumed to be constant over that interval. By means of this point-by-point method the paths of rays through the field can be computed through the use of field plots and computation charts, and the refractive properties of the field can be determined. Graphical and machine methods equivalent to the point-by-point method are also used for a determination of the trajectory.[20]

7. Motion of charged particles in magnetostatic fields

A charged particle in motion in a magnetostatic field experiences a force whose direction is perpendicular both to the direction of motion of the particle and to the direction of the field. The magnitude of the

[20] V. K. Zworykin and J. A. Rajchman, "The Electrostatic Electron Multiplier," *I.R.E. Proc.*, *27* (1939), 558–566; J. P. Blewett, G. Kron, F. J. Maginiss, H. A. Peterson, J. R. Whinnery, and H. W. Jamison, "Tracing of Electron Trajectories Using the Differential Analyzer," *I.R.E. Proc.*, *36* (1948), 69–83.

force is proportional to, first, the magnitude of the magnetic flux density; second, the charge on the particle; and, third, the component of the velocity of the particle that is perpendicular to the direction of the field. The relationship among the directions of the quantities for a positively charged particle is shown in Fig. 8, where **F**, **v**, and **B** are vectors representing the force, velocity, and magnetic flux density, respectively. The direction of the force on a positively charged particle is the same as the direction a right-hand screw having its axis perpendicular to both **v** and **B** would advance if it were rotated in the same angular direction that would rotate **v** into **B** through the smaller of the two angles between those vectors. Expressed symbolically, the magnitude and sense of the force is given by

$$F = BQv \sin \phi , \qquad ▶[84]$$

where

B is the magnitude of the magnetic flux density,

Q is the charge on the particle,

v is the speed of the particle,

ϕ is the smaller of the two angles between the direction of the magnetic field and the direction of the motion of the particle.

Since $\sin \phi$ is zero when the charge moves in the direction of the magnetic field, the force is then zero; but, when the charge moves in a direction perpendicular to the magnetic field, $\sin \phi$ is unity, and the force is a maximum given by BQv. Equation 84 indicates that for a negatively charged particle the sense of the force is negative. In other words, the direction of the force is opposite to that of the right-hand screw just explained, and is hence opposite to the direction shown in Fig. 8.

Since a current is equivalent to a movement of charged particles, the relationship expressed in Eq. 84 should be equivalent to that observed experimentally for a conductor carrying a current in a magnetic field. The equivalence may be shown as follows. When a conductor carrying a current i is placed in a magnetic field of flux density B, the conductor experiences a force per unit length given by the relation

$$F = Bi \sin \phi, \qquad [85]$$

where ϕ is the smaller of the two angles between the direction of the field and the direction of the current. The direction of this force is perpendicular to the plane containing the field vector **B** and the conductor. This relationship is also illustrated by Fig. 8. An electric current in a conductor of small, uniform cross section is equivalent to a stream of electric charges moving along the path of the current, all with the same speed, and is given by

$$i = Qnv, \qquad [86]$$

where

Q is the charge on each particle,

n is the number of charged particles per unit length of the path of i,

v is the speed of the particles,

nv is the number of charged particles that pass a given cross section of the path of i per unit time.

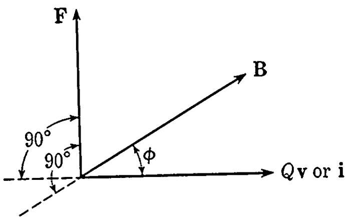


Fig. 8. Direction of the force exerted by a magnetic field on a current or a positively charged moving particle.

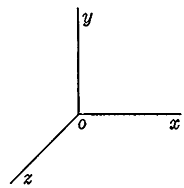


Fig. 9. Rectangular co-ordinate system for Eqs. 92, 93, and 94.

The substitution of i from Eq. 86 in Eq. 85 gives

$$F = BQnv \sin \phi \qquad [87]$$

for the force on a unit length of conductor, or the force on n particles. Since there is no reason why the force on one particle should be greater than that on another, the force on one particle is given by division of Eq. 87 by n; whence,

$$F = BQv \sin \phi \qquad [88]$$

for one particle,[21] an expression identical with Eq. 84.

[21] This derivation of Eq. 88 from the equation for the force on a current-carrying conductor applies only to the force on a single charged particle moving as a part of a uniform stream of charged particles. For a discussion of the force on a single moving charge, see W. R. Smythe, *Static and Dynamic Electricity* (2nd ed.; New York: McGraw-Hill Book Company, Inc., 1950), 565–567 and 574–576.

In the notation of vector analysis,[22] the vector force **F** as given by Eq. 84 is

$$\mathbf{F} = Q(\mathbf{v} \times \mathbf{B}), \tag{89}$$

and as given by Eq. 85 is*

$$\mathbf{F} = \mathbf{i} \times \mathbf{B}. \tag{90}$$

As a result of the force, acceleration of the charged particle takes place, and

$$\mathbf{F} = m\mathbf{a} = Q(\mathbf{v} \times \mathbf{B}). \tag{91}$$

In the rectangular co-ordinate system of Fig. 9, Eq. 91 becomes the set of three differential equations

$$\frac{d^2x}{dt^2} = \frac{Q}{m}\left[B_z \frac{dy}{dt} - B_y \frac{dz}{dt}\right], \qquad ▶[92]$$

$$\frac{d^2y}{dt^2} = \frac{Q}{m}\left[B_x \frac{dz}{dt} - B_z \frac{dx}{dt}\right], \qquad ▶[93]$$

$$\frac{d^2z}{dt^2} = \frac{Q}{m}\left[B_y \frac{dx}{dt} - B_x \frac{dy}{dt}\right], \qquad ▶[94]$$

where B_x, B_y, and B_z are the components of the magnetic flux density along the corresponding three co-ordinate axes. These equations are readily derived from the fact that, when the direction of motion of the charge is perpendicular to the direction of the magnetic field, the force is given by BQv and is perpendicular to these two directions as shown in Fig. 8. The foregoing equations hold in any consistent system of units. For example, in the mks system the force is given in newtons when Q is in coulombs, B is in webers per square meter, v is in meters per second, and x, y, and z are in meters.

An electric field exerts a force on a charged particle whether the particle is at rest or in motion, and the force is always in the direction of the field. On the other hand, according to Eq. 84 or Eq. 89, a magnetic field exerts a force on a charged particle only if the particle is in motion and the direction of the motion is not parallel to the direction of the field. If a force does act on a particle because of a magnetic field,

[22] For an introduction to the cross-product notation, see N. H. Frank, *Introduction to Electricity and Optics* (2nd ed.; New York: McGraw-Hill Book Company, Inc., 1950), 107–108.

* Current is ordinarily defined as a scalar quantity, and current density as a vector. Since the current considered here is supposed to be analogous to a stream of charged particles, the current-density vector associated with the current crossing a given cross section of path is directed along the path at every point on the cross section, and it is therefore possible to define the vector **i** whose magnitude is the current, and whose direction is that of the current path.

the force is perpendicular to the field. Since the force is also perpendicular to the direction of motion of the particle, a magnetic field can do no work on the particle. Thus it is not possible for a magnetic field to change either the kinetic energy or the speed of a charged particle. A magnetic field is capable of changing only the direction of motion of the particle.

In the remainder of this article, the equation for the force on a moving charged particle in a magnetic field, Eq. 84 or Eq. 89, together with the concept that a magnetic field can change the direction but not the magnitude of the velocity of a charged particle, is applied to determine the path of a charged particle in a uniform magnetic field for different initial-velocity conditions.

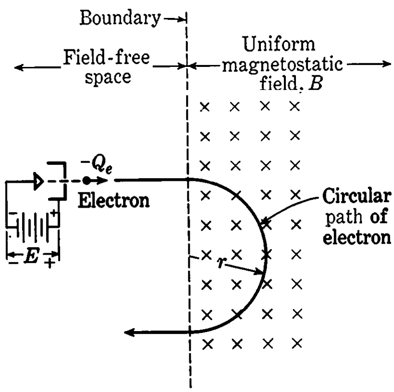


Fig. 10. Circular path of an electron that enters perpendicularly into a magnetostatic field.

7a. *Circular Path.* In Fig. 10 a charged particle, assumed to be an electron, is projected into a magnetostatic field having a direction perpendicular to the paper and directed into it. The force exerted by the field is always perpendicular to the direction of motion, and the particle therefore follows a curved path. The force producing the curvature is

$$F = BQv, \tag{95}$$

where v is the speed of the particle at any point. The acceleration caused by this force is, from kinematics, v^2/r, where r is the radius of curvature of the path. By Newton's second law,

$$F = ma = BQv = \frac{mv^2}{r}\,; \tag{96}$$

whence,

$$r = \frac{mv}{BQ}\,. \tag{97}$$

In this equation v is a constant speed unchanged by the magnetic field, and Q and m are constants. If, further, the magnetic flux density B is uniform everywhere along the path, the radius is then constant and the path is a circle.

If the particle acquires its speed v from motion through an accelerating electrostatic field of potential difference E, as is indicated in

Fig. 10, the speed is given by Eq. 30, and substitution of this relation in Eq. 97 gives

$$r = \frac{1}{B}\sqrt{-2\frac{m}{Q}E}. \qquad [98]$$

In Eq. 98, Q is negative if the particle carries a negative charge. Thus if the particle is an electron, the value of Q_e/m_e from Eq. 3 may be placed in Eq. 98, and the expression for r becomes

$$r = \frac{1}{B}\sqrt{\frac{2E}{1.76 \times 10^{11}}}$$

$$= 3.37 \times 10^{-6}\frac{\sqrt{E}}{B} \text{ meters,} \qquad ▶[99]$$

where E is in volts and B is in webers per square meter.

The relation given in Eq. 98 may be applied for the experimental measurement[23] of the ratio Q/m for electrons or ions. A beam of charged particles produced by a "gun" in the form of a filament and pierced anode, or by some other suitable source, is directed into a ring-shaped tube properly arranged in the magnetic field of Helmholtz coils so that the beam is bent in a circle along the axis of the tube. To determine the path of the beam, the charge that impinges on a collector electrode located in the tube at a fixed distance away from the source of the particles may be measured, sufficient gas may be left in the tube to render the path visible through excitation of the gas, or the beam may be allowed to fall on a fluorescent or photographic plate. The measurements of r, E, and B constitute data for the calculation of Q/m. The magnetic field of the earth may contribute to the bending of the beam, and this possibility must be taken into account in the calculation unless experimental precautions are taken to counteract it.

The principle that the radius depends on the mass-to-charge ratio of the particle as indicated by Eq. 98 has been applied in a device called a *mass spectrometer* for the identification and separation of particles having different values of this ratio. In this apparatus a beam composed of a mixture of ions having different ratios of mass to charge separates along paths of different radii when it passes through a magnetic field. The particle having the desired ratio is then collected on an electrode at the terminus of its path.

Isotopes, which are atoms that have different masses but identical chemical properties and are therefore different forms of the same

[23] A. J. Dempster, "Positive-Ray Analysis of Potassium, Calcium, and Zinc," *Phys. Rev.*, *20* (1922), 631–638; F. W. Aston, *Mass Spectra and Isotopes* (2nd ed.; London: Edward Arnold & Co., 1942).

chemical element, were first discovered and identified by use of a mass spectrometer. Separation of natural uranium to obtain the isotope uranium 235 was first accomplished in quantity by application of the same principle.[24]

The mass spectrometer also finds wide industrial use as a means for locating leaks in vacuum systems.[25] The device is connected to the evacuated system so as to sample the gas in it while a small jet of rare gas such as helium is passed over the outside surface. Presence

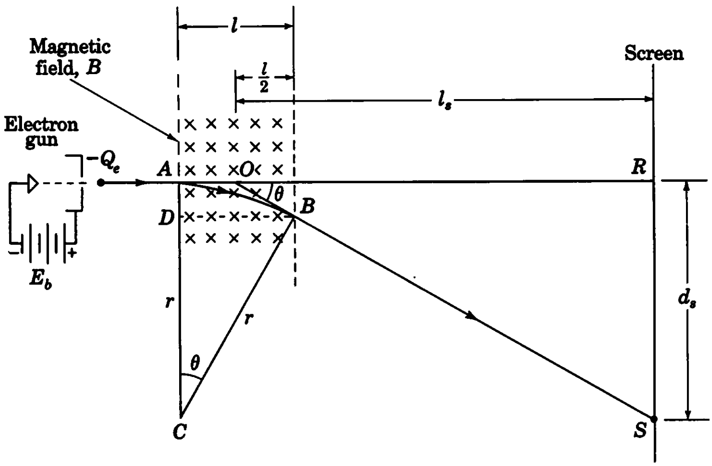


Fig. 11. Magnetic deflection of the beam in a cathode-ray tube.

of that particular gas in the mass-spectrometer indication shows that the leak has been reached by the jet.

7b. *Magnetic Deflection in a Cathode-Ray Tube.* A magnetic field is frequently used as a means of deflecting the electron beam in a cathode-ray tube. Usually the magnetic field is produced by the instantaneous current in a pair of coils located near the electron gun and oriented so as to direct the field perpendicular to the electron path, as is illustrated in Fig. 11.

If at any instant the field is uniform between the boundaries indicated and remains constant while the electrons pass through it, the beam, which enters the field at A, is deflected to B along a circular path, having a radius r. Thereafter it continues along a straight path

[24] H. D. Smyth, *Atomic Energy for Military Purposes* (Princeton, N.J.: Princeton University Press, 1947), 187–205. J. H. Pomeroy, "Electromagnetic Methods of Separating Isotopes," *The Science and Engineering of Nuclear Power, Vol. II*, Clark Goodman, Editor (Cambridge, Mass.: Addison-Wesley Press, Inc., 1948), Appendix A-2, 301–306.

[25] A. O. Nier, C. M. Stevens, A. Hustrulid, and T. A. Abbot, "Mass Spectrometer for Leak Detection," *J. App. Phys.*, *18* (1947), 30–48.

to produce a spot on the screen at the deflected position S. The circular path AB subtends an angle θ at C. The triangle SOR formed by the screen, the undeflected path, and the deflected path projected backward to O is similar to the triangle BCD because their respective sides are perpendicular. Hence, the angle at O is also θ, and

$$\frac{d_s}{SO} = \frac{l}{r}. \qquad [100]$$

For small values of θ, point O lies at the center of l as is indicated on the diagram, and SO is approximately equal to l_s, the distance from the center point to the screen. When the deflection is small, therefore, its value as given by Eq. 100 is

$$d_s = \frac{ll_s}{r}. \qquad [101]$$

Substitution for r from Eq. 98 with the values of Q and m for an electron gives

$$\text{Magnetic deflection sensitivity} = \frac{d_s}{B} = ll_s\sqrt{Q_e/(2m_e)}\,\frac{1}{\sqrt{E_b}}, \qquad [102]$$

where E_b is the accelerating voltage in the electron gun. Thus the instantaneous deflection is directly proportional to the magnetic flux density B, and, in turn, to the instantaneous current in the coils that produce the field.

Note that, for magnetic deflection, the sensitivity is inversely proportional to $\sqrt{E_b}$. For this reason, magnetic deflection is advantageous in tubes with high accelerating voltages. It is used extensively in television and radar applications.

Magnetic deflection of the beam through large angles requires a more complete analysis than the foregoing because the point O is then not exactly at the center of the field and the distance SO is not closely approximated by l_s. In any practical application, however, the distance l is indefinite because the magnetic field does not have sharp boundaries. Hence, use of a more exact expression than Eq. 102 is seldom justified.

7c. *Cyclotron, Betatron, and Synchrotron.* The angular velocity (about the center of curvature of its path) of a particle moving with a velocity v along a path of radius of curvature r is

$$\frac{d\theta}{dt} = \frac{v}{r}. \qquad [103]$$

From Eq. 97, this angular velocity for a moving electron in a magnetic field becomes

$$\frac{d\theta}{dt} = \frac{v}{r} = \frac{BQ}{m}. \qquad \blacktriangleright[104]$$

The angular velocity is therefore *independent* of the translational velocity. Thus, a fast-moving particle, which travels in a path of large radius of curvature in a given uniform magnetostatic field in accordance with Eq. 97, requires the same time for a complete revolution as

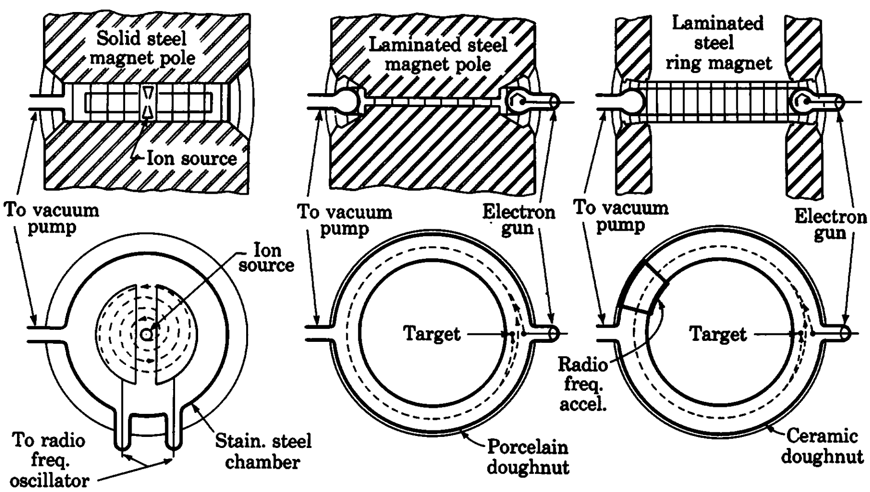


Fig. 12. Schematic diagrams of the three principal types of cyclic accelerators for elementary particles.*

does a slow-moving particle, which travels in a path of small radius of curvature.

This principle is applied in the *cyclotron*, in which it is used to impart high kinetic energy to charged particles.[26] Figure 12 shows a schematic diagram of the cyclotron. A short cylindrical tank, cut in two along a diameter, forms two metals *dees*—semicylindrical chambers resembling the letter **D** in shape. The dees are insulated from each other in a vacuum chamber located between the poles of a powerful electro-

* This diagram is adapted from I. A. Getting, "Artificial Cosmic Radiation," *The Technology Review, 50* (June, 1948), 436, edited at the Massachusetts Institute of Technology, with permission.

[26] E. O. Lawrence and M. S. Livingston, "A Method for Producing High-Speed Hydrogen Ions, without the Use of High Voltages (Abstract)," *Phys. Rev., 37* (1931), 1707; M. S. Livingston, "The Cyclotron," *J. App. Phys., 15* (1944), 2-19, 128-147; M. S. Livingston, "Particle Accelerators," *Advances in Electronics, Vol. I*, L. Marton, Editor (New York: Academic Press, Inc., 1948), 269–316.

magnet weighing many tons in most cyclotrons, and the direction of the magnetic field is perpendicular to the semicircular faces of the dees. A high-frequency oscillator is connected to the dees, and a source of charged particles such as protons or deuterons is provided near their center as shown on the diagram. When a charged particle leaves the source it is accelerated into the dees in one direction or the other by the electric field set up by the oscillator. Inside the dees, the electric field is small, because it is concentrated mainly across the gap, and the charged particle travels in a circular path at constant speed under the influence of the magnetic field. The frequency of the oscillator and the magnetic field strength are so adjusted that, when the charged particle again reaches the gap, the electric field has reversed. The electric field therefore gives the charged particle a second acceleration across the gap, adding to its speed. The charged particle then travels a path of larger radius through the opposite dee, but reaches the gap after the same time interval. During this interval the electric field has again reversed and is in a direction to accelerate the particle further. This process is repeated many times until finally the charged particle travels in a path near the periphery of the dee. As long as the speed of the particle remains so low that the mass stays essentially constant, a fixed-frequency oscillator can be used because the time of a revolution is constant. Subjected to a hundred or more revolutions with an increase in speed corresponding to several thousand volts twice each revolution, the particles can be given energies corresponding to several million volts, yet the voltage of the source need be only a few thousand volts. The particles are finally led off through a side tube to a target for experimental purposes, such as the nuclear disintegration of elements.

When the speed of the particles such as protons in a cyclotron exceeds that corresponding to energies of 10 or 20 million electron volts, the mass increases appreciably, and the angular velocity decreases as the energy increases. An equilibrium is reached at which the particles cross the gaps at essentially the instants of reversal of the electric field. The particles then receive no further energy from the field and hence travel in a path of constant radius. If the frequency is then decreased, however, the radius and the mass continue to increase in such a manner as to maintain the equilibrium. A cyclotron with such a changing frequency is called a *frequency-modulated cyclotron* or *synchrocyclotron*.[27] It has produced particles having energies of 400 million electron volts.

[27] W. M. Brobeck, E. O. Lawrence, K. R. MacKenzie, E. M. McMillan, R. Serber, D. C. Sewell, K. M. Simpson, and R. L. Thornton, "Initial Performance of the 184-inch Cyclotron of the University of California," *Phys. Rev.*, *71* (1947), 449–450.

Magnetic induction, rather than a radio-frequency electric field, is used for accelerating electrons in a device also illustrated in Fig. 12 called a *betatron*.[28] A changing magnetic flux in it induces a uniform electromotive force tangentially along a circular path for the electrons, and accelerates them to high energies. The electrons are held in an orbit of fixed radius by a second changing magnetic field located at the orbit but perpendicular to it. The flux density of the second field is maintained equal to one-half the average flux density inside the orbit. An evacuated annular chamber in the shape of a hollow doughnut surrounds the circular path. The magnetic flux for inducing the electromotive force along the path is produced in a laminated steel electromagnet by an alternating or pulsed power source, and the second magnetic field for guiding the electrons is produced in an air gap in the same or a different electromagnet excited by the same source. Bunches of electrons released in the chamber in synchronism with the supply travel many times around the chamber, and gain a few hundred electron volts of energy each time around. Finally the magnetic field is suddenly altered so as to cause the electrons to emerge with energies of more than 100 million electron volts.

The *synchrotron*[29] shown in Fig. 12 is a particle accelerator that utilizes some of the principles of both the cyclotron and the betatron. For accelerating electrons, the machine first acts as a betatron while imparting an initial energy to the electrons. After they reach energies of a few million electron volts, so that their speed is near that of light, additional energy is given them by a radio-frequency electric field existing between electrodes situated along their path in the doughnut-shaped chamber. The frequency of the electric field can be constant because the electrons continue to travel at essentially the speed of light despite further increase in energy. Thus their angular velocity around the center of their orbit is essentially constant. The guiding magnetic field at their orbit must continue to increase in proportion to their mass, however. Increase of the magnetic field inside the orbit is no longer necessary once the betatron action is complete, hence the steel inside the orbit may be allowed to saturate magnetically. The weight and cost of the machine may therefore be reduced below the corresponding values for a betatron. Electron energies of more than 300 million electron volts are produced by electron synchrotrons.

[28] D. W. Kerst, "The Acceleration of Electrons by Magnetic Induction," *Phys. Rev.*, *60* (1941), 47–53; E. E. Charlton and W. F. Westendorf, "A 100-Million Volt Induction Electron Accelerator," *J. App. Phys.*, *16* (1945), 581–593.

[29] V. Veksler, "A New Method of Acceleration of Relativistic Particles," *J. Phys. U.S.S.R.*, *9* (1945), 153–158; E. M. McMillan, "The Synchrotron—A Proposed High Energy Particle Accelerator," *Phys. Rev.*, *68* (1945), 143–144; E. M. McMillan, "The Origin of the Synchrotron," *Phys. Rev.*, *69* (1946), 534.

A synchrotron for accelerating protons must include a changing radio frequency as well as a changing magnetic field at the orbit, because an auxiliary means for accelerating protons to speeds near that of light before injection for synchrotron action is not available. Increase of the frequency by a factor of ten while the guiding magnetic field increases from zero to about 10,000 gausses is expected to produce proton energies of several billion electron volts. Such a *proton synchrotron*[30] is hence also known as a *bevatron* from the initials *bev* for billion electron volts.

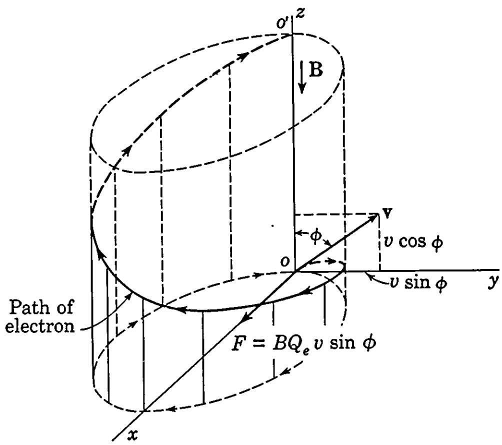


Fig. 13. Helical path of a moving electron in a uniform magnetostatic field.

7d. *Helical Path.* In each of the foregoing illustrative examples a charged particle moves in a direction perpendicular to that of the magnetic field. If the particle has a constant speed but does not move perpendicularly into the magnetic field, it describes a helical path.[31] This is shown in Fig. 13, where a particle, assumed in the figure to be an electron, starting from the origin has a velocity **v**, which, at the initial instant, lies in the y–z plane and makes an angle ϕ with the z axis. Since Q is negative for this diagram, the angle ϕ on it has the same significance as that between the magnetic flux density **B** and Q**v** on Fig. 8. The component of velocity of the particle parallel to the field and along the z axis, $v \cos \phi$, is unaltered by the field. Hence the

[30] M. S. Livingston, J. P. Blewett, G. K. Green, and L. J. Haworth, "Design Study for a Three-Bev Proton Accelerator," *R.S.I.*, *21* (1950), 7–22; E. J. Lofgren, "Berkeley Proton-Synchrotron," *Science*, *11* (March 25, 1950), 295–300.

[31] H. Busch, "Eine neue Methode zur e/m Bestimmung," *Phys. Zeits.*, *23* (1922), 438–441; H. Busch, "Berechnung der Bahn von Kathodenstrahlen im axialsymmetrischen elektromagnetischen Felde," *Ann. d. Phys.*, *81* (1926), 974–993.

particle continues to move in the direction of the z axis with a speed $v \cos \phi$. However, the velocity component $v \sin \phi$ gives rise to a force on the particle, and, although the magnitude of this component remains constant, its direction is altered continuously. By Eq. 97, the projection of the path in the x–y plane is a circle whose radius is

$$r = \frac{mv \sin \phi}{BQ} \qquad \blacktriangleright[105]$$

The time of one revolution of the projection is

$$t_0 = \frac{2\pi r}{v \sin \phi} = \frac{2\pi m}{BQ}. \qquad [106]$$

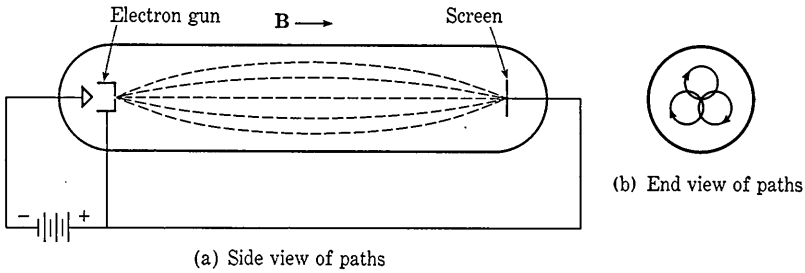


Fig. 14. Focusing by a longitudinal magnetic field.

The superposition of the uniform circular motion in the plane normal to **B** and the uniform translational motion parallel to **B** is the condition for a helical path. The vertical distance covered in one revolution is the pitch of the helix, which is given by

$$\text{Pitch} = t_0 v \cos \phi = \frac{2\pi mv \cos \phi}{BQ}. \qquad \blacktriangleright[107]$$

7e. *Magnetic Focusing.* The helical paths described above are sometimes utilized for focusing cathode-ray tubes and high-voltage x-ray tubes, and are typical of the trajectories in some magnetic electron lenses.[32] Figure 14 shows the method of focusing a divergent beam of electrons produced by an electron gun. The divergent electrons in the beam are made to follow helical paths by the magnetic field directed along the axis of the tube. The electrons follow paths whose projections on a plane perpendicular to the axis of the tube are circles

[32] V. K. Zworykin and G. A. Morton, *Television* (New York: John Wiley & Sons, Inc., 1940), 117–120; C. J. Davisson and C. J. Calbick, "Electron Lenses," *Phys. Rev.*, *38* (1931), 585; and *42* (1932), 580.

that pass through the axis, as is shown in Fig. 14b. The time of a revolution is independent of the angle of divergence, and, if this angle is small enough for its cosine to be essentially unity, all the electrons have the same translational velocity parallel to the axis of the tube. Thus all the electrons are at the same phase of their revolution at any distance from the source. By adjustment of the magnetic field, which may be produced by a solenoid, all the electrons can be allowed to complete one or any integral number of revolutions during their travel down the tube, and they then focus at the screen.

8. Motion of charged particles in concurrent electrostatic and magnetostatic fields

When electrostatic and magnetostatic fields simultaneously influence the motion of charged particles, the net force on the particle is the sum of the electrostatic and magnetostatic forces which would appear if each field acted independently. Both forces may be represented as vector quantities; hence, in vector notation, the net force is

$$\mathbf{F} = Q[\mathcal{E} + (\mathbf{v} \times \mathbf{B})] = m\mathbf{a}, \qquad [108]$$

this expression being the sum of Eqs. 9 and 89. On resolution of the vectors into components along the axes of the rectangular co-ordinate system of Fig. 9, Eq. 108 becomes the set of three equations:

$$\frac{d^2x}{dt^2} = \frac{Q}{m}\left[\mathcal{E}_x + B_z\frac{dy}{dt} - B_y\frac{dz}{dt}\right], \qquad ▸[109]$$

$$\frac{d^2y}{dt^2} = \frac{Q}{m}\left[\mathcal{E}_y + B_x\frac{dz}{dt} - B_z\frac{dx}{dt}\right], \qquad ▸[110]$$

$$\frac{d^2z}{dt^2} = \frac{Q}{m}\left[\mathcal{E}_z + B_y\frac{dx}{dt} - B_x\frac{dy}{dt}\right]. \qquad ▸[111]$$

These equations completely describe the motion of a particle in combined electrostatic and magnetostatic fields. Their solution, however, is a simple one only when special field configurations having symmetry are involved.

Since, as is shown in Art. 7, the component of force on a moving charged particle caused by a magnetic field is always perpendicular to the direction of motion, the magnetic field cannot alter the kinetic energy of the particle. If a charged particle moves in a combined electrostatic and magnetostatic field, the changes in its kinetic energy and speed are the result of the electric field alone, and Eqs. 29 and 30 are applicable just as though the magnetic field were not present. The paths in concurrent fields are not, in general, circles, since the speed of a particle is altered as the particle moves through the electric field. A discussion of several important examples of such motion follows.

8a. *Motion in Parallel Fields.* Perhaps the simplest example of motion in combined fields occurs when the fields are parallel. A particle liberated with zero velocity is then accelerated along the electric field, but experiences no force from the magnetic field, because the particle has no velocity component perpendicular to that field. This

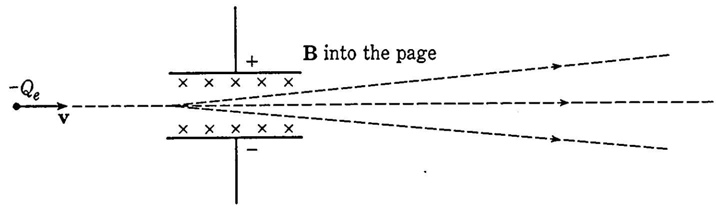


Fig. 15. Arrangement for measurement of particle velocity.

situation also holds if the particle has an initial velocity parallel to the direction of the fields, but not if the initial velocity has a component perpendicular to them.

If the particle has an initial velocity component perpendicular to the parallel fields, the path that results is a "helix" whose pitch changes with time as the particle is accelerated in the direction of the electric field. The diameter of the "helix," however, is constant, because neither of the fields can contribute energy to the component of velocity perpendicular to them.

8b. *Measurement of Particle Velocity.* When the electric and magnetic fields are perpendicular to each other, several useful forms of trajectory arise. One of these forms occurs when particles (for example, the electrons in the beam of a cathode-ray tube) are deflected simultaneously by both fields. In Fig. 15 a stream of particles having a velocity **v** enters the evacuated region between the parallel deflection plates in a direction parallel to the plane of the plates. A uniform electric field intensity $\mathcal{E}$ and a uniform magnetic flux density **B** are produced in the region between the plates, and are so oriented that they tend to deflect the beam in opposite directions; that is, the directions of **v**, **B**, and $\mathcal{E}$ are mutually perpendicular. If the magnitudes of the fields are so adjusted that the forces they exert on the particles are equal, the particles pass through the region without deflection from their original path. For this condition,

$$\mathcal{E}Q + Q(\mathbf{v} \times \mathbf{B}) = 0, \qquad [112]$$

or

$$\mathcal{E}Q = QvB, \qquad [113]$$

since the fields are perpendicular. Hence,

$$v = \mathcal{E}/B. \qquad [114]$$

Since $\mathcal{E}$ and B can be determined, this arrangement of the fields provides a method of measurement of the velocity of the particles, and it was so used in early experiments on the properties of electrons.[33]

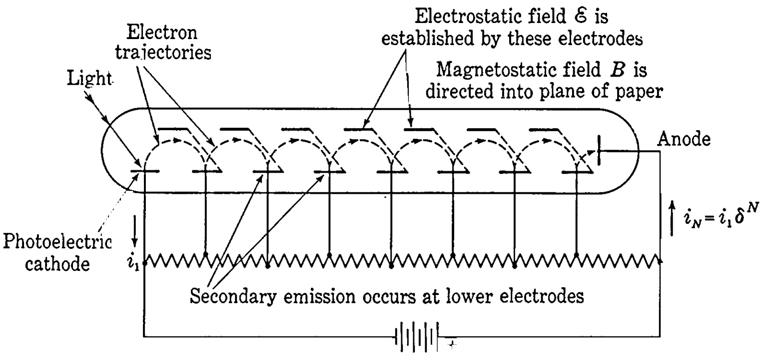


Fig. 16. A secondary-emission electron-multiplier tube.

Special precautions are necessary to limit the fields effectively to the region between the deflecting plates.

8c. *Secondary-Emission Electron Multiplier.* An important engineering application of a particular form of electron trajectory encountered in crossed electrostatic and electromagnetic fields is found in an early form of a device called an *electron multiplier*.[34] The arrangement of the electrodes and the trajectory of the electrons are shown in Fig. 16. The dotted straight lines show connections between the upper and lower parallel plane electrodes. The direct-voltage source and the voltage divider create an electric field in the tube that tends to accelerate electrons from the cathode on the left to the anode on the right through the set of plates. An electron emitted by the action of light on the photoelectric cathode is drawn toward the first upper plate by the electric field. However, a magnetostatic field set up by an external coil or permanent magnet is directed perpendicularly to the electric field, that is, perpendicularly to and into the page of the

[33] J. J. Thomson, "Cathode Rays," *Phil. Mag.*, *44* (1897), 293–316.

[34] V. K. Zworykin, G. A. Morton, and L. Malter, "The Secondary Emission Multiplier —A New Electronic Device," *I.R.E. Proc.*, *24* (1936), 351–375.

diagram. The path of the electron is therefore bent until it hits the second lower plate. This plate and all the other lower horizontal plates have specially prepared surfaces so that, when one electron strikes them with considerable velocity, several electrons are emitted by the process of secondary emission discussed in Art. 12, Ch. II. These *secondary electrons* start toward the second upper plate, but their paths also are bent by the magnetic field, and they strike the third lower plate, where each causes the emission of several secondary electrons. This process is repeated in several similar stages down the length of the tube, the number of electrons in the path increasing in each stage until finally the stream of electrons is made to strike the anode and is led to the external circuit. If the ratio of the number of secondary electrons that leave a plate to the number of primary ones that strike it is δ, and if the device has N stages, the current at the anode is δ^N times the current that leaves the photoelectric cathode. The current is thereby multiplied in the tube. By this means very sensitive photoelectric devices can be constructed.

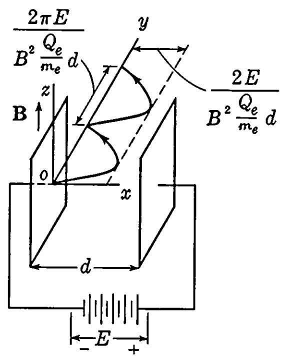


Fig. 17. Cycloidal motion in crossed fields.

The electric field in the electrode configuration of Fig. 16 is not uniform in magnitude or direction; hence an exact derivation of the trajectory of an electron is difficult. However, the trajectory is somewhat similar to that for an electron released with zero initial velocity at the negative electrode of the parallel-plane configuration of Fig. 17, where a uniform magnetic flux density **B** exists in a direction perpendicular to that of the electric field. By adjustment of the magnetic field strength, the electron path can be curved to such an extent that the electron does not reach the positive electrode.

The trajectory of the electron in Fig. 17 is conveniently described by reference to the rectangular co-ordinate axes x, y, and z oriented as shown. Then, if the electrodes are infinite in extent,

$$\mathcal{E}_x = -E/d, \qquad [115]$$

where E is the potential of the right-hand electrode with respect to the left-hand electrode and d is the distance between the electrodes. Also,

$$B_z = B, \qquad [116]$$

and all other components of the crossed electrostatic and magnetostatic fields are zero. Equations 109, 110, and 111 apply in this co-ordinate system, and, upon substitution of the above conditions and

$-(Q_e/m_e)$ for the charge-to-mass ratio of the electron, they become

$$\frac{d^2x}{dt^2} = -\frac{Q_e}{m_e}\left[-\frac{E}{d} + B\frac{dy}{dt}\right], \quad [117]$$

$$\frac{d^2y}{dt^2} = -\frac{Q_e}{m_e}\left[-B\frac{dx}{dt}\right], \quad [118]$$

$$\frac{d^2z}{dt^2} = 0. \quad [119]$$

From Eq. 119 it follows that there is no force on the electron in the z direction; hence the electron moves only in the x–y plane. The position of the electron may therefore be expressed in the notation of complex numbers.* Thus if

$$\boldsymbol{u} \equiv x + jy, \quad [120]$$

Eqs. 117 and 118 may then be combined to express the motion in terms of this single variable, for multiplication of Eq. 118 by j and addition of it to Eq. 117 give

$$\frac{d^2x}{dt^2} + j\frac{d^2y}{dt^2} = \frac{Q_e}{m_e}\left[\frac{E}{d} - B\frac{dy}{dt} + jB\frac{dx}{dt}\right]; \quad [121]$$

whence,

$$\frac{d^2\boldsymbol{u}}{dt^2} = \frac{Q_e}{m_e}\left[\frac{E}{d} + jB\frac{d\boldsymbol{u}}{dt}\right], \quad [122]$$

or

$$\frac{d^2\boldsymbol{u}}{dt^2} - jB\frac{Q_e}{m_e}\frac{d\boldsymbol{u}}{dt} = \frac{Q_e}{m_e}\frac{E}{d}. \quad [123]$$

The solution† of this differential equation subject to the initial conditions that at

$$t = 0, \ \ \boldsymbol{u} = 0, \text{ and } v = 0 \text{ or } \frac{d\boldsymbol{u}}{dt} = 0, \quad [124]$$

is

$$\boldsymbol{u} = j\frac{E}{Bd}\left[t - j\frac{1}{B(Q_e/m_e)}\left(1 - \epsilon^{jB(Q_e/m_e)t}\right)\right]. \quad [125]$$

* ***Boldface italic*** (slanting) type is used for letters denoting complex numbers.

† This solution may be recognized as similar in form to that for the charge in the problem of an electric circuit containing resistance and inductance in series and subject to a suddenly applied electromotive force E. Thus, if

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = E,$$

the current is

$$i = \frac{dq}{dt} = \frac{E}{R}\left(1 - \epsilon^{-(R/L)t}\right),$$

and by integration,

$$q = \frac{E}{R}\left[t - \frac{L}{R}\left(1 - \epsilon^{-(R/L)t}\right)\right].$$

Solutions for the variables x and y contained in $\boldsymbol{u}$ may be obtained from Eq. 125 through expansion of both sides into their real and imaginary parts. As a first step, the equation may be put in the form

$$\boldsymbol{u} = \frac{E}{B^2(Q_e/m_e)d}\left[jB\frac{Q_e}{m_e}t + \left(1 - \epsilon^{jB(Q_e/m_e)t}\right)\right]. \qquad [126]$$

Expansion of the exponential term into its equivalent trigonometric expressions gives

$$\boldsymbol{u} = x + jy$$
$$= \frac{E}{B^2(Q_e/m_e)d}\left[jB\frac{Q_e}{m_e}t + \left(1 - \cos B\frac{Q_e}{m_e}t - j\sin B\frac{Q_e}{m_e}t\right)\right]. \qquad [127]$$

This equation of complex numbers is equivalent to two equations of real numbers given by equating separately the reals and the imaginaries on the two sides of the equation. Thus

$$x = \frac{E}{B^2(Q_e/m_e)d}\left[1 - \cos B\frac{Q_e}{m_e}t\right], \qquad ▶[128]$$

and

$$y = \frac{E}{B^2(Q_e/m_e)d}\left[B\frac{Q_e}{m_e}t - \sin B\frac{Q_e}{m_e}t\right]. \qquad ▶[129]$$

These equations for the components of motion parallel to the x and y axes may be recognized as being those for a cycloid. Such a path for the electron is shown in Fig. 17. The x co-ordinate of the electron always lies between zero and

$$\frac{2E}{B^2(Q_e/m_e)d}.$$

The y co-ordinate, however, increases continuously with time. The average velocity along the y axis is E/Bd. The distance along the y axis between the points for which x is zero is

$$\frac{2\pi E}{B^2(Q_e/m_e)d}.$$

If d is greater than the maximum x co-ordinate,

$$\frac{2E}{B^2(Q_e/m_e)d},$$

an electron cannot reach the positive plate. Therefore, for a fixed separation, a magnetic flux density greater than the critical magnetic flux density,

$$B_0 = \frac{1}{d}\sqrt{2Em_e/Q_e}, \qquad [130]$$

is sufficient to interrupt the electronic current between the electrodes.

The relation given by Eq. 130 was used to measure[35] the ratio of charge to mass of the particles emitted by the photoelectric effect, which were thus identified as electrons.

Because uniform fields are seldom established with the electrode configuration of limited size employed in practical electronic devices, the expressions derived above are useful only as a guide to the conditions for critical control or production of the desired paths, such as those in the electron multiplier. In the electron multiplier the electric field is nonuniform, and the arrangement of the electrodes produces longitudinal as well as transverse components of varying magnitude with respect to position in the tube. The path of the electrons in that device therefore only approximately resembles a cycloid.

Most present-day electron multipliers do not require an auxiliary magnetic field to produce the desired path of the electrons.[36] The electrodes are shaped so that the electrostatic field alone causes the electrons to impinge upon the electrodes in succession. Figure 28 of Ch. II shows one configuration of electrodes used. Discussion of the operating characteristics of multiplier phototubes is given in Art. 13, Ch. II.

8d. *Magnetron.* Another practical application of the motion of electrons in perpendicular electrostatic and magnetostatic fields occurs in an electron tube called the *magnetron*.[37] Its chief application is as an oscillator to generate high radio-frequency power for such purposes as microwave radar.[38] In its elementary form the magnetron consists of a

[35] J. J. Thomson, "On the Masses of Ions in Gases at Low Pressures," *Phil. Mag.*, *48* (1899), 547–567.

[36] V. K. Zworykin and J. A. Rajchman, "The Electrostatic Electron Multiplier," *I.R.E. Proc.*, *27* (1939), 558–566; C. C. Larson and H. Salinger, "Photocell Multiplier Tubes," *R.S.I.*, *11* (1940), 226–229; J. S. Allen, "Recent Applications of Electron Multiplier Tubes," *I.R.E. Proc.*, *38* (1950), 346–358.

[37] For early work along these lines, see: H. Gerdien, United States Patent, 1,004,012 (September 26, 1911); A. W. Hull, "The Effect of a Uniform Magnetic Field on the Motion of Electrons between Coaxial Cylinders," *Phys. Rev.*, *18* (1921), 31–57; E. G. Linder, "Description and Characteristics of the End-Plate Magnetron," *I.R.E. Proc.*, *29* (1936), 633–653.

[38] G. B. Collins, Editor, *Microwave Magnetrons*, Massachusetts Institute of Technology Radiation Laboratory Series, Vol. 6 (New York: McGraw-Hill Book Company, Inc., 1948); K. R. Spangenberg, *Vacuum Tubes* (New York: McGraw-Hill Book Company, Inc., 1948).

cylindrical anode and a coaxial cylindrical cathode inside an evacuated chamber as shown in Fig. 18a. The electrons emitted by the cathode are attracted to the anode by the radial electric field set up by an external source of voltage connected between the anode and cathode. If a sufficiently strong longitudinal magnetic field is impressed parallel to the axis, the paths of the electrons are curved, and they can be prevented from reaching the anode. The magnetic field is therefore a means

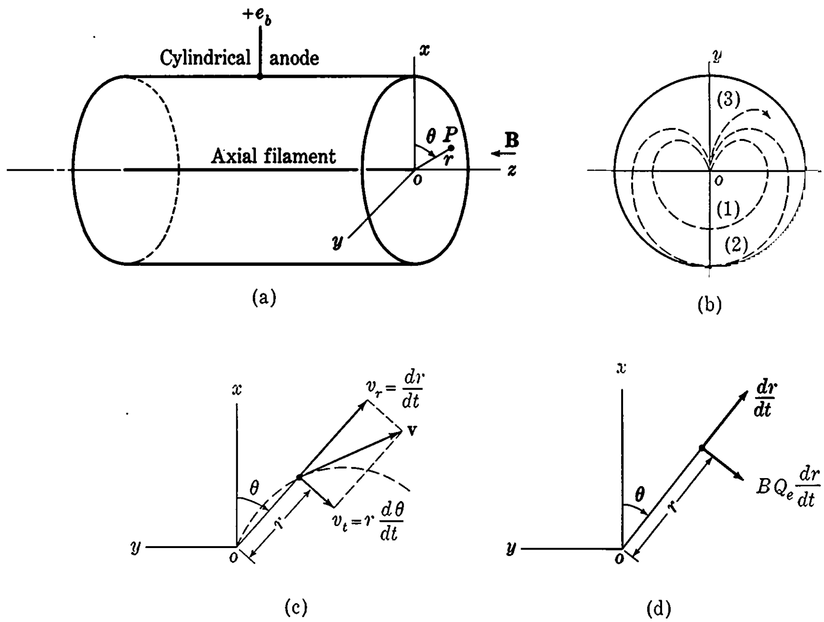


Fig. 18. Electron trajectories in a magnetron.

of current control. Furthermore, under certain conditions mentioned subsequently, the tube may exhibit negative resistance, and the magnetron may then be employed in conjunction with a resonant circuit to generate oscillations of a very high frequency.

Electrons that leave the cathode of the magnetron follow curved trajectories such as paths 1, 2, or 3 in Fig. 18b for decreasing strengths of the magnetic field. The trajectory 2 corresponds to the condition that the electrons graze the anode. For magnetic flux densities greater than the critical value that corresponds to trajectory 2, the electrons miss the anode and are turned back to the filament; for magnetic flux densities smaller than this critical value, all the electrons reach the anode. A determination of this critical value of magnetic flux density, denoted by B_0, is therefore of interest.

In the derivation of the relation involving the plate voltage e_b, the critical magnetic flux density B_0, and the dimensions of the tube, it is necessary to make certain simplifying assumptions in order to obtain a solution with reasonable effort. The tube has cylindrical symmetry, and cylindrical co-ordinates can be used to advantage if the fringing of the electric field at the ends of the cylinder is considered negligible. Accordingly, it is assumed that (a) the effect of the fringing of the electric field is negligible, (b) the electrons are emitted from the cathode with zero initial velocity, (c) the radius of the cathode is small compared with the radius of the plate, and (d) the voltage drop along the cathode is small compared with the plate voltage e_b. The relations derived on the basis of these assumptions are found to be in good agreement with the results of experimental measurement of the characteristics of such tubes.

To utilize the cylindrical symmetry, it is convenient to express the position of an electron between the filament and anode, say at point P in Fig. 18a, in terms of the radius r and the angle θ. The solution is independent of z, since the fields are oriented so that the electron experiences no component of force along the z axis. The radius of the plate is denoted by r_p, and that of the cathode by r_k, where r_k is small compared with r_p. A solution for the trajectory might be obtained by employment of Eq. 108 in cylindrical–co-ordinate form and substitution of boundary conditions. However, in the following analysis the relations are built up as they are needed from fundamental physical principles.

For the critical magnetic flux density, the electron grazes the plate, and the radial component of its velocity dr/dt is therefore zero when r is equal to r_p. In order to utilize this important boundary condition, the first objective of this analysis is to obtain an expression involving dr/dt; and the second objective is to evaluate each term of this expression for the point on the grazing trajectory at which it touches the plate. A critical relation involving the plate voltage e_b, the critical flux density B_0, and the dimensions of the tube for the grazing condition is thereby obtained.

The velocity of the electron at any point may be resolved into a radial component

$$v_r = \frac{dr}{dt} \qquad [131]$$

and a tangential component

$$v_t = r\frac{d\theta}{dt}, \qquad [132]$$

as is shown in Fig. 18c, and the square of the speed of the electron is then

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2. \qquad [133]$$

Each term of Eq. 133 may be evaluated for the condition of an electron grazing the plate. Because of the boundary condition already stated,

$$\frac{dr}{dt} = 0 \qquad [134]$$

for a grazing electron at the plate, and, from Eq. 30, if the initial emission velocity of the electron is neglected,

$$v^2 = 2\frac{Q_e}{m_e}e_b \qquad [135]$$

when the electron reaches the plate. Evaluation of the third term of Eq. 133 is then the only remaining step.

This third term may be determined through equating the torque tending to rotate the electron about the axis of symmetry of the tube to the rate of change of the angular momentum of the electron about the same center. The torque is the result of the radial velocity component dr/dt and the magnetic flux density B producing the tangential force $BQ_e(dr/dt)$ and hence the torque $rBQ_e(dr/dt)$. The angular momentum is the product of the moment of inertia m_er^2 and the angular velocity $d\theta/dt$, and therefore

$$\frac{d}{dt}\left(m_er^2\frac{d\theta}{dt}\right) = rBQ_e\frac{dr}{dt}, \qquad [136]$$

which, upon integration, gives

$$m_er^2\frac{d\theta}{dt} = \frac{1}{2}r^2BQ_e + C. \qquad [137]$$

The constant of integration C may be evaluated from the initial conditions. When

$$t = 0,\ r = r_k \ \text{ and } \ \frac{d\theta}{dt} = 0; \qquad [138]$$

thus

$$C = -\frac{1}{2}r_k^2BQ_e, \qquad [139]$$

and

$$m_er^2\frac{d\theta}{dt} = \frac{1}{2}BQ_e(r^2 - r_k^2). \qquad [140]$$

For an electron to graze the plate, B must have its critical value B_0 and r must be r_p. Since r_k is assumed to be much smaller than

r_p, r_k^2 may be neglected in comparison with r_p^2. Thus, from Eq. 140 is obtained

$$r\frac{d\theta}{dt} = \frac{1}{2} B_0 \frac{Q_e}{m_e} r_p \qquad [141]$$

as the value to be used in the third term of Eq. 133 when the electron just grazes the plate.

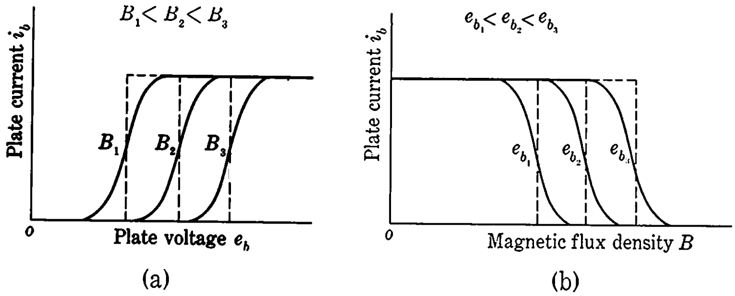


Fig. 19. Characteristic curves of a magnetron.

The values from Eqs. 134, 135, and 141 may be substituted into Eq. 133 to obtain the relation

$$2\frac{Q_e}{m_e} e_b = \frac{1}{4} B_0^2 \left(\frac{Q_e}{m_e}\right)^2 r_p^2 \qquad [142]$$

or

$$B_0 = \frac{1}{r_p}\sqrt{8\frac{m_e}{Q_e} e_b}\,. \qquad [143]$$

Substitution of the numerical value of $\frac{Q_e}{m_e}$ from Eq. 3 yields, in the mks system,

$$B_0 = 6.74 \times 10^{-6} \frac{\sqrt{e_b}}{r_p} \text{ webers per square meter.} \qquad ▶[144]$$

Equations 143 and 144 give the critical value of the magnetic flux density at which the anode current is interrupted as the flux density is increased. They show that the current in a tube with a large anode radius and small plate voltage may be interrupted by a smaller magnetic flux density than for the opposite conditions. Fig. 19 presents the general form of the characteristic curves of a cylindrical-anode magnetron. The dotted lines indicate the idealized characteristics based on Eq. 143; the solid lines the experimental ones. The difference between the idealized and the experimental curves results from the influence of

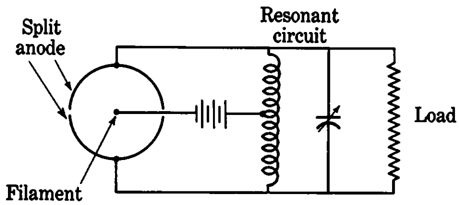


Fig. 20. Magnetron oscillator circuit.

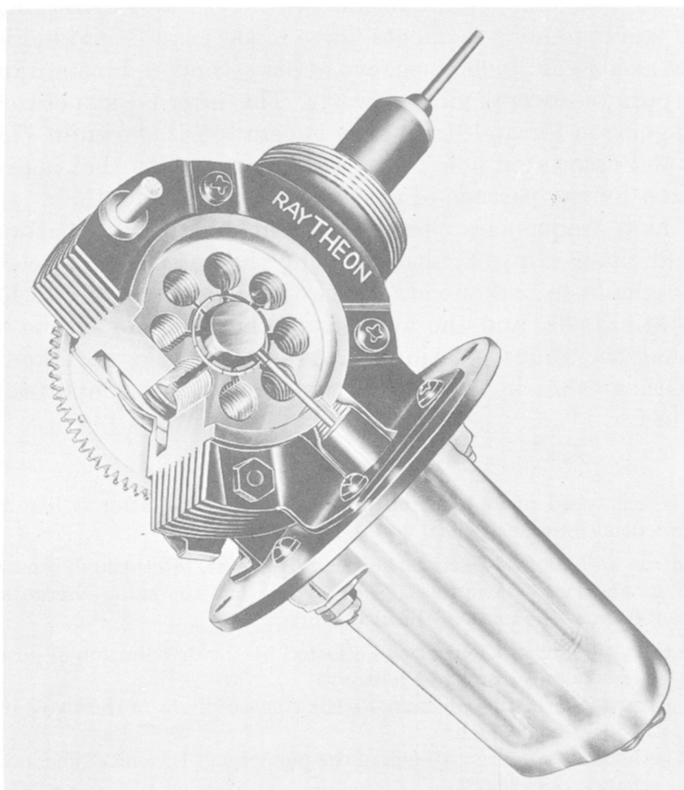


Fig. 21. Cutaway view showing internal construction of a magnetron capable of generating 50 kilowatts of peak power at 3200 megacycles per second under pulsed conditions. (*Courtesy Raytheon Manufacturing Company.*)

such causes as the initial velocities of the electrons at the cathode, the fringing of the electric field, the voltage drop along the cathode, and deviations of the cathode from true axial alignment.

An early form of the magnetron for use as an oscillator to generate high-frequency power involves separation of the anode into two segments[39] as is shown in Fig. 20. The two semicylinders are connected to the opposite ends of a resonant circuit with a return path through a direct-voltage source to the cathode. Frequencies of the order of 300 megacycles per second and higher can be generated in this way with relatively high efficiency.

Placing the resonant circuit inside the tube,[40] and increasing the number of pairs of anode segments, have made possible production of frequencies as high as 30,000 megacycles per second and instantaneous power outputs of several million watts. The internal structure of a typical magnetron for high-frequency generation is shown in Fig. 21. Each slot and associated hole forms a resonant cavity that is excited to oscillation by the passage of electrons across its mouth.

At the high frequencies produced by magnetron oscillators the electric field changes in magnitude during the passage of an electron from the cathode to the anode. Hence the conditions are no longer those of static fields, and the analysis of the behavior of the tubes requires that the time variation of the electric field be taken into account. Other tubes in which this time variation is important are discussed in Ch. IV.

PROBLEMS

1. What is the speed of an electron in miles per second after it has moved through a potential difference of 10 volts?

2. An electron and another particle, starting from rest, fall through a potential difference E in an electrostatic field. The particle has the same charge as the electron and 400 times the mass of the electron.

(a) Is the time of flight of the particles affected by the distribution of potential which they encounter on their journey?

(b) For a given potential distribution, do the times of flight of the two particles differ? If so, by how much?

(c) What is the kinetic energy of each of the particles at the end of the journey?

3. Assume the parallel plane plates of Fig. 2 to be large and spaced 1 cm apart in vacuum. Instead of the battery shown, an impulse generator with polarity as

[39] G. R. Kilgore, "Magnetron Oscillators for the Generation of Frequencies between 300 and 600 Megacycles," *I.R.E. Proc.*, *24* (1936), 1140–1157; L. Rosen, "Characteristics of a Split-Anode Magnetron Oscillator as a Function of Slot Angle," *R.S.I.*, *9* (1938), 372–373.

[40] E. G. Linder, "The Anode-Tank-Circuit Magnetron," *I.R.E. Proc.*, *27* (1939), 732–738; J. B. Fisk, H. D. Hagstram, and P. L. Hartman, "The Magnetron as a Generator of Centimeter Waves," *B.S.T.J.*, *25* (1946), 1-188.

indicated supplies a potential difference which increases at a uniform rate from zero to one volt in 10^{-8} sec. At the end of this time the plates are short-circuited.

(a) If an electron is at rest very near the negative plate at the beginning of the voltage pulse, where is it at the time of short circuit?
(b) What is the total electron transit time to the positive plate?

4. Two large plane plates separated by a distance of 1 cm in a high vacuum are arranged as shown in Fig. 2. Instead of the battery, an alternating square wave of voltage having an amplitude of 1 volt and a period of 2×10^{-8} sec is impressed between the plates.

If an electron is at rest near the negative plate at the beginning of the first cycle, where will it be at the end of that cycle?

5. Two large plates separated by a distance d in a high vacuum are arranged as shown in Fig. 2. Instead of the battery a voltage source of instantaneous value

$$e_b = E_m \cos (\omega t + \psi)$$

is applied between the plates. The electric field between the plates is essentially uniform at any instant of time. Electrons are emitted from the left-hand plate and may be assumed to have zero initial velocity.

(a) Derive a general expression for the position of an emitted electron with respect to the left-hand plate as a function of time.
(b) Is it possible that some of the emitted electrons may never reach the right-hand plate? Either show that this condition cannot occur, or derive the conditions for which it can occur.

6. An electron is initially at rest at the surface of the left-hand plate of the parallel-plate configuration shown in Fig. 2. Instead of the battery a sinusoidal voltage source having a 1-volt peak value and a frequency of 50 megacycles per second is applied to the plates. The voltage is passing through its zero value at the instant of application and is increasing in the direction which makes the right-hand plate positive. The plates are infinite in extent and are placed 1 cm apart in vacuum.

(a) What is the maximum speed attained by the electron, and at what positions does this speed occur?
(b) What is the speed of the electron at time t equals 2×10^{-8} sec?
(c) Describe completely the motion of the electron.

7. The screen of an electron oscilloscope is 20 cm from the anode. Two electrons pass through the aperture in the anode traveling in parallel paths 0.1 mm apart.

(a) The electrons are 0.11 mm apart when they reach the screen. What is their transit time from anode to screen?
(b) If it is assumed that the electrons started from rest at the cathode, what is the accelerating voltage of the anode?
(c) Determine the anode accelerating voltage corresponding to an electron separation of 0.101 mm at the screen.

8. The voltages E_1 and E in the diagram of Fig. 22 each have a magnitude of 100 volts. An electron starts from rest at the left-hand electrode, passes through the hole in the center diaphragm, and finally reaches the anode on the right. The left-hand and right-hand electrodes are each 1 cm from the centrally located diaphragm.

(a) With what energy in electron volts does the electron strike the electrode at the right?

(b) On the assumption that the electrodes are infinite parallel plates, what is the time of transit of the electron between the left-hand electrode and the diaphragm?

(c) On the same assumption, what is the time of transit of the electron between the diaphragm and the right-hand electrode?

(d) If the voltage E is changed to 200 volts, what then is the answer to (c)?

9. An electron is moving at a speed of 10^8 cm per sec. Through how many volts of potential difference must it pass to double its speed?

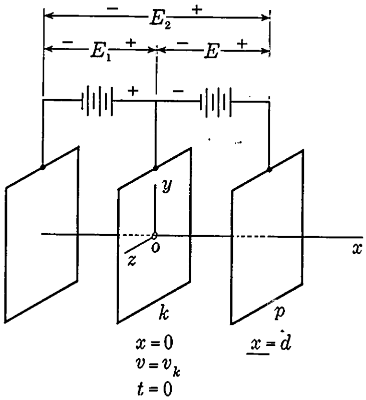


Fig. 22. Electrode configuration for Prob. 8.

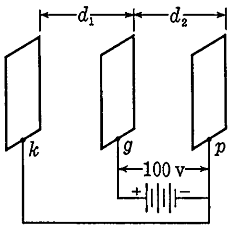


Fig. 23. Electrode configuration for idealized Barkhausen oscillator of Prob. 11.

10. A cathode-ray oscilloscope is built with a fluorescent screen deposited on a metallic conducting plate. A lead is available from the conducting screen. Three different connections are tried as follows:

(1) The screen is left without connection, that is, insulated from the anode.

(2) The screen is connected through a high resistance to the anode.

(3) The screen is connected directly through a low-resistance wire to the anode.

If secondary emission from the screen is neglected:

(a) How do the velocities of the electrons as they reach the screen differ in the three cases?

(b) Which of the connections should be employed in practical use of the tube?

11. In a Barkhausen oscillator, the grid of a triode is maintained at a positive potential, and the plate is maintained at essentially the cathode potential. The transit time of the electrons in such an electrode configuration is to be found.

In order to obtain an approximate solution, consider that the vacuum triode has parallel plane electrodes as shown in the diagram, Fig. 23, with interelectrode spacings d_1 and d_2, each equal to 1 cm. The plate p and cathode k are connected together, and the grid g is maintained at a positive potential of 100 volts with respect to them. The space charge is negligible, so that the potential distribution is an inverted **V** with its apex at the grid. An electron leaves the cathode with zero velocity. It may be assumed that the electron passes freely through the grid structure.

(a) What time elapses before the electron reaches the anode?
(b) On the assumption that the grid may be moved with respect to the plate and cathode, show what effect the location of the grid has on the transit time of an electron from cathode to anode. Is there a position of the grid that gives a minimum transit time? If so, where is it?

12. Find the electrostatic deflection sensitivity of the cathode-ray tube of Fig. 5 in volts per centimeter, if d is 1 cm, l is 4.5 cm, and l_s is 33 cm, for accelerating voltages E_b of 300, 1,000, 5,000, and 10,000 volts.

13. Electrons accelerated by a potential difference of 1,000 volts are shot tangentially into the interspace between two concentric semicylinders of radii 4 cm and 5 cm respectively. What potential difference must be applied between the plates to make the electron paths circular and concentric with the plates?

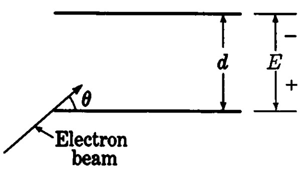


Fig. 24. Configuration for Prob. 14.

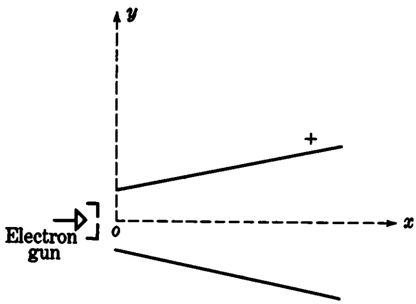


Fig. 25. Diverging plate arrangement of Prob. 15.

14. Two large capacitor plates are spaced a distance d apart and charged to a potential difference E as shown in Fig. 24. It may be assumed that the field intensity between the plates is uniform and that there is no fringing. A stream of electrons is projected at a velocity corresponding to an accelerating voltage E_0 into the electric field at the edge of the lower plate and at an angle θ with the plates.

(a) Find the relation among the voltages and θ necessary in order that the electrons may just miss the upper plate.
(b) If θ is 30°, d is 1 cm, and E is 50 volts, what initial kinetic energy should the electrons in the beam have in order that they may just miss the upper plate? Express the result in electron volts and joules.

15. An electron gun shoots an electron horizontally into an electric field established by a pair of diverging plane electrodes, with a velocity corresponding to a kinetic energy of 400 electron volts, as shown in Fig. 25. At any distance x from the origin, the electric field intensity is assumed to be given by the following approximate expression

$$\mathcal{E}_y = \frac{200}{x + 4} \text{ volts per cm,}$$

where x is in centimeters, and the field may be considered to be entirely in the vertical direction and to be directed upward within the space under consideration.

Using the axes indicated in the figure, find the equation of the path of the electron in the form

$$y = f(x).$$

16. The mass of an alpha particle is 7,344 times that of an electron. Its charge is twice that of an electron. If its velocity corresponds to a fall through a potential of 5×10^6 volts, what magnetic field is required to bend its path into a circle of radius 1 meter?

17. Find the magnetic deflection sensitivity of the cathode-ray oscillograph tube of Fig. 11 in centimeters per gauss for accelerating voltages of 100, 1,000, 5,000, and 10,000 volts. The distance l is 5 cm, and l_s is 35 cm.

18. An electron and an ionized hydrogen atom are projected into a uniform magnetic field of 10 gausses. Their velocities correspond to 300 volts, and they enter in a direction normal to the magnetic field.

(a) Find the ratio of the diameters of the circular paths traced by the particles.
(b) Find the time required for each of the particles to make a complete revolution.

19. An electron oscilloscope (see Fig. 5) is located with its axis parallel to and spaced 2 meters from a long conductor carrying direct current. The return conductor for the direct current is very far away, so that the magnetic field from it is negligible. The distance l_s from the anode to the screen is 50 cm, and the accelerating voltage E_b is 1,500 volts.

(a) What current in the conductor will cause a deflection of 1 cm?
(b) Is it possible to reorient the tube so that the deflection is zero for all values of current in the conductor?

20. Electrons are projected into a uniform magnetic field of 100 gausses. The speed of the electrons corresponds to 300 volts, and the beam makes an angle of 30° with the direction of the magnetic field. Find the diameter and pitch of the spiral path described by the electrons.

21. It is desired to find some of the relations among the size of a cyclotron and the voltage, frequency, and magnetic flux density applied to it.

In the cyclotron a proton is whirled in a succession of semicircular paths of increasing radius, and is alternately under the influence of an electric field which imparts linear acceleration and under that of a magnetic field which bends its path of flight. As indicated in the plan view of Fig. 12, the electric field lies in the plane of the paper and is normal to the magnetic field, which is perpendicular to and into the paper. The path of the proton is also in the plane of the paper. The magnetic field may be considered uniform in this plane, and its effect within the slit between the dees may be neglected because of the narrowness of the slit. The electric field, however, may be assumed to exist only in this slit and to be produced by an alternating potential difference

$$e = E \cos \omega t.$$

Owing to the narrowness of the slit, the time of flight of the proton through the electric field is small compared to the time it is acted upon by the magnetic field alone, so that if the proton is assumed to cross the slit at the times t equals 0, π/ω, $2\pi/\omega$, etc., the corresponding values of e are $+E$, $-E$, $+E$, etc.

(a) Derive the analytic relation between the angular frequency ω and the magnetic flux density B which must exist in order that the proton may cross the slit at the specified time intervals.

(b) If the speed of the proton in its final semicircular path is to correspond to 10^6 volts, and the magnetic flux density is 1,000 gausses, what will be the radius of the final path and what must be the angular frequency ω of the voltage e?

(c) Is the direction of travel of the proton counterclockwise as shown or clockwise?

22. Electrons emerge from a small hole in the anode of a cathode-ray tube in all directions contained in a cone of small angle. The accelerating potential difference between cathode and anode is 200 volts. The distance from the anode to the screen is 20 cm, and the cathode is 5 mm from the anode.

The tube is placed in a long solenoid of 10 cm diameter having six turns per centimeter along its entire length, the tube and solenoid axes coinciding.

What is the smallest value of current in amperes in the coil that will produce a magnetic effect on the beam of electrons sufficient to cause them to focus on a spot the same size as the hole but at the screen?

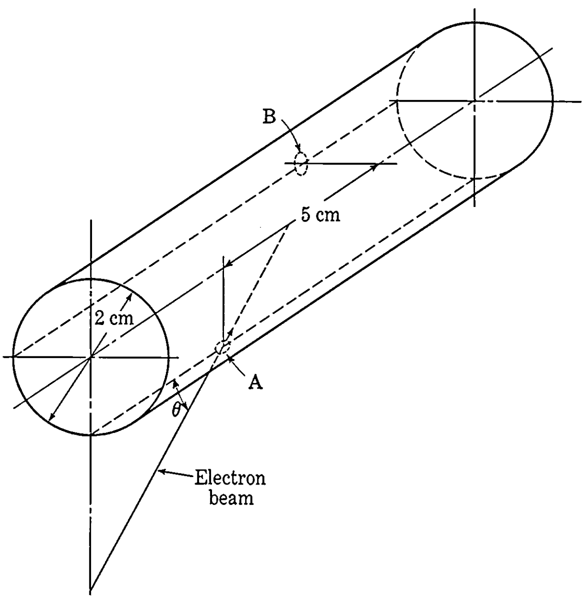


Fig. 26. Hollow cylinder of Prob. 23.

23. An electron gun directs a beam of 1,000-volt electrons through a hole A in the wall of a hollow cylinder 2 cm in diameter, as indicated in Fig. 26. The axis of the gun intersects that of the cylinder at an angle θ. The cylinder wall has another hole B, removed 90° circumferentially and 5 cm axially from the hole A.

Determine the angle θ, and the density B and direction of the uniform axial magnetic field required to cause the electron beam to emerge from the hole B.

24. A region in space is permeated by a constant electric field parallel to the x axis and a constant magnetic field parallel to the y axis in a system of rectangular co-ordinates.

If an electron is projected into this space with a finite velocity, what are the necessary and sufficient conditions that must be fulfilled if is it to remain undeflected?

25. An electron leaves the origin with an initial velocity $\mathbf{v}_0$ in the positive direction along the x axis. There are a uniform electric field of intensity $\mathcal{E}$ in the positive x direction and a uniform magnetic field of flux density B in the positive z direction.

(a) What differential equations determine the motion of the particle in the x–y plane?
(b) How can these equations be solved?

26. A magnetron contains an axial tungsten filament of 0.03 cm diameter. The plate is a 1-cm diameter right-circular cylinder concentric with the filament, and it is 100 volts positive with respect to the filament. A magnetic flux density of 50 gausses is impressed parallel to the filament. Consider that the potential of all parts of the filament is the same and that the lengths of the filament and plate are equal and are large compared with the diameter of the plate.

(a) What is the tangent of the angle between the velocity vector and the radius at the point where an electron strikes the plate?
(b) What is the ratio of the speed with which an electron strikes the plate when the magnetic field is on and the speed with which it strikes when the field is off?

27. Find the approximate path of an electron emitted from the filament of a cylindrical-anode magnetron. The filament radius is r_k and is very small compared to the anode radius r_p. Assume no space charge and assume that the initial velocity of the electron at the filament surface is negligible.